Algebraic methods 1A

1 B At least one multiple of three is odd.

2 **a** At least one rich person is not happy.

b There is at least one prime number between 10 million and 11 million.

c If p and q are prime numbers there exists a number of the form $(pq + 1)$ that is not prime.

d There is a number of the form $2^n - 1$ that is either not prime or not a multiple of 3.

e None of the above statements is true.

3 **a** There exists a number n such that n^2 is odd but n is even.

b n is even so write $n = 2k$
$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$
So n^2 is even.
This contradicts the assumption that n^2 is odd.
Therefore, if n^2 is odd then n must be odd.

4 **a** Assumption: there is a greatest even integer, $2n$.
$2(n + 1)$ is also an integer and
$2(n + 1) > 2n$
$2n + 2 =$ even + even = even
So there exists an even integer greater that $2n$.
This contradicts the assumption that the greatest even integer is $2n$.
Therefore there is no greatest even integer.

b Assumption: there exists a number n such that n^3 is even but n is odd.
n is odd so write $n = 2k + 1$
$n^3 = (2k + 1)^3$
$= 8k^3 + 12k^2 + 6k + 1$
$= 2(4k^3 + 6k^2 + 3k) +1$
So n^3 is odd.
This contradicts the assumption that n^3 is even.
Therefore, if n^3 is even then n must be even.

c Assumption: if pq is even then neither p nor q is even.
p is odd, $p = 2k + 1$
q is odd, $q = 2m + 1$
$pq = (2k +1)(2m + 1)$
$= 2km + 2k +2m +1$
$= 2(km + k + m) +1$
So pq is odd.
This contradicts the assumption that pq is even.
Therefore, if pq is even then at least one of p and q is even.

4 d Assumption: if $p + q$ is odd than neither p nor q is odd.
p is even, $p = 2k$
q is even, $q = 2m$
$p + q = 2k + 2m = 2(k + m)$
So $p + q$ is even.
This contradicts the assumption that $p + q$ is odd.
Therefore, if $p + q$ is odd then at least one of p and q is odd.

5 a Assumption: if ab is an irrational number then neither a nor b is irrational.

a is rational, $a = \frac{c}{d}$ where c and d are integers.

b is rational, $b = \frac{e}{f}$ where e and f are integers.

$ab = \frac{ce}{df}$, ce is an integer, df is an integer.
Therefore ab is a rational number.
This contradicts assumption that ab is irrational.
Therefore, if ab is an irrational number then at least one of a and b is an irrational number.

b Assumption: if $a + b$ is an irrational number then neither a nor b is irrational.

a is rational, $a = \frac{c}{d}$ where c and d are integers.

b is rational, $b = \frac{e}{f}$ where e and f are integers.

$a + b = \frac{cf + de}{df}$, cf, de and df are integers.
So $a + b$ is rational. This contradicts the assumption that $a + b$ is irrational.
Therefore if $a + b$ is irrational then at least one of a and b is irrational.

c Many possible answers
e.g. $a = 2 - \sqrt{2}$, $b = \sqrt{2}$.

6 Assumption: there exist integers a and b such that $21a + 14b = 1$.
Since 21 and 14 are multiples of 7, divide both sides by 7.

So now $3a + 2b = \frac{1}{7}$

$3a$ is also an integer. $2b$ is also an integer.
The sum of two integers will always be an integer, so $3a + 2b$ is an integer.
This contradicts the statement that

$3a + 2b = \frac{1}{7}$

Therefore there exist no integers a and b for which $21a + 14b = 1$.

7 **a** Assumption: There exists a number n such that n^2 is a multiple of 3, but n is not a multiple of 3.
All multiples of 3 can be written in the form $n = 3k$ where k is an integer, therefore $3k + 1$ and $3k + 2$ are not multiples of 3.
Let $n = 3k + 1$
$n^2 = (3k + 1)^2$
$= 9k^2 + 6k + 1$
$= 3(3k^2 + 2k) + 1$
In this case n^2 is not a multiple of 3.
Let $n = 3k + 2$
$n^2 = (3k + 2)^2$
$= 9k^2 + 12k + 4$
$= 3(3k^2 + 4k + 1) + 1$
In this case n^2 is also not a multiple of 3.
This contradicts the assumption that n^2 is a multiple of 3.
Therefore if n^2 is a multiple of 3, n is a multiple of 3.

b Assumption: $\sqrt{3}$ is a rational number.
Then $\sqrt{3} = \frac{a}{b}$ for some integers a and b.
Further assume that this fraction is in its simplest terms:
there are no common factors between a and b.
So $3 = \frac{a^2}{b^2}$ or $a^2 = 3b^2$
Therefore a^2 must be a multiple of 3.
We know from part **a** that this means a must also be a multiple of 3.
Write $a = 3c$, which means $a^2 = (3c)^2 = 9c^2$.
Now $9c^2 = 3b^2$, or $3c^2 = b^2$.
Therefore b^2 must be a multiple of 3, which means b is also a multiple of 3.
If a and b are both multiples of 3, this contradicts the statement that there are no common factors between a and b.
Therefore, $\sqrt{3}$ is an irrational number.

8 Assumption: there is an integer solution to the equation $x^2 – y^2 = 2$.
Remember that $x^2 – y^2 = (x – y)(x + y) = 2$.
To make a product of 2 using integers, the possible pairs are: (2, 1), (1, 2), (–2, –1) and (–1, –2).
Consider each possibility in turn:
$x – y = 2$ and $x + y = 1 \Rightarrow x = \frac{3}{2}, y = -\frac{1}{2}$
$x – y = 1$ and $x + y = 2 \Rightarrow x = \frac{3}{2}, y = \frac{1}{2}$
$x – y = –2$ and $x + y = –1 \Rightarrow x = -\frac{3}{2}, y = \frac{1}{2}$
$x – y = –1$ and $x + y = –2 \Rightarrow x = -\frac{3}{2}, y = -\frac{1}{2}$
This contradicts the statement that there is an integer solution to the equation $x^2 – y^2 = 2$.
Therefore the original statement must be true:

8 (continued)
There are no integer solutions to the equation $x^2 - y^2 = 2$.

9 Assumption: $\sqrt[3]{2}$ is a rational number

Then $\sqrt[3]{2} = \frac{a}{b}$ for some integers a and b.

Further assume that this fraction is in its simplest terms:
there are no common factors between a and b.
This means that if a^3 is even, a must also be even.
If a is even, $a = 2n$.
So $a^3 = 2b^3$ becomes $(2n)^3 = 2b^3$ which means $8n^3 = 2b^3$ or $4n^3 = b^3$ or $2(2n^3) = b^3$
This means that b^3 must be even, so b is also even.
If a and b are both even, they will have a common factor of 2.
This contradicts the statement that a and b have no common factors.
We can conclude the original statement is true: $\sqrt[3]{2}$ is an irrational number.

10 a The number $\frac{a-b}{b}$ could be negative.

e.g. If $n = \frac{1}{2}$, $n - 1$ is non-positive.

b Assumption: There is a least positive rational number, n.

$n = \frac{a}{b}$ where a and b are integers.

Let $m = \frac{a}{2b}$. Since a and b are integers, m is rational and $m < n$.
This contradicts the statement that n is the least positive rational number.
Therefore, there is no least positive rational number.

Algebraic methods 1B

1 a $$\frac{a}{d}\times\frac{a}{c}=\frac{a\times a}{d\times c}=\frac{a^2}{cd}$$

b $$\frac{a^{\cancel{2}}}{\cancel{c}_1}\times\frac{\cancel{c}^1}{\cancel{a}_1}=\frac{a\times 1}{1\times 1}=a$$

c $$\frac{\cancel{2}^1}{\cancel{x}_1}\times\frac{\cancel{x}^1}{\cancel{4}_2}=\frac{1\times 1}{1\times 2}=\frac{1}{2}$$

d $$\frac{3}{x}\div\frac{6}{x}=\frac{\cancel{3}^1}{\cancel{x}_1}\times\frac{\cancel{x}^1}{\cancel{6}_2}=\frac{1\times 1}{1\times 2}=\frac{1}{2}$$

e $$\frac{4}{xy}\div\frac{x}{y}=\frac{4}{x\cancel{y}_1}\times\frac{\cancel{y}^1}{x}=\frac{4\times 1}{x\times x}=\frac{4}{x^2}$$

f $$\frac{2r^2}{5}\div\frac{4}{r^3}=\frac{{}^1\cancel{2}r^2}{5}\times\frac{r^3}{\cancel{4}_2}=\frac{r^5}{10}$$

2 **a** $$(x+2)\times\frac{1}{x^2-4}=\frac{\cancel{(x+2)}}{\cancel{(x+2)}(x-2)}$$
$$=\frac{1}{1\times(x-2)}$$
$$=\frac{1}{x-2}$$

b $$\frac{1}{a^2+6a+9}\times\frac{a^2-9}{2}$$
$$=\frac{1}{\cancel{(a+3)}(a+3)}\times\frac{\cancel{(a+3)}(a-3)}{2}$$
$$=\frac{a-3}{2(a+3)}$$

c $$\frac{x^2-3x}{y^2+y}\times\frac{y+1}{x}$$
$$=\frac{\cancel{x}^1(x-3)}{y\cancel{(y+1)}_1}\times\frac{\cancel{(y+1)}^1}{\cancel{x}_1}$$
$$=\frac{x-3}{y}$$

d $$\frac{y}{y+3}\div\frac{y^2}{y^2+4y+3}$$
$$=\frac{y}{y+3}\times\frac{y^2+4y+3}{y^2}$$
$$=\frac{\cancel{y}}{\cancel{y+3}}\times\frac{(y+1)\cancel{(y+3)}}{y^{\cancel{2}}}$$
$$=\frac{y+1}{y}$$

2 e $\frac{x^2}{3} \div \frac{2x^3-6x^2}{x^2-3x} = \frac{x^2}{3} \times \frac{x^2-3x}{2x^3-6x^2}$

$$= \frac{\cancel{x^2}}{3} \times \frac{x\,\cancel{(x-3)}^{1}}{2\cancel{x^2}\,\cancel{(x-3)}_{1}}$$

$$= \frac{1\times x}{3\times 2}$$

$$= \frac{x}{6}$$

f $\frac{4x^2-25}{4x-10} \div \frac{2x+5}{8}$

$$= \frac{4x^2-25}{4x-10} \times \frac{8}{(2x+5)}$$

$$= \frac{\cancel{(2x+5)}^{1}\,\cancel{(2x-5)}^{1}}{2\,\cancel{(2x-5)}_{1}} \times \frac{8}{\cancel{(2x+5)}_{1}}$$

$$= \frac{1\times 8}{2\times 1}$$

$$= 4$$

g $\frac{x+3}{x^2+10x+25} \times \frac{x^2+5x}{x^2+3x}$

$$= \frac{\cancel{x+3}^{1}}{\cancel{(x+5)}_{1}(x+5)} \times \frac{x^{1}\,\cancel{(x+5)}^{1}}{\cancel{x}_{1}\,\cancel{(x+3)}_{1}}$$

$$= \frac{1}{x+5}$$

h $\frac{3y^2+4y-4}{10} \div \frac{3y+6}{15}$

$$= \frac{3y^2+4y-4}{10} \times \frac{15}{3y+6}$$

$$= \frac{(3y-2)\,\cancel{(y+2)}^{1}}{\cancel{10}_{2}} \times \frac{\cancel{15}^{3}}{\cancel{3}\,\cancel{(y+2)}_{1}}$$

$$= \frac{3y-2}{2}$$

2 i $\frac{x^2+2xy+y^2}{2}\times\frac{4}{(x-y)^2}$

$$=\frac{(x+y)^2}{\cancel{2}_1}\times\frac{\cancel{4}^2}{(x-y)^2}$$

$$=\frac{2(x+y)^2}{(x-y)^2}$$

3 $\frac{x^2-64}{x^2-36}\div\frac{64-x^2}{x^2-36}$

$$=\frac{x^2-64}{x^2-36}\times\frac{x^2-36}{64-x^2}$$

$$=\frac{\cancel{(x+8)}(x-8)}{\cancel{(x+6)}\cancel{(x-6)}}\times\frac{\cancel{(x+6)}\cancel{(x-6)}}{\cancel{(8+x)}(8-x)}$$

$$=\frac{(x-8)}{(8-x)}$$

$$=\frac{(x-8)}{-(x-8)}$$

$$=-1$$

4 $\frac{2x^2-11x-40}{x^2-4x-32}\times\frac{x^2+8x+16}{6x^2-3x-45}\div\frac{8x^2+20x-48}{10x^2-45x+45}$

$$=\frac{2x^2-11x-40}{x^2-4x-32}\times\frac{x^2+8x+16}{6x^2-3x-45}\times\frac{10x^2-45x+45}{8x^2+20x-48}$$

$$=\frac{\cancel{(2x+5)}\cancel{(x-8)}}{\cancel{(x+4)}\cancel{(x-8)}}\times\frac{\cancel{(x+4)}\cancel{(x+4)}}{3\cancel{(2x+5)}\cancel{(x-3)}}\times\frac{5\cancel{(2x-3)}\cancel{(x-3)}}{4\cancel{(2x-3)}\cancel{(x+4)}}$$

$$=1\times\frac{1}{3}\times\frac{5}{4}$$

$$=\frac{5}{12}$$

$a=5$, $b=12$

5 a $\frac{x^2+2x-24}{2x^2+10x}\times\frac{x^2-3x}{x^2+3x-18}$

$$=\frac{\cancel{(x+6)}(x-4)}{2\cancel{x}(x+5)}\times\frac{\cancel{x}\cancel{(x-3)}}{\cancel{(x+6)}\cancel{(x-3)}}$$

$$=\frac{(x-4)}{2(x+5)}$$

$$=\frac{x-4}{2x+10}$$

5 b $$\ln\left(\left(x^2+2x-24\right)\left(x^2-3x\right)\right)$$
$$-\ln\left(\left(2x^2+10x\right)\left(x^2+3x-18\right)\right)=2$$

$$\ln\left(\frac{\left(x^2+2x-24\right)\left(x^2-3x\right)}{\left(2x^2+10x\right)\left(x^2+3x-18\right)}\right)=2$$

$$\ln\left(\frac{\cancel{(x+6)}\,(x-4)\,\cancel{x}\,\cancel{(x-3)}}{\cancel{x}\,(2x+10)\,\cancel{(x+6)}\,\cancel{(x-3)}}\right)=2$$

$$\ln\left(\frac{x-4}{2x+10}\right)=2$$

$$\frac{x-4}{2x+10}=\mathrm{e}^2$$

$$x-4=2x\mathrm{e}^2+10\mathrm{e}^2$$

$$x\left(1-2\mathrm{e}^2\right)=10\mathrm{e}^2+4$$

$$x=\frac{10\mathrm{e}^2+4}{1-2\mathrm{e}^2}$$

6 a $$\mathrm{f}(x)=\frac{2x^2-3x-2}{6x-8}\div\frac{x-2}{3x^2+14x-24}$$

$$=\frac{2x^2-3x-2}{6x-8}\times\frac{3x^2+14x-24}{x-2}$$

$$=\frac{(2x+1)\,\cancel{(x-2)}}{2\,\cancel{(3x-4)}}\times\frac{\cancel{(3x-4)}\,(x+6)}{\cancel{x-2}}$$

$$=\frac{(2x+1)(x+6)}{2}$$

$$=\frac{2x^2+13x+6}{2}$$

b $$\mathrm{f}(x)=x^2+\frac{13}{2}x+3$$

$$\mathrm{f}'(x)=2x+\frac{13}{2}$$

$$\mathrm{f}'(4)=2\times4+\frac{13}{2}=\frac{29}{2}$$

Algebraic Methods 1C

1 a $$\frac{1}{3}+\frac{1}{4}=\frac{4}{12}+\frac{3}{12}=\frac{7}{12}$$

b $$\frac{3}{4}-\frac{2}{5}=\frac{15}{20}-\frac{8}{20}=\frac{7}{20}$$

c $$\frac{1}{p}+\frac{1}{q}=\frac{q}{pq}+\frac{p}{pq}=\frac{p+q}{pq}$$

d $$\frac{3}{4x}+\frac{1}{8x}=\frac{6}{8x}+\frac{1}{8x}=\frac{7}{8x}$$

e $$\frac{3}{x^2}-\frac{1}{x}=\frac{3}{x^2}-\frac{x}{x^2}=\frac{3-x}{x^2}$$

f $$\frac{a}{5b}-\frac{3}{2b}=\frac{2a}{10b}-\frac{15}{10b}=\frac{2a-15}{10b}$$

2 a $$\frac{3}{x}-\frac{2}{x+1}=\frac{3(x+1)}{x(x+1)}-\frac{2x}{x(x+1)}=\frac{3x+3-2x}{x(x+1)}=\frac{x+3}{x(x+1)}$$

b $$\frac{2}{x-1}-\frac{3}{x+2}=\frac{2(x+2)}{(x-1)(x+2)}-\frac{3(x-1)}{(x-1)(x+2)}=\frac{2(x+2)-3(x-1)}{(x-1)(x+2)}=\frac{2x+4-3x+3}{(x-1)(x+2)}=\frac{-x+7}{(x-1)(x+2)}$$

c $$\frac{4}{2x+1}+\frac{2}{x-1}=\frac{4(x-1)}{(2x+1)(x-1)}+\frac{2(2x+1)}{(2x+1)(x-1)}=\frac{4(x-1)+2(2x+1)}{(2x+1)(x-1)}=\frac{4x-4+4x+2}{(2x+1)(x-1)}=\frac{8x-2}{(2x+1)(x-1)}$$

d $$\frac{1}{3}(x+2)-\frac{1}{2}(x+3)=\frac{2}{6}(x+2)-\frac{3}{6}(x+3)=\frac{2(x+2)-3(x+3)}{6}=\frac{2x+4-3x-9}{6}=\frac{-x-5}{6}$$

e $$\frac{3x}{(x+4)^2}-\frac{1}{x+4}=\frac{3x}{(x+4)^2}-\frac{x+4}{(x+4)^2}=\frac{3x-x-4}{(x+4)^2}=\frac{2x-4}{(x+4)^2}$$

2 f $$\frac{5}{2(x+3)}+\frac{4}{3(x-1)}$$
$$=\frac{15(x-1)}{6(x+3)(x-1)}+\frac{8(x+3)}{6(x+3)(x-1)}$$
$$=\frac{15(x-1)+8(x+3)}{6(x+3)(x-1)}$$
$$=\frac{15x-15+8x+24}{6(x+3)(x-1)}$$
$$=\frac{23x+9}{6(x+3)(x-1)}$$

3 a $$\frac{2}{x^2+2x+1}+\frac{1}{x+1}$$
$$=\frac{2}{(x+1)^2}+\frac{1}{x+1}$$
$$=\frac{2}{(x+1)^2}+\frac{x+1}{(x+1)^2}$$
$$=\frac{2+x+1}{(x+1)^2}$$
$$=\frac{x+3}{(x+1)^2}$$

b $$\frac{7}{x^2-4}+\frac{3}{x+2}$$
$$=\frac{7}{(x+2)(x-2)}+\frac{3}{x+2}$$
$$=\frac{7}{(x+2)(x-2)}+\frac{3(x-2)}{(x+2)(x-2)}$$
$$=\frac{7+3(x-2)}{(x+2)(x-2)}$$
$$=\frac{7+3x-6}{(x+2)(x-2)}$$
$$=\frac{3x+1}{(x+2)(x-2)}$$

c $$\frac{2}{x^2+6x+9}-\frac{3}{x^2+4x+3}$$
$$=\frac{2}{(x+3)^2}-\frac{3}{(x+3)(x+1)}$$
$$=\frac{2(x+1)}{(x+3)^2(x+1)}-\frac{3(x+3)}{(x+3)^2(x+1)}$$
$$=\frac{2(x+1)-3(x+3)}{(x+3)^2(x+1)}$$
$$=\frac{2x+2-3x-9}{(x+3)^2(x+1)}$$
$$=\frac{-x-7}{(x+3)^2(x+1)}$$

d $$\frac{2}{y^2-x^2}+\frac{3}{y-x}$$
$$=\frac{2}{(y+x)(y-x)}+\frac{3}{y-x}$$
$$=\frac{2}{(y+x)(y-x)}+\frac{3(y+x)}{(y+x)(y-x)}$$
$$=\frac{2+3(y+x)}{(y+x)(y-x)}$$
$$=\frac{3x+3y+2}{(y+x)(y-x)}$$

e $$\frac{3}{x^2+3x+2}-\frac{1}{x^2+4x+4}$$
$$=\frac{3}{(x+1)(x+2)}-\frac{1}{(x+2)^2}$$
$$=\frac{3(x+2)}{(x+1)(x+2)^2}-\frac{(x+1)}{(x+1)(x+2)^2}$$
$$=\frac{3(x+2)-(x+1)}{(x+1)(x+2)^2}$$
$$=\frac{3x+6-x-1}{(x+1)(x+2)^2}$$
$$=\frac{2x+5}{(x+1)(x+2)^2}$$

3 f

$$\frac{x+2}{x^2-x-12}-\frac{x+1}{x^2+5x+6}$$

$$=\frac{x+2}{(x-4)(x+3)}-\frac{x+1}{(x+2)(x+3)}$$

$$=\frac{(x+2)^2}{(x-4)(x+2)(x+3)}-\frac{(x+1)(x-4)}{(x-4)(x+2)(x+3)}$$

$$=\frac{(x+2)^2-(x+1)(x-4)}{(x-4)(x+2)(x+3)}$$

$$=\frac{x^2+4x+4-x^2+3x+4}{(x-4)(x+2)(x+3)}$$

$$=\frac{7x+8}{(x-4)(x+2)(x+3)}$$

4

$$\frac{6x+1}{x^2+2x-15}-\frac{4}{x-3}$$

$$=\frac{6x+1}{(x+5)(x-3)}-\frac{4}{x-3}$$

$$=\frac{6x+1}{(x+5)(x-3)}-\frac{4(x+5)}{(x+5)(x-3)}$$

$$=\frac{6x+1-4(x+5)}{(x+5)(x-3)}$$

$$=\frac{6x+1-4x-20}{(x+5)(x-3)}$$

$$=\frac{2x-19}{(x+5)(x-3)}$$

5 a

$$\frac{3}{x}+\frac{2}{x+1}+\frac{1}{x+2}$$

$$=\frac{3(x+1)(x+2)}{x(x+1)(x+2)}+\frac{2x(x+2)}{x(x+1)(x+2)}+\frac{x(x+1)}{x(x+1)(x+2)}$$

$$=\frac{3(x+1)(x+2)+2x(x+2)+x(x+1)}{x(x+1)(x+2)}$$

$$=\frac{3x^2+9x+6+2x^2+4x+x^2+x}{x(x+1)(x+2)}$$

$$=\frac{6x^2+14x+6}{x(x+1)(x+2)}$$

b

$$\frac{4}{3x}-\frac{2}{x-2}+\frac{1}{2x+1}$$

$$=\frac{4(x-2)(2x+1)}{3x(x-2)(2x+1)}-\frac{6x(2x+1)}{3x(x-2)(2x+1)}+\frac{3x(x-2)}{3x(x-2)(2x+1)}$$

$$=\frac{4(x-2)(2x+1)-6x(2x+1)+3x(x-2)}{3x(x-2)(2x+1)}$$

$$=\frac{8x^2-12x-8-12x^2-6x+3x^2-6x}{3x(x-2)(2x+1)}$$

$$=\frac{-x^2-24x-8}{3x(x-2)(2x+1)}$$

c

$$\frac{3}{x-1}+\frac{2}{x+1}+\frac{4}{x-3}$$

$$=\frac{3(x+1)(x-3)}{(x-1)(x+1)(x-3)}+\frac{2(x-1)(x-3)}{(x-1)(x+1)(x-3)}+\frac{4(x-1)(x+1)}{(x-1)(x+1)(x-3)}$$

$$=\frac{3(x+1)(x-3)+2(x-1)(x-3)+4(x-1)(x+1)}{(x-1)(x+1)(x-3)}$$

$$=\frac{3x^2-6x-9+2x^2-8x+6+4x^2-4}{(x-1)(x+1)(x-3)}$$

$$=\frac{9x^2-14x-7}{(x-1)(x+1)(x-3)}$$

6

$$\frac{4(2x-1)}{36x^2-1}+\frac{7}{6x-1}$$

$$=\frac{4(2x-1)}{(6x-1)(6x+1)}+\frac{7}{6x-1}$$

$$=\frac{4(2x-1)}{(6x-1)(6x+1)}+\frac{7(6x+1)}{(6x-1)(6x+1)}$$

$$=\frac{4(2x-1)+7(6x+1)}{(6x-1)(6x+1)}$$

$$=\frac{8x-4+42x+7}{(6x-1)(6x+1)}$$

$$=\frac{50x+3}{(6x-1)(6x+1)}$$

7 **a** $$\begin{aligned} g(x) &= x+\frac{6}{x+2}+\frac{36}{x^2-2x-8} \\ &= x+\frac{6}{x+2}+\frac{36}{(x-4)(x+2)} \\ &= \frac{x(x+2)(x-4)}{(x+2)(x-4)}+\frac{6(x-4)}{(x+2)(x-4)} \\ &\qquad\qquad +\frac{36}{(x+2)(x-4)} \\ &= \frac{x(x+2)(x-4)+6(x-4)+36}{(x+2)(x-4)} \\ &= \frac{x^3-2x^2-8x+6x-24+36}{(x+2)(x-4)} \\ &= \frac{x^3-2x^2-2x+12}{(x+2)(x-4)} \end{aligned}$$

b Using the factor theorem,

$$(-2)^3 - 2(-2)^2 - 2(-2) + 12 = 0$$

So $(x+2)$ is a factor of

$$x^3-2x^2-2x+12$$

Hence, you can write
$x^3-2x^2-2x+12=(x+2)\times \text{p}(x)$

for some quadratic polynomial
p(x).

You can find p(x) by long division:

$$\begin{array}{r} x^2-4x+6 \\ x+2\overline{\big)\,x^3-2x^2-2x+12} \\ \underline{x^3+2x^2}\qquad\qquad\quad \\ -4x^2-2x\qquad\;\; \\ \underline{-4x^2-8x}\qquad\;\; \\ 6x+12 \\ \underline{6x+12} \\ 0 \end{array}$$

Hence, $\text{p}(x)=x^2-4x+6$ and so

$$\begin{aligned} g(x) &= \frac{(x+2)(x^2-4x+6)}{(x+2)(x-4)} \\ &= \frac{x^2-4x+6}{x-4} \end{aligned}$$

Algebraic Methods 1D

Note that all questions in this exercise can be solved either by the method of substitution, or by equating coefficients. Questions **1b** and **1e** have been solved by equating coefficients. All others have been solved using substitution.

1 a $$\frac{6x-2}{(x-2)(x+3)} \equiv \frac{A}{(x-2)} + \frac{B}{(x+3)}$$
$$\equiv \frac{A(x+3)+B(x-2)}{(x-2)(x+3)}$$
$$6x-2 \equiv A(x+3)+B(x-2)$$

Let $x = 2$:
$$6\times 2-2 = A(2+3)+B(2-2)$$
$$10 = 5A$$
$$A = 2$$

Let $x = -3$:
$$6\times(-3)-2 = A(-3+3)+B(-3-2)$$
$$-20 = B\times -5$$
$$B = 4$$

Hence $$\frac{6x-2}{(x-2)(x+3)} \equiv \frac{2}{(x-2)} + \frac{4}{(x+3)}$$

b $$\frac{2x+11}{(x+1)(x+4)} \equiv \frac{A}{(x+1)} + \frac{B}{(x+4)}$$
$$\equiv \frac{A(x+4)+B(x+1)}{(x+1)(x+4)}$$
$$2x+11 \equiv A(x+4)+B(x+1)$$
$$\equiv Ax+4A+Bx+B$$
$$\equiv (A+B)x+(4A+B)$$

Equate coefficients of x:
$$2 = A+B \qquad (1)$$
Equate constant terms:
$$11 = 4A+B \qquad (2)$$
$(2)-(1)$:
$$9 = 3A$$
$$A = 3$$
Substitute $A = 3$ in (1): $2 = 3+B$
$$B = -1$$

Hence $$\frac{2x+11}{(x+1)(x+4)} \equiv \frac{3}{(x+1)} - \frac{1}{(x+4)}$$

1 c $\frac{-7x-12}{2x(x-4)} \equiv \frac{A}{2x} + \frac{B}{(x-4)}$

$$\equiv \frac{A(x-4)+B\times 2x}{2x(x-4)}$$

$$-7x-12 \equiv A(x-4)+2Bx$$

Let $x=4$:

$$-7\times 4-12 = A(4-4)+2B\times 4$$
$$-40 = 8B$$
$$B = -5$$

Let $x=0$:

$$-7\times 0-12 = A(0-4)+2B\times 0$$
$$-12 = -4A$$
$$A = 3$$

Hence $\frac{-7x-12}{2x(x-4)} \equiv \frac{3}{2x} - \frac{5}{(x-4)}$

d $\frac{2x-13}{(2x+1)(x-3)} \equiv \frac{A}{(2x+1)} + \frac{B}{(x-3)}$

$$\equiv \frac{A(x-3)+B(2x+1)}{(2x+1)(x-3)}$$

$$2x-13 \equiv A(x-3)+B(2x+1)$$

Let $x=3$:

$$2\times 3-13 = A(3-3)+B(2\times 3+1)$$
$$-7 = B\times 7$$
$$B = -1$$

Let $x=-\frac{1}{2}$:

$$2\times\left(\frac{1}{2}\right)-13 = A\left(-\frac{1}{2}-3\right)+B\left(2\times\left(-\frac{1}{2}\right)+1\right)$$
$$-14 = A\times -3\frac{1}{2}$$
$$A = 4$$

Hence $\frac{2x-13}{(2x+1)(x-3)} \equiv \frac{4}{(2x+1)} - \frac{1}{(x-3)}$

1 **e** First factorise the denominator:

$$\frac{6x+6}{x^2+9} \equiv \frac{6x+6}{(x+3)(x-3)}$$

Then
$$\begin{aligned}\frac{6x+6}{(x+3)(x-3)} &\equiv \frac{A}{(x+3)} + \frac{B}{(x-3)}\\ &\equiv \frac{A(x-3)+B(x+3)}{(x+3)(x-3)}\\ 6x+6 &\equiv A(x-3)+B(x+3)\\ &\equiv Ax-3A+Bx+3B\\ &\equiv (A+B)x+(3B-3A)\end{aligned}$$

Equate coefficients of x:

$6 = A+B \qquad (1)$

Equate constant terms:

$6 = 3B-3A \qquad (2)$

$(2)+3\times(1)$:

$24 = 6B$

$B = 4$

Substitute $B = 4$ in (1): $6 = A+4$

$A = 2$

Hence
$$\frac{6x+6}{x^2-9} \equiv \frac{2}{(x+3)} + \frac{4}{(x-3)}$$

f First factorise the denominator:

$$\frac{7-3x}{x^2-3x-4} \equiv \frac{7-3x}{(x-4)(x+1)}$$

Then
$$\begin{aligned}\frac{7-3x}{(x-4)(x+1)} &\equiv \frac{A}{(x-4)} + \frac{B}{(x+1)}\\ &\equiv \frac{A(x+1)+B(x-4)}{(x-4)(x+1)}\\ 7-3x &\equiv A(x+1)+B(x-4)\end{aligned}$$

Let $x = -1$:

$$\begin{aligned}7-3\times(-1) &= A(-1+1)+B(-1-4)\\ 10 &= B\times -5\\ B &= -2\end{aligned}$$

1 f (continued)

Let $x = 4$:

$$7 - 3\times 4 = A(4+1) + B(4-4)$$
$$-5 = A\times 5$$
$$A = -1$$

Hence $\dfrac{7-3x}{x^2-3x-4} \equiv -\dfrac{1}{(x-4)} - \dfrac{2}{(x+1)}$

g First factorise the denominator:

$$\frac{8-x}{x^2+4x} \equiv \frac{8-x}{x(x+4)}$$

Then $\dfrac{8-x}{x(x+4)} \equiv \dfrac{A}{x} + \dfrac{B}{(x+4)}$

$$\equiv \frac{A(x+4)+Bx}{x(x+4)}$$
$$8 - x \equiv A(x+4) + Bx$$

Let $x = 0$:

$$8 - 0 = A(0+4) + B\times 0$$
$$8 = 4A$$
$$A = 2$$

Let $x = -4$:

$$8 - (-4) = A(-4+4) + B\times(-4)$$
$$12 = -4B$$
$$B = -3$$

Hence $\dfrac{8-x}{x^2+4x} \equiv \dfrac{2}{x} - \dfrac{3}{(x+4)}$

1 h First factorise the denominator:

$$\frac{2x-14}{x^2+2x-15} \equiv \frac{2x-14}{(x+5)(x-3)}$$

Then $\dfrac{2x-14}{(x+5)(x-3)} \equiv \dfrac{A}{(x+5)} + \dfrac{B}{(x-3)}$

$$\equiv \frac{A(x-3)+B(x+5)}{(x+5)(x-3)}$$

$$2x-14 \equiv A(x-3)+B(x+5)$$

Let $x=3$:

$$2\times3-14 = A(3-3)+B(3+5)$$
$$-8 = B\times8$$
$$B = -1$$

Let $x=-5$:

$$2\times(-5)-14 = A(-5-3)+B(-5+5)$$
$$-24 = A\times(-8)$$
$$A = 3$$

Hence $\dfrac{2x-14}{x^2+2x-15} \equiv \dfrac{3}{(x+5)} - \dfrac{1}{(x-3)}$

2 $\dfrac{-2x-5}{(4+x)(2-x)} \equiv \dfrac{A}{4+x} + \dfrac{B}{2-x}$

$$\equiv \frac{A(2-x)+B(4+x)}{(4+x)(2-x)}$$

$$-2x-5 \equiv A(2-x)+B(4+x)$$

Let $x=2$:

$$-2\times2-5 = A(2-2)+B(4+2)$$
$$-9 = B\times6$$
$$B = \frac{-3}{2}$$

Let $x=-4$:

$$-2\times(-4)-5 = A(2-(-4)+B(4+(-4))$$
$$3 = A\times6$$
$$\frac{1}{2} = A$$

Hence $\dfrac{-2x-5}{(4+x)(2-x)} = \dfrac{1}{2(4+x)} - \dfrac{3}{2(2-x)}$

3 $$\frac{A}{(x-4)(x+8)} \equiv \frac{2}{x-4}+\frac{B}{x+8}$$
$$\equiv \frac{2(x+8)+B(x-4)}{(x-4)(x+8)}$$
$$A \equiv 2(x+8)+B(x-4)$$

Let $x=4$:
$$A=2(4+8)+B(4-4)$$
$$=24$$

Let $x=-8$:
$$24=2(-8+8)+B(-8-4)$$
$$=-12B$$
$$\Rightarrow B=-2$$

$A = 24, B = -2$

4 $$\frac{2x^2-12x-26}{(x+1)(x-2)(x+5)} \equiv \frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{x+5}$$
$$\equiv \frac{A(x-2)(x+5)+B(x+1)(x+5)+C(x+1)(x-2)}{(x+1)(x-2)(x+5)}$$
$$2x^2-12x-26 \equiv A(x-2)(x+5)+B(x+1)(x+5)+C(x+1)(x-2)$$

Let $x=-1$:
$$2+12-26=A\times(-3)\times4+0+0$$
$$-12=-12A$$
$$A=1$$

Let $x = 2$:
$$8-24-26=0+B\times3\times7+0$$
$$-42=21B$$
$$B=-2$$

Let $x=-5$:
$$50+60-26=0+0+C\times(-4)\times(-7)$$
$$84=28C$$
$$C=3$$

$A = 1, B = -2, C = 3$

5 $$\frac{-10x^2-8x+2}{x(2x+1)(3x-2)} \equiv \frac{D}{x}+\frac{E}{2x+1}+\frac{F}{3x-2}$$
$$\equiv \frac{D(2x+1)(3x-2)+Ex(3x-2)+Fx(2x+1)}{x(2x+1)(3x-2)}$$
$$-10x^2-8x+2 \equiv D\left(2x+\ 1\right)\left(3x-2\right)+Ex\left(3x-2\right)+Fx\left(2x+1\right)$$

Let $x = 0$:
$$2 = D\times1\times\left(-2\right)+0+0$$
$$= -2D$$
$$\Rightarrow D = -1$$

Let $x = -\frac{1}{2}$:
$$-\frac{5}{2}+4+2 = 0+E\times\left(-\frac{1}{2}\right)\times\left(-\frac{7}{2}\right)+0$$
$$\frac{7}{2}=\frac{7}{4}E$$
$$\Rightarrow E = 2$$

Let $x = \frac{2}{3}$:
$$-\frac{40}{9}-\frac{16}{3}+2 = 0+0+F\times\left(\frac{2}{3}\right)\times\left(\frac{7}{3}\right)$$
$$-\frac{70}{9}=\frac{14}{9}F$$
$$F = -5$$

$D = -1$, $E = 2$, $F = -5$

6 Let $\dfrac{-5x^2-19x-32}{(x+1)(x+2)(x-5)} \equiv \dfrac{A}{(x+1)} + \dfrac{B}{(x+2)} + \dfrac{C}{(x-5)}$

$$\equiv \frac{A(x+2)(x-5)+B(x+1)(x-5)+C(x+1)(x+2)}{(x+1)(x+2)(x-5)}$$

$$-5x^2-19x-32 \equiv A(x+2)(x-5)+B(x+1)(x-5)+C(x+1)(x+2)$$

Let $x=-1$:

$$-5+19-32 = A\times1\times(-6)+B\times0+C\times0$$
$$-18=-6A$$
$$A=3$$

Let $x=5$:

$$-125-95-32 = A\times0+B\times0+C\times6\times7$$
$$-252=42C$$
$$C=-6$$

Let $x=-2$:

$$-20+38-32 = A\times0+B\times(-1)\times(-7)+C\times0$$
$$-14=7B$$
$$B=-2$$

Hence $\dfrac{-5x^2-19x-32}{(x+1)(x+2)(x-5)} \equiv \dfrac{3}{(x+1)} - \dfrac{2}{(x+2)} - \dfrac{6}{(x-5)}$

7 a First factorise the denominator:

$$\frac{6x^2+7x-3}{x^3-x} = \frac{6x^2+7x-3}{x(x+1)(x-1)}$$

Then $\dfrac{6x^2+7x-3}{x(x+1)(x-1)} \equiv \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1}$

$$\equiv \frac{A(x+1)(x-1)+Bx(x-1)+Cx(x+1)}{x(x+1)(x-1)}$$

$$6x^2+7x-3 \equiv A\left(x+1\right)(x-1)+Bx(x-1)+Cx\left(x+1\right)$$

7 a (continued)

Let $x = 0$:

$$-3 = A \times 1 \times (-1) + 0 + 0$$
$$= -A$$
$$\Rightarrow A = 3$$

Let $x = -1$:

$$6 - 7 - 3 = 0 + B \times (-1) \times (-2) + 0$$
$$-4 = 2B$$
$$\Rightarrow B = -2$$

Let $x = 1$:

$$6 + 7 - 3 = 0 + 0 + C \times 1 \times 2$$
$$10 = 2C$$
$$\Rightarrow C = 5$$

So $\dfrac{6x^2+7x-3}{x(x+1)(x-1)} \equiv \dfrac{3}{x} - \dfrac{2}{x+1} + \dfrac{5}{x-1}$

b First factorise the denominator:

$$\frac{8x+9}{10x^2+3x-4} = \frac{8x+9}{(5x+4)(2x-1)}$$

Then $\dfrac{8x+9}{(5x+4)(2x-1)} \equiv \dfrac{A}{5x+4} + \dfrac{B}{2x-1}$

$$\equiv \frac{A(2x-1)}{(5x+4)(2x-1)} + \frac{B(5x+4)}{(5x+4)(2x-1)}$$

$$8x+9 \equiv A(2x-1) + B\left(5x+4\right)$$

Let $x = -\dfrac{4}{5}$:

$$-\frac{32}{5} + 9 = A \times \left(-\frac{13}{5}\right) + 0$$
$$\frac{13}{5} = -\frac{13}{5}A$$
$$\Rightarrow A = -1$$

Let $x = \dfrac{1}{2}$:

$$4 + 9 = 0 + B \times \left(\frac{13}{2}\right)$$
$$13 = \frac{13}{2}B$$
$$\Rightarrow B = 2$$

So $\dfrac{8x+9}{(5x+4)(2x-1)} \equiv -\dfrac{1}{5x+4} + \dfrac{2}{2x-1}$

Challenge

Evaluating the denominator at $x = 2$:

$$2^3 - 4(2^2) + 2 + 6 = 0$$

By the factor theorem, $(x-2)$ is a factor of $x^3 - 4x^2 + x + 6$

So we can write $x^3 - 4x^2 + x + 6 = (x-2) \times \mathrm{p}(x)$ for some quadratic polynomial p.

We can find p using long division:

$$\begin{array}{r l}
 & \quad\; x^2 - 2x - 3 \\
x-2 \big) \!\!\! & \overline{x^3 - 4x^2 + x + 6} \\
 & \underline{x^3 - 2x^2} \\
 & \quad -2x^2 + x \\
 & \quad \underline{-2x^2 + 4x} \\
 & \qquad\quad -3x + 6 \\
 & \qquad\quad \underline{-3x + 6} \\
 & \qquad\qquad\quad 0
\end{array}$$

$$\begin{aligned}
\frac{5x^2 - 15x - 8}{x^3 - 4x^2 + x + 6} &\equiv \frac{5x^2 - 15x - 8}{(x-2)(x^2 - 2x - 3)} \\
&\equiv \frac{5x^2 - 15x - 8}{(x-2)(x+1)(x-3)} \\
&\equiv \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{x-3} \\
&\equiv \frac{A(x+1)(x-3) + B(x-2)(x-3) + C(x-2)(x+1)}{(x-2)(x+1)(x-3)}
\end{aligned}$$

$$5x^2 - 15x - 8 \equiv A(x+1)(x-3) + B(x-2)(x-3) + C(x-2)(x+1)$$

Let $x = 2$:

$$20 - 30 - 8 = A \times 3 \times (-1) + 0 + 0$$
$$-18 = -3A$$
$$\Rightarrow A = 6$$

Let $x = -1$:

$$5 + 15 - 8 = 0 + B \times (-3) \times (-4) + 0$$
$$12 = 12B$$
$$\Rightarrow B = 1$$

Let $x = 3$:

$$45 - 45 - 8 = 0 + 0 + C \times 1 \times 4$$
$$-8 = 4C$$
$$C = -2$$

So $\dfrac{5x^2 - 15x - 8}{x^3 - 4x^2 + x + 6} \equiv \dfrac{6}{x-2} + \dfrac{1}{x+1} - \dfrac{2}{x-3}$

Algebraic Methods 1E

1 $$\frac{3x^2+x+1}{x^2(x+1)} \equiv \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$$

$$\equiv \frac{Ax(x+1)+B(x+1)+Cx^2}{x^2(x+1)}$$

$$3x^2+x+1 \equiv Ax(x+1)+B(x+1)+Cx^2$$

Let $x = 0$:

$0 + 0 + 1 = 0 + B \times 1 + 0$

$B = 1$

Let $x = -1$:

$3 - 1 + 1 = 0 + 0 + C \times (-1)^2$

$C = 3$

Equating terms in x^2:

$3 = A + C$

$3 = A + 3$

$A = 0$

$A = 0$, $B = 1$, $C = 3$

2 $$\frac{-x^2-10x-5}{(x+1)^2(x-1)} \equiv \frac{D}{x+1}+\frac{E}{(x+1)^2}+\frac{F}{x-1}$$

$$\equiv \frac{D(x+1)(x-1)+E(x-1)+F(x+1)^2}{(x+1)^2(x-1)}$$

$$-x^2-10x-5 \equiv D(x+1)(x-1)+E(x-1)+F(x+1)^2$$

Let $x = -1$:

$-1 + 10 - 5 = 0 + E \times (-2) + 0$

$4 = -2E$

$E = -2$

Let $x = 1$:

$-1 - 10 - 5 = 0 + 0 + F \times 2^2$

$-16 = 4F$

$F = -4$

Equating terms in x^2:

$-1 = D + F$

$-1 = D - 4$

$D = 3$

$D = 3$, $E = -2$, $F = -4$

3 $$\frac{2x^2+2x-18}{x(x-3)^2} \equiv \frac{P}{x}+\frac{Q}{x-3}+\frac{R}{(x-3)^2}$$

$$\equiv \frac{P(x-3)^2+Qx(x-3)+Rx}{x(x-3)^2}$$

$$2x^2+2x-18 \equiv P\left(x-3\right)^2+Qx\left(x-3\right)+Rx$$

Let $x = 0$:

$-18 = P \times (-3)^2 + 0 + 0$

$-18 = 9P$

$P = -2$

Let $x = 3$:

$18 + 6 - 18 = 0 + 0 + R \times 3$

$6 = 3R$

$R = 2$

Equating terms in x^2:

$2 = P + Q$

$2 = -2 + Q$

$Q = 4$

$P = -2$, $Q = 4$, $R = 2$

4 First factorise the denominator:

$$\frac{5x^2-2x-1}{x^3-x^2} \equiv \frac{5x^2-2x-1}{x^2(x-1)}$$

Then $$\frac{5x^2-2x-1}{x^2(x-1)} \equiv \frac{C}{x}+\frac{D}{x^2}+\frac{E}{x-1}$$

$$\equiv \frac{Cx(x-1)+D(x-1)+Ex^2}{x^2(x-1)}$$

$$5x^2-2x-1 \equiv Cx(x-1)+D(x-1)+Ex^2$$

Let $x = 0$:

$-1 = 0 + D \times (-1) + 0$

$-1 = -D$

$D = 1$

Let $x = 1$:

$5 - 2 - 1 = 0 + 0 + E \times 1^2$

$E = 2$

Equating terms in x^2:

$5 = C + E$

$5 = C + 2$

$C = 3$

$C = 3$, $D = 1$, $E = 2$

5 $$\frac{2x}{(x+2)^2} \equiv \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$$\equiv \frac{A(x+2)+B}{(x+2)^2}$$

$$2x \equiv A\left(x+\ 2\right)+B$$

Let $x = -2$:

$-4 = 0 + B$

$B = -4$

Let $x = 0$:

$0 = 2A + B$

$0 = 2A - 4$

$A = 2$

$A = 2$, $B = -4$

6 $$\frac{10x^2-10x+17}{(2x+1)(x-3)^2} \equiv \frac{A}{2x+1} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

$$\equiv \frac{A(x-3)^2+B(2x+1)(x-3)+C(2x+1)}{(2x+1)(x-3)^2}$$

$$10x^2-10x+17 \equiv A(x-3)^2+B(2x+1)(x-3)+C(2x+1)$$

Let $x = -\frac{1}{2}$:

$$\frac{10}{4}+5+17 = A\times\left(-\frac{7}{2}\right)^2+0+0$$

$$\frac{98}{4} = \frac{49}{4}A$$

$A = 2$

Let $x = 3$:

$90 - 30 + 17 = 0 + 0 + C \times 7$

$77 = 7C$

$C = 11$

Equating terms in x^2:

$10 = A + 2B$

$10 = 2 + 2B$

$B = 4$

$A = 2$, $B = 4$, $C = 11$

7 $$\frac{39x^2+2x+59}{(x+5)(3x-1)^2} \equiv \frac{A}{x+5}+\frac{B}{3x-1}+\frac{C}{(3x-1)^2}$$

$$\equiv \frac{A(3x-1)^2+B(x+5)(3x-1)+C(x+5)}{(x+5)(3x-1)^2}$$

$$39x^2 + 2x + 59 \equiv A(3x-1)^2+B(x+5)(3x-1)+C(x+5)$$

Let $x = \frac{1}{3}$:

$$\frac{39}{9}+\frac{2}{3}+59=0+0+C\times\frac{16}{3}$$

$$64=\frac{16}{3}C$$

$$C = 12$$

Let $x = -5$:

$$975 - 10 + 59 = A \times (-16)^2 + 0 + 0$$

$$1024 = 256A$$

$$A = 4$$

Equating terms in x^2:

$$39 = 9A + 3B$$

$$39 = 36 + 3B$$

$$B = 1$$

$A = 4$, $B = 1$, $C = 12$

8 **a** $$\frac{4x+1}{(x+5)^2} \equiv \frac{A}{x+5}+\frac{B}{(x+5)^2}$$

$$\equiv \frac{A(x+5)+B}{(x+5)^2}$$

$$4x + 1 \equiv A(x + 5) + B$$

Let $x = -5$:

$$-20 + 1 = 0 + B$$

$$B = -19$$

Let $x = 0$:

$$1 = 5A + B$$

$$1 = 5A - 19$$

$$A = 4$$

$$\frac{4x+1}{(x+5)^2} \equiv \frac{4}{x+5}-\frac{19}{(x+5)^2}$$

8 b $\frac{6x^2-x+2}{x(2x-1)^2} \equiv \frac{A}{x}+\frac{B}{2x-1}+\frac{C}{(2x-1)^2}$

$$\equiv \frac{A(2x-1)^2+Bx(2x-1)+Cx}{x(2x-1)^2}$$

$6x^2 - x + 2 \equiv A(2x-1)^2+Bx(2x-1)+Cx$

Let $x = 0$:

$2 = A \times (-1)^2 + 0 + 0$

$A = 2$

Let $x = \frac{1}{2}$:

$\frac{3}{2}-\frac{1}{2}+2=0+0+C\times\frac{1}{2}$

$C = 6$

Equating terms in x^2:

$6 = 4A + 2B$

$6 = 8 + 2B$

$B = -1$

So $\frac{6x^2-x+2}{x(2x-1)^2} \equiv \frac{2}{x}-\frac{1}{2x-1}+\frac{6}{(2x-1)^2}$

Algebraic Methods 1F

1 $\dfrac{x^3+2x^2+3x-4}{x+1} \equiv Ax^2+Bx+C+\dfrac{D}{x+1}$

$$\begin{array}{rllll}
 & & x^2 & +x & +2 \\
x+1 \big) & \overline{x^3+2x^2} & \overline{+3x} & \overline{-4} & \\
 & \underline{x^3+x^2} & & & \\
 & x^2 & +3x & & \\
 & \underline{x^2} & \underline{+x} & & \\
 & & 2x & -4 & \\
 & & \underline{2x} & \underline{+2} & \\
 & & & -6 &
\end{array}$$

$$\frac{x^3+2x^2+3x-4}{x+1} \equiv x^2+x+2-\frac{6}{x+1}$$

So $A=1, B=1, C=2, D=-6$

2 Using algebraic long division:

$$\begin{array}{rllll}
 & & 2x^2 & -3x & +5 \\
x+3 \big) & \overline{2x^3+3x^2} & \overline{-4x} & \overline{+5} & \\
 & \underline{2x^3+6x^2} & & & \\
 & -3x^2 & -4x & & \\
 & \underline{-3x^2} & \underline{-9x} & & \\
 & & 5x & +5 & \\
 & & \underline{5x} & \underline{+15} & \\
 & & & -10 &
\end{array}$$

$$\frac{2x^3+3x^2-4x+5}{x+3} = 2x^2-3x+5-\frac{10}{x+3}$$

So $a = 2$, $b = -3$, $c = 5$ and $d = -10$

3 Using algebraic long division:

$$\begin{array}{r}
x^2+2x+4\\
x-2\,\overline{\big)\,x^3+0x^2+0x-8}\\
\underline{x^3-2x^2}\qquad\qquad\quad\\
2x^2+0x\qquad\;\;\\
\underline{2x^2-4x}\qquad\;\;\\
4x-8\\
\underline{4x-8}\\
0
\end{array}$$

So $\frac{x^3-8}{x-2}=x^2+2x+4$

p = 1, q = 2 and r = 4

4 Using algebraic long division:

$$\begin{array}{r}
2\\
x^2-1\,\overline{\big)\,2x^2+4x+5}\\
\underline{2x^2+0x-2}\\
4x+7
\end{array}$$

$$\frac{2x^2+4x+5}{x^2-1}\equiv 2+\frac{4x+7}{x^2-1}$$

So m = 2, n = 4 and p = 7

5 Using algebraic long division:

$$\begin{array}{r}
4x+1\\
2x^2+2\,\overline{\big)\,8x^3+2x^2+0x+5}\\
\underline{8x^3\qquad\quad+8x}\quad\;\;\\
2x^2-8x+5\\
\underline{2x^2\qquad\;\;+2}\\
-8x+3
\end{array}$$

$8x^2 + 2x^2 + 5 \equiv (4x + 1)(2x^2 + 2) - 8x + 3$

So A = 4, B = 1, C = −8 and D = 3.

6 Using algebraic long division:

$$
\begin{array}{rl}
 & 4x-13 \\
x^2+2x-1 & \overline{\smash{)}4x^3-5x^2+3x-14} \\
 & \underline{4x^3+8x^2-4x} \\
 & -13x^2+7x-14 \\
 & \underline{-13x^2-26x+13} \\
 & 33x-27
\end{array}
$$

$$\frac{4x^3-5x^2+3x-14}{x^2+2x-1} \equiv 4x-13+\frac{33x-27}{x^2+2x-1}$$

So $A = 4$, $B = -13$, $C = 33$ and $D = -27$.

7 Using algebraic long division:

$$
\begin{array}{rl}
 & x^2+2 \\
x^2+1 & \overline{\smash{)}x^4+3x^2-4} \\
 & \underline{x^4+x^2} \\
 & 2x^2-4 \\
 & \underline{2x^2+2} \\
 & -6
\end{array}
$$

$$\frac{x^4+3x^2-4}{x^2+1} \equiv x^2+2-\frac{6}{x^2+1}$$

So $p = 1$, $q = 0$, $r = 2$, $s = 0$ and $t = -6$.

8 Using algebraic long division:

$$
\begin{array}{rl}
 & 2x^2+x+1 \\
x^2+x-2 & \overline{\smash{)}2x^4+3x^3-2x^2+4x-6} \\
 & \underline{2x^4+2x^3-4x^2} \\
 & x^3+2x^2+4x \\
 & \underline{x^3+x^2-2x} \\
 & x^2+6x-6 \\
 & \underline{x^2+x-2} \\
 & 5x-4
\end{array}
$$

$$\frac{2x^4+3x^3-2x^2+4x-6}{x^2+x-2} \equiv 2x^2+x+1+\frac{5x-4}{x^2+x-2}$$

So $a = 2$, $b = 1$, $c = 1$, $d = 5$ and $e = -4$.

9 $3x^4 - 4x^3 - 8x^2 + 16x - 2 \equiv (Ax^2 + Bx + C)(x^2 - 3) + Dx + E$

Compare coefficients of x^4 :

$$A = 3$$

Compare coefficients of x^3:

$$B = -4$$

Compare coefficients of x^2:

$$\begin{aligned} -8 &= -3A + C \\ -8 &= -9 + C \qquad \text{(substituting } A = 3\text{)} \\ C &= 1 \end{aligned}$$

Compare coefficients of x :

$$\begin{aligned} 16 &= -3B + D \\ 16 &= 12 + D \qquad \text{(substituting } B = -4\text{)} \\ D &= 4 \end{aligned}$$

Equate constant terms:

$$\begin{aligned} -2 &= -3C + E \\ -2 &= -3 + E \qquad \text{(substituting } C = 1\text{)} \\ E &= 1 \end{aligned}$$

Hence $3x^4 - 4x^3 - 8x^2 + 16x - 2 \equiv (3x^2 - 4x + 1)(x^2 - 3) + 4x + 1$

Note: After lots of comparing coefficients, it is a good idea to check your answer by substituting a simple value of $\boldsymbol{x}$ into both sides of the identity to check that your answers are correct. For example,

Substitute $x = 1$ into LHS

$$\Rightarrow \; 3 - 4 - 8 + 16 - 2 = 5$$

Substitute $x = 1$ into RHS

$$\Rightarrow \; (3 - 4 + 1) \times (1 - 3) + 4 + 1$$
$$= 0 \times -2 + 4 + 1 = 5$$

LHS = RHS, so you can be fairly sure the identity is correct.

10 a $x^4 - 1 \equiv (x^2 - 1)(x^2 + 1)$

$$\equiv (x - 1)(x + 1)(x^2 + 1)$$

b $\dfrac{x^4 - 1}{x + 1} \equiv \dfrac{(x - 1)(x + 1)(x^2 + 1)}{(x + 1)}$

$$\equiv (x - 1)(x^2 + 1)$$

So a = 1, b = −1, c = 1, d = 0 and e = 1.

Algebraic Methods 1G

1 $$\frac{x^2+3x-2}{(x-1)(x-2)} \equiv \frac{x^2+3x-2}{x^2-3x+2}$$

$$\begin{array}{r} 1 \\ x^2-3x+2\overline{)x^2+3x-2} \\ \underline{x^2-3x+2} \\ 6x-4 \end{array}$$

Therefore $$\frac{x^2+3x-2}{(x-1)(x-2)} \equiv 1+\frac{6x-4}{x^2-3x+2}$$
$$\equiv 1+\frac{6x-4}{(x-1)(x-2)}$$

Let $$\frac{6x-4}{(x-1)(x-2)} \equiv \frac{B}{x-1}+\frac{C}{x-2}$$
$$\equiv \frac{B(x-2)+C(x-1)}{(x-1)(x-2)}$$
$$6x-4 \equiv B(x-2)+C(x-1)$$

Let $x = 2$:
$12 - 4 = 0 + C \times 1$
$C = 8$

Let $x = 1$:
$6 - 4 = B \times (-1) + 0$
$2 = -B$
$B = -2$

$$\frac{x^2+3x-2}{(x-1)(x-2)} \equiv 1+\frac{6x-4}{(x-1)(x-2)}$$
$$\equiv 1-\frac{2}{x-1}+\frac{8}{x-2}$$

So $A = 1$, $B = -2$ and $C = 8$.

2 $\dfrac{x^2-10}{(x-2)(x+1)} \equiv \dfrac{x^2-10}{x^2-x-2}$

$$\begin{array}{r}
1 \\
x^2-x-2\,\overline{)\,x^2+0x-10} \\
\underline{x^2-\ x-\ 2} \\
x-\ 8
\end{array}$$

Therefore $\dfrac{x^2-10}{(x-2)(x+1)} \equiv 1+\dfrac{x-8}{x^2-x-2}$

$$\equiv 1+\frac{x-8}{(x-2)(x+1)}$$

Let $\dfrac{x-8}{(x-2)(x+1)} \equiv \dfrac{B}{x-2}+\dfrac{C}{x+1}$

$$\equiv \frac{B(x+1)+C(x-2)}{(x-2)(x+1)}$$

$$x-8 \equiv B(x+1)+C(x-2)$$

Let $x = -1$:

$-1 - 8 = 0 + C \times (-3)$

$-9 = -3C$

$C = 3$

Let $x = 2$:

$2 - 8 = B \times 3 + 0$

$-6 = 3B$

$B = -2$

$$\frac{x^2-10}{(x-2)(x+1)} \equiv 1+\frac{x-8}{(x-2)(x+1)}$$

$$\equiv 1-\frac{2}{x-2}+\frac{3}{x+1}$$

So $A = 1$, $B = -2$ and $C = 3$.

3 $$\frac{x^3-x^2-x-3}{x(x-1)} \equiv \frac{x^3-x^2-x-3}{x^2-x}$$

$$\begin{array}{rl} & \phantom{x^3-x^2-{}}x \\ x^2-x & \overline{\big)\, x^3-x^2-x-3} \\ & \underline{\;x^3-x^2\;} \\ & -x-3 \end{array}$$

Therefore $$\frac{x^3-x^2-x-3}{x(x-1)} \equiv x+\frac{-x-3}{x^2-x}$$

$$\equiv x+\frac{-x-3}{x(x-1)}$$

Let $$\frac{-x-3}{x(x-1)} \equiv \frac{C}{x}+\frac{D}{x-1}$$

$$\equiv \frac{C(x-1)+Dx}{x(x-1)}$$

$$-x-3 \equiv C(x-1)+Dx$$

Let $x = 0$:

$$0-3 = C \times (-1)+0$$
$$-3 = -C$$
$$C = 3$$

Let $x = 1$:

$$-1-3 = 0+D \times 1$$
$$D = -4$$

$$\frac{x^3-x^2-x-3}{x(x-1)} \equiv x+\frac{-x-3}{x(x-1)}$$

$$\equiv x+\frac{3}{x}-\frac{4}{x-1}$$

So $A = 1$, $B = 0$, $C = 3$ and $D = -4$.

4

$$
\begin{array}{rl}
 & -3x+2 \\
x^2+2x-3 \big) & \overline{-3x^3-4x^2+19x+8} \\
 & \underline{-3x^3-6x^2+9x} \\
 & 2x^2+10x+8 \\
 & \underline{2x^2+\ \ 4x-6} \\
 & 6x+14
\end{array}
$$

Therefore $\dfrac{-3x^3-4x^2+19x+8}{x^2+2x-3} \equiv -3x+2+\dfrac{6x+14}{x^2+2x-3}$

$$\equiv -3x+2+\frac{6x+14}{(x-1)(x+3)}$$

Let $\dfrac{6x+14}{(x-1)(x+3)} \equiv \dfrac{C}{(x-1)} + \dfrac{D}{(x+3)}$

$$\equiv \frac{C(x+3)+D(x-1)}{(x-1)(x+3)}$$

$$6x+14 \equiv C(x+3)+D(x-1)$$

Let $x = 1$:

$$6+14 = C\times 4 + D\times 0$$
$$20 = 4C$$
$$5 = C$$

Let $x = -3$:

$$6\times(-3)+14 = C\times 0 + D\times(-4)$$
$$-4 = -4D$$
$$D = 1$$

$$\frac{-3x^3-4x^2+19x+8}{x^2+2x-3} \equiv 2-3x+\frac{6x+14}{(x-1)(x+3)}$$

$$\equiv 2-3x+\frac{5}{(x-1)}+\frac{1}{(x+3)}$$

So $A = 2,\ B = -3,\ C = 5$ and $D = 1$

5 $$\begin{array}{r} 1 \\ 4x^2-25\overline{\smash{\big)}4x^2+25} \\ \underline{4x^2-25} \\ 50 \end{array}$$

Therefore $\mathrm{p}(x) \equiv \dfrac{4x^2+25}{4x^2-25}$

$$\equiv 1+\frac{50}{4x^2-25}$$

$$\equiv 1+\frac{50}{(2x-5)(2x+5)}$$

Let $\dfrac{50}{(2x-5)(2x+5)} \equiv \dfrac{B}{2x-5}+\dfrac{C}{2x+5}$

$$\equiv \frac{B(2x+5)+C(2x-5)}{(2x-5)(2x+5)}$$

$$50 \equiv B(2x+5)+C(2x-5)$$

Let $x = \dfrac{5}{2}$:

$50 = B \times 10 + 0$
$50 = 10B$
$B = 5$

Let $x = -\dfrac{5}{2}$:

$50 = 0 + C \times (-10)$
$50 = -10C$
$C = -5$

$\mathrm{p}(x) \equiv \dfrac{4x^2+25}{4x^2-25}$

$$\equiv 1+\frac{50}{(2x-5)(2x+5)}$$

$$\equiv 1+\frac{5}{2x-5}-\frac{5}{2x+5}$$

So $A = 1$, $B = 5$ and $C = -5$.

6 $$\begin{array}{r} 2 \\ x^2+2x+1\overline{\smash{\big)}2x^2+0x-1} \\ \underline{2x^2+4x+2} \\ -4x-3 \end{array}$$

Therefore $\dfrac{2x^2-1}{x^2+2x+1} \equiv 2+\dfrac{-4x-3}{x^2+2x+1}$

$$\equiv 2+\frac{-4x-3}{(x+1)^2}$$

Let $\dfrac{-4x-3}{(x+1)^2} \equiv \dfrac{B}{x+1}+\dfrac{C}{(x+1)^2}$

$$\equiv \frac{B(x+1)+C}{(x+1)^2}$$

$$-4x-3 \equiv B(x+1)+C$$

Let $x = -1$:
$4 - 3 = 0 + C$
$C = 1$

Let $x = 0$:
$-3 = B \times 1 + C$
$-3 = B + 1$
$B = -4$

$$\frac{2x^2-1}{x^2+2x+1} \equiv 2+\frac{-4x-3}{(x+1)^2}$$

$$\equiv 2-\frac{4}{x+1}+\frac{1}{(x+1)^2}$$

So $A = 2$, $B = -4$ and $C = 1$.

7 a $$\begin{array}{r} 4 \\ x^2+3x-4\overline{\smash{\big)}4x^2+17x-11} \\ \underline{4x^2+12x-16} \\ 5x+5 \end{array}$$

Therefore $$\frac{4x^2+17x-11}{x^2+3x-4} \equiv 4+\frac{5x+5}{x^2+3x-4}$$
$$\equiv 4+\frac{5x+5}{(x+4)(x-1)}$$

Let $$\frac{5x+5}{(x+4)(x-1)} \equiv \frac{A}{(x+4)}+\frac{B}{(x-1)}$$
$$\equiv \frac{A(x-1)+B(x+4)}{(x+4)(x-1)}$$
$$5x+5 \equiv A(x-1)+B(x+4)$$

Let $x=1$:
$$5\times 1+5 = A\times 0+B\times 5$$
$$10=5B$$
$$B=2$$

Let $x=-4$:
$$5\times(-4)+5 = A\times(-5)+B\times 0$$
$$-15=-5A$$
$$A=3$$

Hence $$\frac{4x^2+17x-11}{x^2+3x-4} \equiv 4+\frac{5x+5}{(x+4)(x-1)}$$
$$\equiv 4+\frac{3}{(x+4)}+\frac{2}{(x-1)}$$

7 b $$\begin{array}{r} x \\ x^3-4x^2+4x\overline{\big)x^4-4x^3+9x^2-17x+12} \\ \underline{x^4-4x^3+4x^2} \\ 5x^2-17x+12 \end{array}$$

Therefore $$\frac{x^4-4x^3+9x^2-17x+12}{x^3-4x^2+4x} \equiv x+\frac{5x^2-17x+12}{x^3-4x^2+4x}$$
$$\equiv x+\frac{5x^2-17x+12}{x(x-2)^2}$$

Let $$\frac{5x^2-17x+12}{x(x-2)^2} \equiv \frac{A}{x}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$$
$$\equiv \frac{A(x-2)^2+Bx(x-2)+Cx}{x(x-2)^2}$$
$$5x^2-17x+12 \equiv A(x-2)^2+Bx(x-2)+Cx$$

Let $x=0$:

$12=A\times(-2)^2$

$12=4A$

$A=3$

Let $x=2$:

$5\times(2)^2-17\times2+12=2C$

$-2=2C$

$C=-1$

Compare terms in x^2:

$5=A+B$

$5=3+B$

$B=2$

$$\frac{x^4-4x^3+9x^2-17x+12}{x^3-4x^2+4x} \equiv x+\frac{5x^2-17x+12}{x(x-2)^2}$$
$$\equiv x+\frac{3}{x}+\frac{2}{x-2}-\frac{1}{(x-2)^2}$$

8
$$\begin{array}{r} 2x-3 \\ 3x^2+x-10\overline{)6x^3-7x^2\;+\;0x+3} \\ \underline{6x^3+2x^2-20x\quad\;\;} \\ -9x^2+20x+3 \\ \underline{-9x^2-3x+30} \\ 23x-27 \end{array}$$

Therefore $\dfrac{6x^3-7x^2+3}{3x^2+x-10} \equiv 2x-3+\dfrac{23x-27}{3x^2+x-10}$

$$\equiv 2x-3+\frac{23x-27}{(3x-5)(x+2)}$$

Let $\dfrac{23x-27}{(3x-5)(x+2)} \equiv \dfrac{C}{3x-5}+\dfrac{D}{x+2}$

$$\equiv \frac{C(x+2)+D(3x-5)}{(3x-5)(x+2)}$$

$$23x-27 \equiv C(x+2)+D(3x-5)$$

Let $x = \dfrac{5}{3}$:

$$\frac{115}{3} - 27 = C \times \frac{11}{3} + 0$$

$$\frac{34}{3} = \frac{11}{3}C$$

$$C = \frac{34}{11}$$

Let $x = -2$:

$$-46 - 27 = 0 + D \times (-11)$$

$$D = \frac{73}{11}$$

$$\frac{6x^3-7x^2+3}{3x^2+x-10} \equiv 2x-3+\frac{23x-27}{(3x-5)(x+2)}$$

$$\equiv 2x-3+\frac{34}{11(3x-5)}+\frac{73}{11(x+2)}$$

So $A = 2$, $B = -3$, $C = \dfrac{34}{11}$ and $D = \dfrac{73}{11}$.

9
$$
\begin{array}{r}
2x+2 \\
4x^2-4x+1\overline{\smash{)}8x^3+0x^2+0x+1} \\
\underline{8x^3-8x^2+2x} \\
8x^2-2x+1 \\
\underline{8x^2-8x+2} \\
6x-1
\end{array}
$$

Therefore
$$
\begin{aligned}
\frac{8x^3+1}{4x^2-4x+1} &\equiv 2x+2+\frac{6x-1}{4x^2-4x+1} \\
&\equiv 2x+2+\frac{6x-1}{(2x-1)^2}
\end{aligned}
$$

Let
$$
\begin{aligned}
\frac{6x-1}{(2x-1)^2} &\equiv \frac{C}{2x-1}+\frac{D}{(2x-1)^2} \\
&\equiv \frac{C(2x-1)+D}{(2x-1)^2} \\
6x-1 &\equiv C(2x-1)+D
\end{aligned}
$$

Let $x=\frac{1}{2}$:

$3-1=0+D$

$D=2$

Let $x=0$:

$0-1=C\times(-1)+D$

$-1=-C+2$

$C=3$

$$
\begin{aligned}
\frac{8x^3+1}{4x^2-4x+1} &\equiv 2x+2+\frac{6x-1}{(2x-1)^2} \\
&\equiv 2x+2+\frac{3}{2x-1}+\frac{2}{(2x-1)^2}
\end{aligned}
$$

So $A=2$, $B=2$, $C=3$ and $D=2$.

10

$$\begin{array}{r} x^2 - x + 5 \\ x^2 + x - 2\overline{\big)x^4 + 0x^3 + 2x^2 - 3x + 8} \\ \underline{x^4 + x^3 - 2x^2} \\ -x^3 + 4x^2 - 3x \\ \underline{-x^3 - x^2 + 2x} \\ 5x^2 - 5x + 8 \\ \underline{5x^2 + 5x - 10} \\ -10x + 18 \end{array}$$

Therefore $\dfrac{x^4 + 2x^2 - 3x + 8}{x^2 + x - 2} \equiv x^2 - x + 5 + \dfrac{-10x + 18}{x^2 + x - 2}$

$$\equiv x^2 - x + 5 + \frac{-10x + 18}{(x + 2)(x - 1)}$$

Let $\dfrac{-10x + 18}{(x + 2)(x - 1)} \equiv \dfrac{D}{x + 2} + \dfrac{E}{x - 1}$

$$\equiv \frac{D(x - 1) + E(x + 2)}{(x + 2)(x - 1)}$$

$$-10x + 18 = D(x - 1) + E\left(x + 2\right)$$

Let $x = -2$:

$20 + 18 = D \times (-3) + 0$

$38 = -3D$

$D = -\dfrac{38}{3}$

Let $x = 1$:

$-10 + 18 = 0 + E \times 3$

$8 = 3E$

$E = \dfrac{8}{3}$

$$\frac{x^4 + 2x^2 - 3x + 8}{x^2 + x - 2} \equiv x^2 - x + 5 + \frac{-10x + 18}{(x + 2)(x - 1)}$$

$$\equiv x^2 - x + 5 - \frac{38}{3(x + 2)} + \frac{8}{3(x - 1)}$$

So $A = 1$, $B = -1$, $C = 5$, $D = -\dfrac{38}{3}$ and $E = \dfrac{8}{3}$.

Algebraic methods Mixed exercise 1

1 Assumption: $\sqrt{\frac{1}{2}}$ is a rational number.

Then $\sqrt{\frac{1}{2}} = \frac{a}{b}$ for some integers a and b.

There is a further assumption that this fraction is in its simplest terms: there are no common factors between a and b.

So $0.5 = \frac{a^2}{b^2}$ or $2a^2 = b^2$.

Therefore b^2 must be a multiple of 2.
We know that this means b must also be a multiple of 2.
Write $b = 2c$, which means $b^2 = (2c)^2 = 4c^2$.
Now $4c^2 = 2a^2$, or $2c^2 = a^2$.
Therefore a^2 must be a multiple of 2, which implies a is also a multiple of 2.
If a and b are both multiples of 2, this contradicts the statement that there are no common factors between a and b.

Therefore, $\sqrt{\frac{1}{2}}$ is an irrational number.

2 Assumption: There exists a rational number q where q^2 is irrational

So write $q = \frac{a}{b}$, where a and b are integers.

$q^2 = \frac{a^2}{b^2}$

As a and b are integers, a^2 and b^2 are integers.
So q^2 is rational.
This contradicts assumption that q^2 is irrational.
Therefore if q^2 is irrational then q is irrational.

3 a
$$\frac{x-4}{6}\times\frac{2x+8}{x^2-16}=\frac{x-4}{6}\times\frac{2(x+4)}{(x-4)(x+4)}$$
$$=\frac{1}{3}$$

b
$$\frac{x^2-3x-10}{3x^2-21}\times\frac{6x^2+24}{x^2+6x+8}=\frac{(x-5)(x+2)}{3(x^2-7)}\times\frac{6(x^2+4)}{(x+2)(x+4)}$$
$$=\frac{2(x^2+4)(x-5)}{(x^2-7)(x+4)}$$

3 c

$$\begin{aligned}\frac{4x^2+12x+9}{x^2+6x}\div\frac{4x^2-9}{2x^2+9x-18}&=\frac{4x^2+12x+9}{x^2+6x}\times\frac{2x^2+9x-18}{4x^2-9}\\&=\frac{(2x+3)^2}{x(x+6)}\times\frac{(2x-3)(x+6)}{(2x-3)(2x+3)}\\&=\frac{2x+3}{x}\end{aligned}$$

4 a

$$\begin{aligned}\frac{4x^2-8x}{x^2-3x-4}\times\frac{x^2+6x+5}{2x^2+10x}&=\frac{4x(x-2)}{(x-4)(x+1)}\times\frac{(x+1)(x+5)}{2x(x+5)}\\&=\frac{2(x-2)}{x-4}\\&=\frac{2x-4}{x-4}\end{aligned}$$

b

$$\begin{aligned}6&=\ln\left(\left(4x^2-8x\right)\left(x^2+6x+5\right)\right)-\ln\left(\left(x^2-\ 3x-\ 4\right)\left(2x^2+\ 10x\right)\right)\\&=\ln\left(\frac{\left(4x^2-8x\right)\left(x^2+6x+5\right)}{\left(x^2-\ 3x-\ 4\right)\left(2x^2+\ 10x\right)}\right)\\&=\ln\left(\frac{4x\left(x-2\right)\left(x+1\right)\left(x+5\right)}{\left(x-4\right)\left(x+1\right)2x\left(x+5\right)}\right)\\&=\ln\left(\frac{2x-4}{x-4}\right)\end{aligned}$$

$$\begin{aligned}\frac{2x-4}{x-4}&=\ \mathrm{e}^6\\2x-4&=x\mathrm{e}^6-4\mathrm{e}^6\\4\mathrm{e}^6-4&=x\mathrm{e}^6-2x\\4(\mathrm{e}^6-1)&=x(\mathrm{e}^6-2)\\x&=\frac{4(\mathrm{e}^6-1)}{\mathrm{e}^6-2}\end{aligned}$$

5 a
$$\begin{aligned}
\mathrm{g}(x) &= \frac{4x^3-9x^2-9x}{32x+24} \div \frac{x^2-3x}{6x^2-13x-5}\\
&= \frac{4x^3-9x^2-9x}{32x+24} \times \frac{6x^2-13x-5}{x^2-3x}\\
&= \frac{x(4x+3)(x-3)}{8(4x+3)} \times \frac{(3x+1)(2x-5)}{x(x-3)}\\
&= \frac{(3x+1)(2x-5)}{8}\\
&= \frac{6x^2-13x-5}{8}\\
&= \frac{3}{4}x^2-\frac{13}{8}x-\frac{5}{8}
\end{aligned}$$

$$\mathrm{a} = \frac{3}{4},\ \mathrm{b} = -\frac{13}{8},\ \mathrm{c} = -\frac{5}{8}$$

b
$$\begin{aligned}
\mathrm{g}'(x) &= \frac{3}{2}x-\frac{13}{8}\\
\mathrm{g}'(-2) &= \frac{3}{2}(-2)-\frac{13}{8}\\
&= -\frac{37}{8}
\end{aligned}$$

6
$$\begin{aligned}
\frac{6x+1}{x-5}+\frac{5x+3}{x^2-3x-10} &= \frac{6x+1}{x-5}+\frac{5x+3}{(x-5)(x+2)}\\
&= \frac{(6x+1)(x+2)}{(x-5)(x+2)}+\frac{5x+3}{(x-5)(x+2)}\\
&= \frac{6x^2+13x+2}{(x-5)(x+2)}+\frac{5x+3}{(x-5)(x+2)}\\
&= \frac{6x^2+13x+2+5x+3}{(x-5)(x+2)}\\
&= \frac{6x^2+18x+5}{x^2-3x-10}
\end{aligned}$$

7 $$x+\frac{3}{x-1}-\frac{12}{x^2+2x-3}$$
$$=\frac{x\left(x+3\right)\left(x-1\right)}{\left(x+3\right)\left(x-1\right)}+\frac{3\left(x+3\right)}{\left(x+3\right)\left(x-1\right)}-\frac{12}{\left(x+3\right)\left(x-1\right)}$$
$$=\frac{x\left(x+3\right)\left(x-1\right)+3x-3}{\left(x+3\right)\left(x-1\right)}$$
$$=\frac{\left(x-1\right)\left[x\left(x+3\right)+3\right]}{\left(x+3\right)\left(x-1\right)}$$
$$=\frac{x^2+3x+3}{x+3}$$

8 $$\frac{x-3}{x(x-1)}\equiv\frac{A}{x}+\frac{B}{x-1}$$
$$\equiv\frac{A(x-1)+Bx}{x(x-1)}$$
$$x-3\equiv A(x-1)+Bx$$

Let $x = 0$:
$0 - 3 = A \times (-1) + 0$
$-3 = -A$
$A = 3$

Let $x = 1$:
$1 - 3 = 0 + B \times 1$
$B = -2$

$$\frac{x-3}{x(x-1)}\equiv\frac{3}{x}-\frac{2}{x-1}$$
$A = 3$, $B = -2$

9 $$\frac{-15x+21}{(x-2)(x+1)(x-5)} \equiv \frac{P}{x-2}+\frac{Q}{x+1}+\frac{R}{x-5}$$

$$\equiv \frac{P(x+1)(x-5)+Q(x-2)(x-5)+R(x-2)(x+1)}{(x-2)(x+1)(x-5)}$$

$$-15x+\ 21 \equiv P(x+1)(x-5)+Q(x-2)(x-5)+R(x-2)(x+1)$$

Let $x = 2$:
$-30 + 21 = P \times 3 \times (-3) + 0 + 0$
$-9 = -9P$
$P = 1$

Let $x = -1$:
$15 + 21 = 0 + Q \times (-3) \times (-6) + 0$
$36 = 18Q$
$Q = 2$

Let $x = 5$:
$-75 + 21 = 0 + 0 + R \times 3 \times 6$
$-54 = 18R$
$R = -3$
$P = 1$, $Q = 2$ and $R = -3$

10 $$\frac{16x-1}{(3x+2)(2x-1)} \equiv \frac{D}{3x+2}+\frac{E}{2x-1}$$

$$\equiv \frac{D(2x-1)+E(3x+2)}{(3x+2)(2x-1)}$$

$$16x-1 \equiv D(2x-1)+E\left(3x+2\right)$$

Let $x = -\frac{2}{3}$:

$$-\frac{32}{3} - 1 = D \times \left(-\frac{7}{3}\right) + 0$$

$$-\frac{35}{3} = -\frac{7}{3}D$$

$D = 5$

Let $x = \frac{1}{2}$:

$$8 - 1 = 0 + E \times \left(\frac{7}{2}\right)$$

$$7 = \frac{7}{2}E$$

$E = 2$

10 (continued)

$$\frac{16x-1}{(3x+2)(2x-1)} \equiv \frac{5}{3x+2}+\frac{2}{2x-1}$$

$D = 5$, $E = 2$

11 $$\frac{7x^2+2x-2}{x^2(x+1)} \equiv \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$$

$$\equiv \frac{Ax(x+1)+B(x+1)+Cx^2}{x^2(x+1)}$$

$$7x^2+2x-2 \equiv Ax(x+1)+B(x+1)+Cx^2$$

Let $x = 0$:

$$0 + 0 - 2 = 0 + B \times 1 + 0$$
$$B = -2$$

Let $x = -1$:

$$7 - 2 - 2 = 0 + 0 + C \times (-1)^2$$
$$C = 3$$

Compare terms in x^2 :

$7 = A + C$

$7 = A + 3$

$A = 4$

$A = 4$, $B = -2$ and $C = 3$

12 $$\frac{21x^2-13}{(x+5)(3x-1)^2} \equiv \frac{D}{x+5}+\frac{E}{(3x-1)}+\frac{F}{(3x-1)^2}$$

$$\equiv \frac{D(3x-1)^2+E(x+5)(3x-1)+F(x+5)}{(x+5)(3x-1)^2}$$

$$21x^2-13 \equiv D(3x-1)^2+E(x+5)(3x-1)+F(x+5)$$

Let $x = -5$:

$$525 - 13 = D \times(-16)^2 + 0 + 0$$
$$512 = 256D$$
$$D = 2$$

Let $x = \frac{1}{3}$:

$$\frac{7}{3} - 13 = 0 + 0 + F \times \frac{16}{3}$$
$$-\frac{32}{3} = \frac{16}{3}F$$
$$F = -2$$

Compare terms in x^2 :

12 (continued)

$21 = 9D + 3E$

$21 = 18 + 3E$

$E = 1$

$$\frac{21x^2-13}{(x+5)(3x-1)^2} \equiv \frac{2}{x+5} + \frac{1}{(3x-1)} - \frac{2}{(3x-1)^2}$$

$D = 2$, $E = 1$, $F = -2$

13

$$\begin{array}{r}
x^2-4x+3 \\
x-2\overline{\big)\,x^3-6x^2+11x+2} \\
\underline{x^3-2x^2} \qquad\qquad \\
-4x^2+11x \qquad \\
\underline{-4x^2+8x} \qquad \\
3x+2 \\
\underline{3x-6} \\
8
\end{array}$$

$$x^3-6x^2+11x+2 \equiv (x-2)(x^2-4x+3)+8$$

$A = 1$, $B = -4$, $C = 3$ and $D = 8$

14

$$\begin{array}{r}
2x^2-4x+6 \\
2x+1\overline{\big)\,4x^3-6x^2+8x-5} \\
\underline{4x^3+2x^2} \qquad\qquad \\
-8x^2+8x \qquad \\
\underline{-8x^2-4x} \qquad \\
12x-5 \\
\underline{12x+6} \\
-11
\end{array}$$

$$\frac{4x^3-6x^2+8x-5}{2x+1} \equiv 2x^2-4x+6-\frac{11}{2x+1}$$

$A = 2$, $B = -4$, $C = 6$ and $D = -11$

15
$$\begin{array}{r} x^2+1 \\ x^2-1\overline{)\,x^4+0x^2+2} \\ \underline{x^4-x^2} \\ x^2+\ 2 \\ \underline{x^2-\ 1} \\ 3 \end{array}$$

$$\frac{x^4+2}{x^2-1}\equiv x^2+1+\frac{3}{x^2-1}$$

So A = 1, B = 0, C = 1, and D = 3.

16
$$\begin{array}{r} x^2+2x+3 \\ x^2-2x+1\overline{)\,x^4+0x^3+0x^2+0x+0} \\ \underline{x^4-2x^3+x^2} \\ 2x^3-x^2+0x \\ \underline{2x^3-4x^2+2x} \\ 3x^2-2x+0 \\ \underline{3x^2-6x+3} \\ 4x-3 \end{array}$$

$$\frac{x^4}{x^2-2x+1}\equiv x^2+2x+3+\frac{4x-3}{x^2-2x+1}$$
$$\equiv x^2+2x+3+\frac{4x-3}{(x-1)^2}$$

Let $\dfrac{4x-3}{(x-1)^2}\equiv\dfrac{D}{x-1}+\dfrac{E}{(x-1)^2}$

$$\equiv\frac{D(x-1)+E}{(x-1)^2}$$

$$4x-3=D(x-1)+E$$

Let x = 1:
$4 - 3 = E$
$E = 1$

Let x = 0:
$-3 = D\times(-1) + E$
$-3 = -D + 1$
$D = 4$

$$\frac{x^4}{x^2-2x+1}\equiv x^2+2x+3+\frac{4x-3}{(x-1)^2}$$
$$\equiv x^2+2x+3+\frac{4}{x-1}+\frac{1}{(x-1)^2}$$

A = 1, B = 2, C = 3, D = 4 and E = 1

17 Using algebraic long division:

$$\begin{array}{r}
2 \\
x^2+2x-3\,\overline{)\,2x^2+2x-3} \\
\underline{2x^2+4x-6} \\
-2x+3
\end{array}$$

$$\frac{2x^2+2x-3}{x^2+2x-3}\equiv 2+\frac{-2x+3}{x^2+2x-3}$$

$$\equiv 2+\frac{-2x+3}{(x+3)(x-1)}$$

Let $\dfrac{-2x+3}{(x+3)(x-1)}\equiv\dfrac{B}{x+3}+\dfrac{C}{x-1}$

$$\equiv\frac{B(x-1)+C(x+3)}{x^2+2x-3}$$

$$-2x+3\equiv B(x-1)+C(x+3)$$

Let $x = 1$:

$-2 + 3 = 0 + C \times 4$

$C = \dfrac{1}{4}$

Let $x = -3$:

$6 + 3 = B \times (-4) + 0$

$9 = -4B$

$B = -\dfrac{9}{4}$

$A = 2$, $B = -\dfrac{9}{4}$ and $C = \dfrac{1}{4}$.

18

$$\begin{array}{r}
1 \\
x^2-2x\,\overline{)\,x^2+0x+1} \\
\underline{x^2-2x} \\
2x+1
\end{array}$$

$$\frac{x^2+1}{x(x-2)}\equiv 1+\frac{2x+1}{x(x-2)}$$

Let $\dfrac{2x+1}{x(x-2)}\equiv\dfrac{Q}{x}+\dfrac{R}{x-2}$

$$\equiv\frac{Q(x-2)+Rx}{x(x-2)}$$

$$2x+1=Q(x-2)+Rx$$

18 (continued)

Let $x = 0$:

$$\begin{aligned} 0 + 1 &= Q \times (-2) + 0 \\ 1 &= -2Q \\ Q &= -\frac{1}{2} \end{aligned}$$

Let $x = 2$:

$$\begin{aligned} 4 + 1 &= 0 + R \times 2 \\ 5 &= 2R \\ R &= \frac{5}{2} \end{aligned}$$

$P = 1$, $Q = -\frac{1}{2}$ and $R = \frac{5}{2}$

19 a

$$\begin{aligned} f(-3) &= 2\times(-27)+9\times9+10\times(-3)+3 \\ &= -54+81-30+3 \\ &= 0 \end{aligned}$$

$f(-3)=0$ so $x=-3$ is a root of $f(x)$

OR

$$\begin{array}{r|l} & 2x^2+3x\ +1 \\ \hline x+3 & 2x^3+9x^2+10x+3 \\ & \underline{2x^3+6x^2} \\ & \qquad 3x^2+10x \\ & \qquad \underline{3x^2+\ 9x} \\ & \qquad\qquad x+3 \\ & \qquad\qquad \underline{x+3} \\ & \qquad\qquad\quad 0 \end{array}$$

$(x+3)$ is a factor of $f(x)$, so by the factor theorem $x=-3$ is a root of $f(x)$

b

$$\begin{aligned} \frac{10}{f(x)} &\equiv \frac{10}{2x^3+9x^2+10x+3} \\ &\equiv \frac{10}{(x+3)(2x^2+3x\ +1)} \qquad \text{[by part } \mathbf{a}\text{]} \\ &\equiv \frac{10}{(x+3)(2x+1)(x+1)} \\ &\equiv \frac{A}{(x+3)}+\frac{B}{(2x+1)}+\frac{C}{(x+1)} \\ &\equiv \frac{A(2x+1)(x+1)+B(x+3)(x+1)+C(x+3)(2x+1)}{(x+3)(2x+1)(x+1)} \end{aligned}$$

19 b (continued)

$$10 \equiv A(2x+1)(x+1)+B(x+3)(x+1)+C(x+3)(2x+1)$$

Let $x=-1$:

$$10 = A\times 0 + B\times 0 + C\times 2\times(-1)$$
$$10=-2C$$
$$C=-5$$

Let $x=-3$:

$$10 = A\times(-5)\times(-2)+B\times 0+C\times 0$$
$$10=10A$$
$$A=1$$

Let $x=-\frac{1}{2}$:

$$10 = A\times 0+B\times\left(\frac{5}{2}\right)\times\left(\frac{1}{2}\right)+C\times 0$$
$$10=\frac{5}{4}B$$
$$B=8$$

Hence $\frac{10}{\mathrm{f}(x)} \equiv \frac{1}{(x+3)}+\frac{8}{(2x+1)}-\frac{5}{(x+1)}$

Challenge

Assumption: *L* is not perpendicular to *OA*.
Draw the line through *O* which is perpendicular to *L*.
This line meets *L* at a point *B*, outside the circle.
Triangle *OBA* is right-angled at *B*, so *OA* is the hypotenuse of this triangle, so *OA* > *OB*.
This gives a contradiction, as *B* is outside the circle, so *OA* < *OB*.
Therefore *L* is perpendicular to *OA*.

Functions and graphs 2A

1 a $\left|\frac{3}{4}\right| = \frac{3}{4}$

b $|-0.28| = 0.28$

c $|3-11| = |-8|$
$= 8$

d $\left|\frac{5}{7}-\frac{3}{8}\right| = \left|\frac{40}{56}-\frac{21}{56}\right|$
$= \frac{19}{56}$

e $|20-6\times4| = |20-24|$
$= |-4|$
$= 4$

f $|4^2\times2-3\times7| = |32-21|$
$= 11$

2 a $f(1) = |7-5\times1|+3$
$= |7-5|+3$
$= 5$

b $f(10) = |7-5\times10|+3$
$= |7-50|+3$
$= |-43|+3$
$= 46$

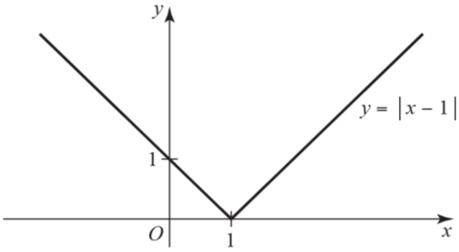
c $f(-6) = |7-5\times(-6)|+3$
$= |7+30|+3$
$= 40$

3 a $g(4) = |4^2-8\times4|$
$= |16-32|$
$= |-16|$
$= 16$

b $g(-5) = |(-5)^2-8\times(-5)|$
$= |25+40|$
$= 65$

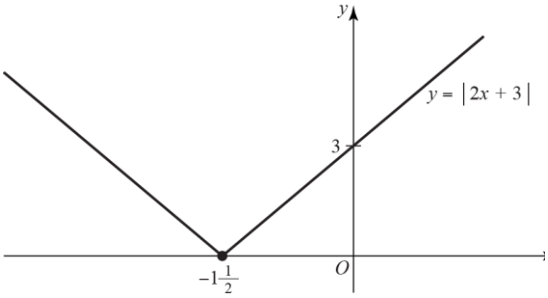
c $g(8) = |8^2-8\times8|$
$= |64-64|$
$= 0$

4 a



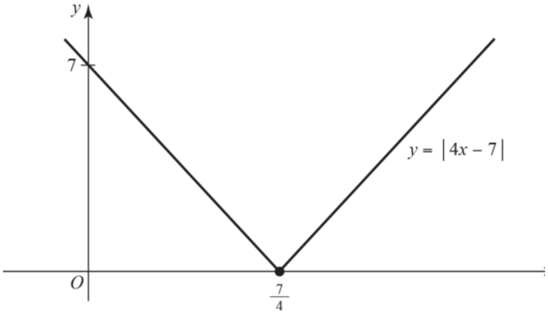
The graph meets the axes at (1, 0) and (0, 1)

4 b



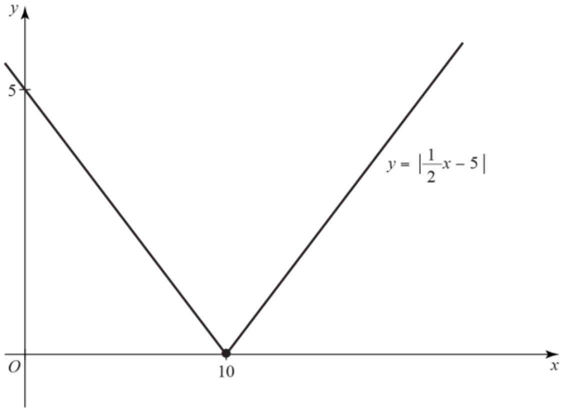
The graph meets the axes at
$\left(-1\frac{1}{2},\ 0\right)$ and (0, 3)

c



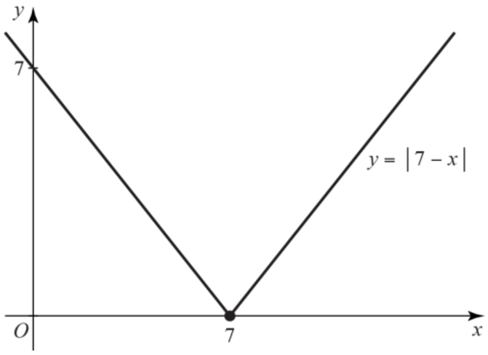
The graph meets the axes at
$\left(\frac{7}{4},\ 0\right)$ and (0, 7)

d



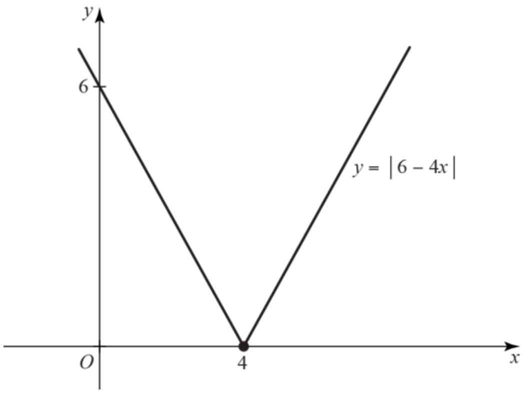
The graph meets the axes at
(10, 0) and (0, 5)

e



The graph meets the axes at
(7, 0) and (0, 7)

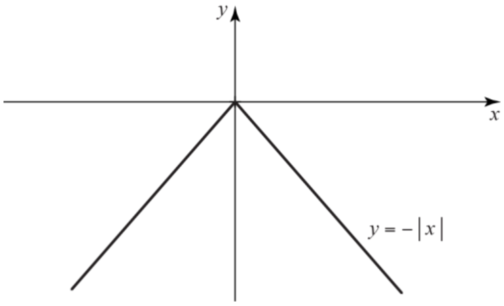
4 f



The graph meets the axes at

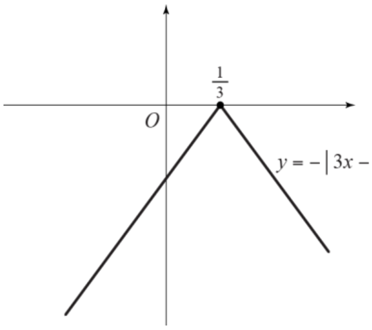
$\left(\frac{3}{2},\ 0\right)$ and (0, 6)

g



The graph meets the axes at (0, 0)

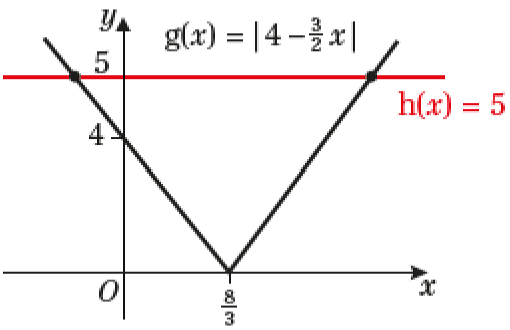
h



The graph meets the axes at

$\left(\frac{1}{3},\ 0\right)$ and (0, −1)

5 a



b At the left-hand point of intersection:

$$4 - \frac{3}{2}x = 5$$

$$\frac{3}{2}x = -1$$

$$x = -\frac{2}{3}$$

At the right-hand point of intersection:

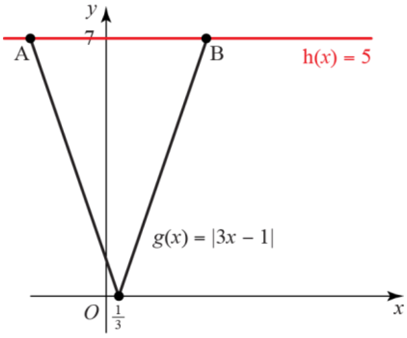
$$-(4 - \frac{3}{2}x) = 5$$

$$\frac{3}{2}x = 9$$

$$x = 6$$

The solutions are $x = -\frac{2}{3}$ and $x = 6$

6 a



At A: $-(3x-1)=5$

$$-3x=4$$

$$x=-\frac{4}{3}$$

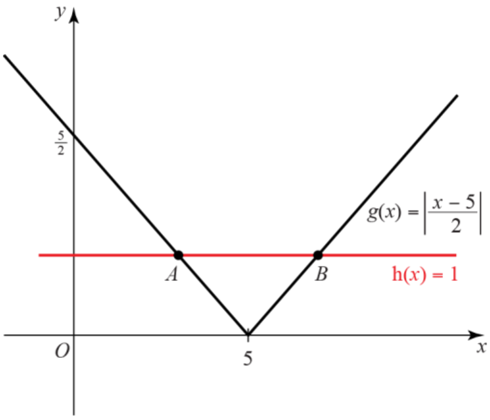
At B: $3x-1=5$

$$3x=6$$

$$x=2$$

The solutions are $x=-\frac{4}{3}$ and $x=2$

b



At A: $-\left(\frac{x-5}{2}\right)=1$

$$x-5=-2$$

$$x=3$$

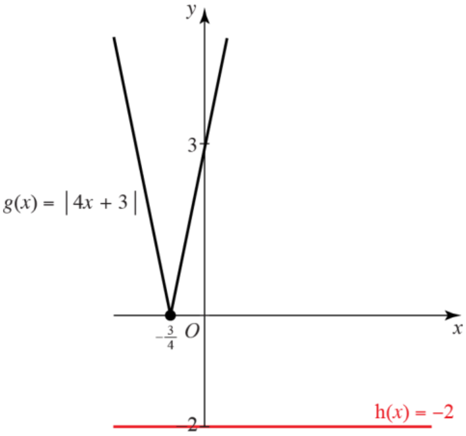
At B: $\frac{x-5}{2}=1$

$$x-5=2$$

$$x=7$$

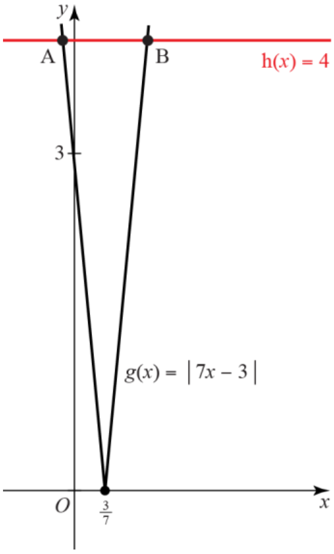
The solutions are $x=3$ and $x=7$

c



The graphs do not intersect so there are no solutions.

d



At A: $-(7x-3)=4$

$$7x=-1$$

$$x=-\frac{1}{7}$$

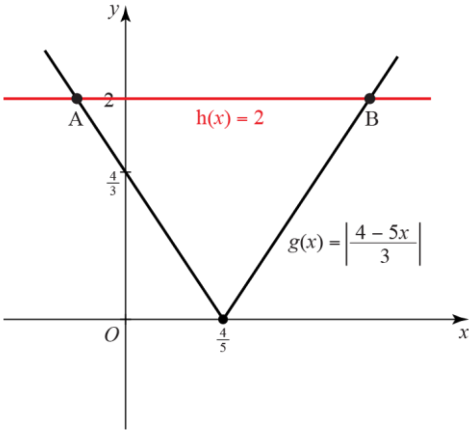
At B: $7x-3=4$

$$7x=7$$

$$x=1$$

The solutions are $x=-\frac{1}{7}$ and $x=1$

6 e



At A: $\frac{4-5x}{3} = 2$

$-5x = 2$

$x = -\frac{2}{5}$

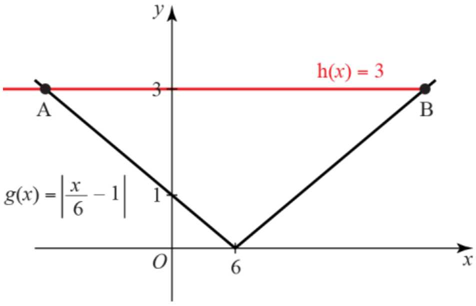
At B: $-\left(\frac{4-5x}{3}\right) = 2$

$-5x = -10$

$x = 2$

The solutions are $x = -\frac{2}{5}$ and $x = 2$

f



At A: $-\left(\frac{x}{6} - 1\right) = 3$

$\frac{x}{6} = -2$

$x = -12$

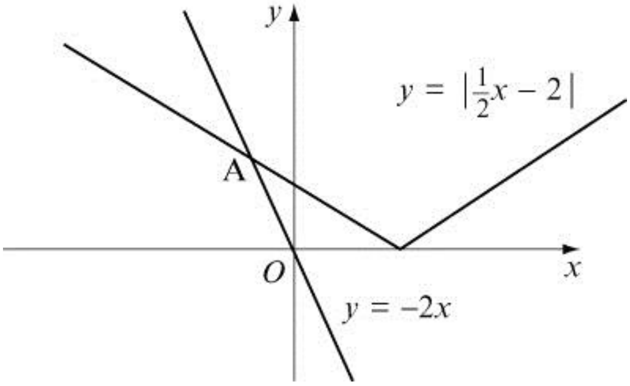
At B: $\frac{x}{6} - 1 = 3$

$\frac{x}{6} = 4$

$x = 24$

The solutions are
$x = -12$ and $x = 24$

7 a



7 b Intersection point *A* is

on the reflected part of $y = \frac{1}{2}x - 2$

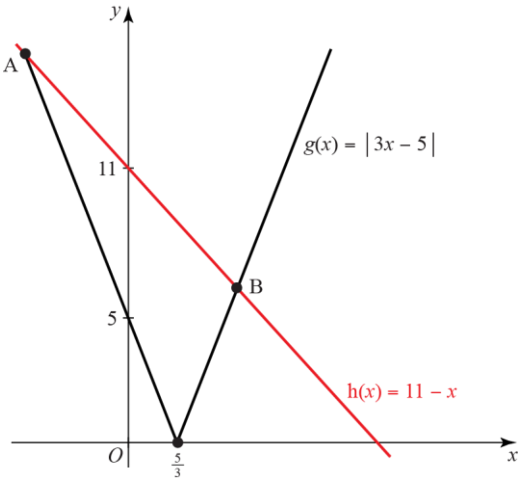
$$-\left(\frac{1}{2}x - 2\right) = -2x$$

$$2x - \frac{1}{2}x = -2$$

$$\frac{3}{2}x = -2$$

$$x = -\frac{4}{3}$$

8



At A: $-(3x - 5) = 11 - x$

$$-6 = 2x$$

$$x = -3$$

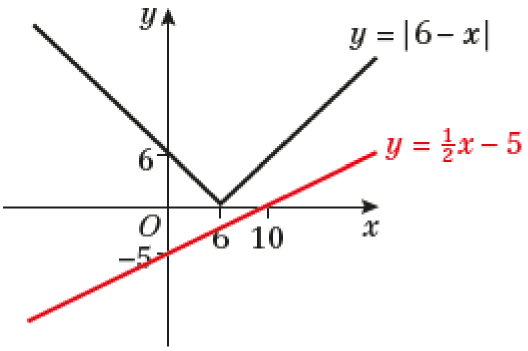
At B: $3x - 5 = 11 - x$

$$4x = 16$$

$$x = 4$$

The solutions are $x = -3$ and $x = 4$

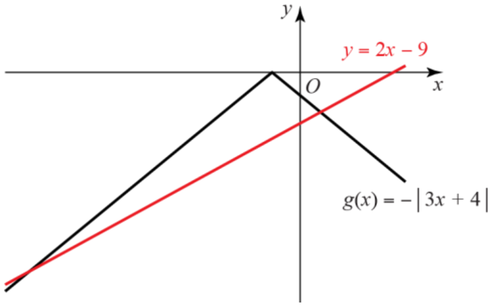
9 a



b The two graphs do not intersect, therefore there are no solutions to the equation $|6 - x| = \frac{1}{2}x - 5$

10 The value for *x* cannot be negative as it equals a modulus which is ≥ 0

11 a



11 b At the left-hand point of intersection:

$$3x+4=2x-9$$
$$x=-13$$

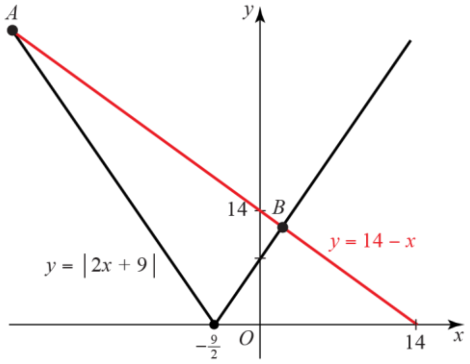
At the right-hand point of intersection:

$$-(3x+4)=2x-9$$
$$-5x=-5$$
$$x=1$$

The points of intersection are $x=-13$ and $x=1$

So the solution to $-|3x+4|<2x-9$ is $x < -13$ and $x > 1$

12



At A: $-(2x+9)=14-x$

$$-x=23$$
$$x=-23$$

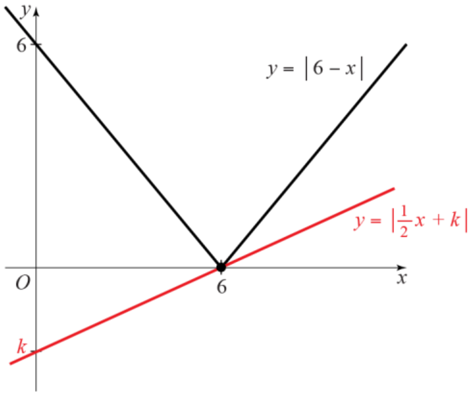
At B: $2x+9=14-x$

$$3x=5$$
$$x=\frac{5}{3}$$

The points of intersection are $x=-23$ and $x=\frac{5}{3}$

So the solution to $|2x+9|<14-x$ is $-23<x<\frac{5}{3}$

13 a For there to be one solution, the graphs $y=|6-x|$ and $y=\frac{1}{2}x+k$ must intersect once at the vertex of $y=|6-x|$



This vertex occurs at (6, 0)

Substituting (6, 0) into $y=\frac{1}{2}x+k$ gives:

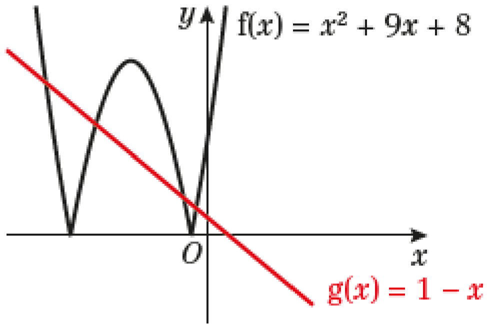
$$0=\frac{1}{2}\times 6+k$$
$$0 = 3 + k$$
$$k = -3$$

b $6-x=\frac{1}{2}x-3$

$$9=\frac{3}{2}x$$
$$x = 6$$

Challenge

a



b At the far left-hand and far right-hand points of intersection:

$$x^2+9x+8=1-x$$
$$x^2+10x+7=0$$

Using the formula:

$$x=\frac{-10\pm\sqrt{10^2-4\times1\times7}}{2\times1}$$
$$x=\frac{-10\pm\sqrt{72}}{2}$$
$$x=\frac{-10\pm6\sqrt{2}}{2}$$
$$x=-5\pm3\sqrt{2}$$

At the two inside points of intersection:

$$-(x^2+9x+8)=1-x$$
$$x^2+9x+8=x-1$$
$$x^2+8x+9=0$$

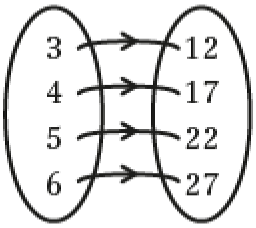
Using the formula:

$$x=\frac{-8\pm\sqrt{8^2-4\times1\times9}}{2\times1}$$
$$x=\frac{-8\pm\sqrt{28}}{2}$$
$$x=\frac{-8\pm2\sqrt{7}}{2}$$
$$x=-4\pm\sqrt{7}$$

The four solutions are $x=-5\pm3\sqrt{2}$ and $x=-4\pm\sqrt{7}$

Functions and graphs 2B

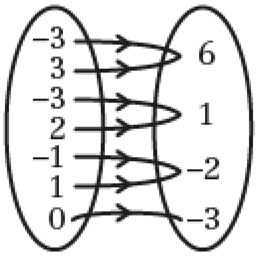
1 a i



ii Every element in set A gets mapped to one element in set B, so the mapping is one-to-one.

iii {f(x) = 12, 17, 22, 27}

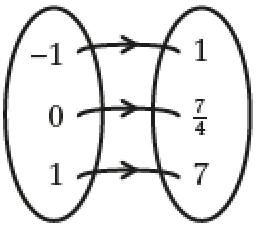
b i



ii Two elements in set A get mapped to one element in set B, so the mapping is many-to-one.

iii {g(x) = −3, −2, 1, 6}

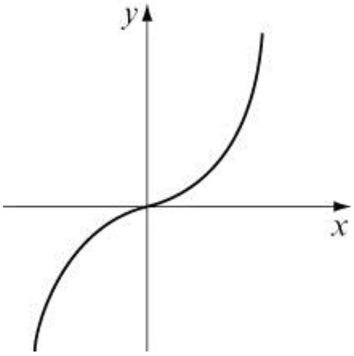
c i



ii Every element in set A gets mapped to one element in set B, so the mapping is one-to-one.

iii {h(x) = 1, $\frac{7}{4}$, 7}

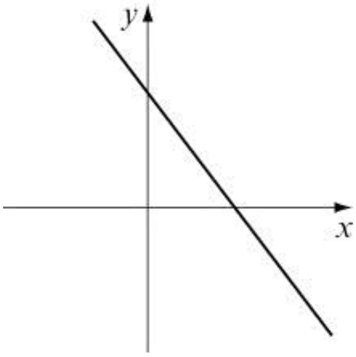
2 a



i One-to-one as each value of x is mapped to a single value of y

ii Yes, this mapping could represent a function.

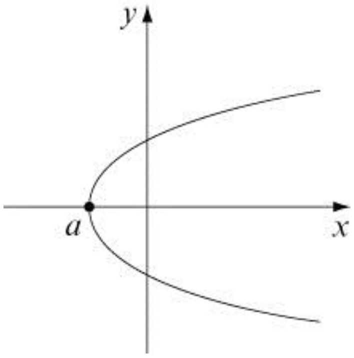
b



i One-to-one as each value of x is mapped to a single value of y

ii Yes, this mapping could represent a function.

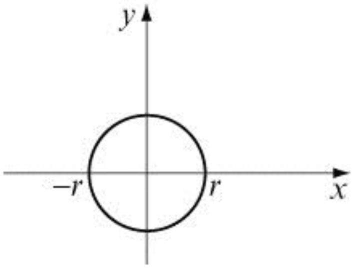
2 c



i One-to-many (see explanation in part **ii**)

ii Not a function.
Values of x which are less than a do not get mapped to a value of y.
Values of x which are greater than a get mapped to two values of y.

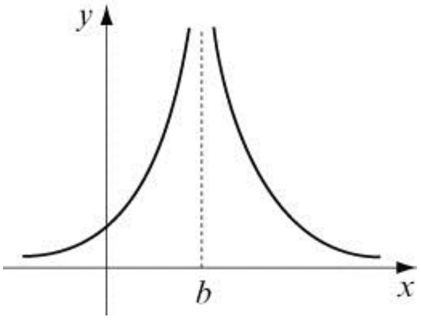
d



i One-to-many (see explanation in part **ii**)

ii Not a function.
Values of x for which $-r < x < r$ get mapped to two values of y.
Values of x for which $x < -r$ or $x > r$ don't get mapped to a value of y.

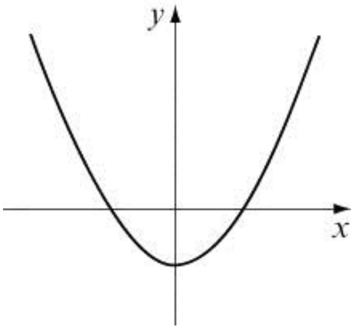
e



i One-to-one as each value of x (except for $x = b$) is mapped to a single value of y.

ii Not a function. The value $x = b$ doesn't get mapped anywhere.

f



i Many-to-one as there are two values of x which map to each value of y.

ii Yes, this mapping could represent a function.

3 **a** Substituting $x = a$ and $p(a) = 16$ into $p : x \mapsto 3x - 2,\ x \in \mathbb{R}$ gives:

$16 = 3a - 2$

$18 = 3a$

$a = 6$

b Substituting $x = b$ and $q(b) = 17$ into $q : x \mapsto x^2 - 3,\ x \in \mathbb{R}$ gives:

$17 = b^2 - 3$

$20 = b^2$

$b = \pm\sqrt{20}$

$b = \pm 2\sqrt{5}$

c Substituting $x = c$ and $r(c) = 34$ into $r : x \mapsto 2 \times 2^x + 2,\ x \in \mathbb{R}$ gives:

$34 = 2\ ´\ 2^c + 2$

$32 = 2\ ´\ 2^c$

$16 = 2^c$

$c = 4$

d Substituting $x = d$ and $s(d) = 0$ into $s : x \mapsto x^2 + x - 6,\ x \in \mathbb{R}$ gives:

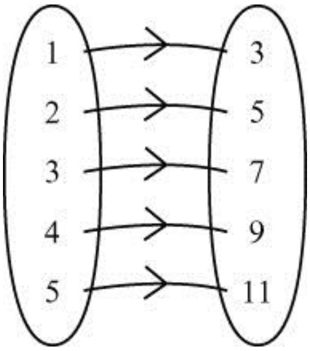
$0 = d^2 + d - 6$

$0 = (d + 3)(d - 2)$

$d = 2,\ -3$

4 **a** $f(x) = 2x + 1$

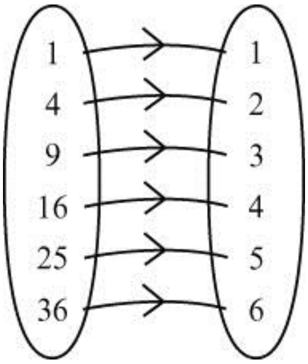
i



ii One-to-one function as each value of *x* maps to a single value of *y*.

4 **b** $g : x \mapsto \sqrt{x}$

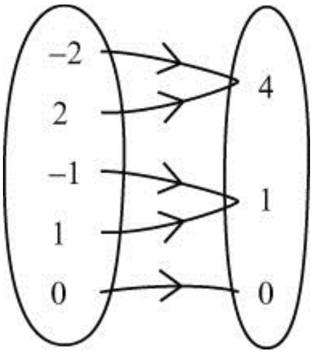
i



ii One-to-one function as each value of *x* maps to a single value of *y*.

c $h(x) = x^2$

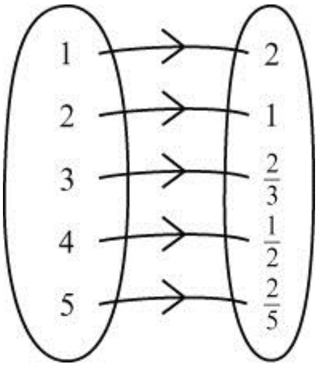
i



ii Many-to-one function as there are four values of *x* which map to two values of *y*.

4 d $\text{j}: x \mapsto \dfrac{2}{x}$

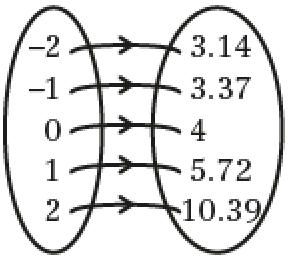
i



ii One-to-one function as each value of *x* maps to a single value of *y*.

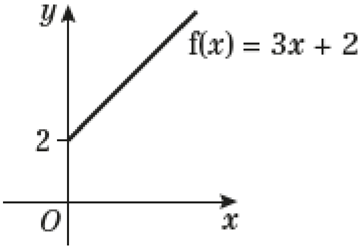
e $k(x) = e^x + 3$

i



ii Every element in set A gets mapped to one element in set B, so the mapping is one-to-one.

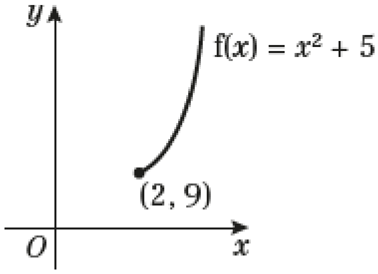
5 a i



ii Range of f(*x*) is $f(x) \geq 2$

iii One-to-one function as each value of *x* maps to a single value of *y*.

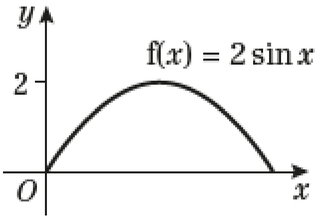
b i



ii Range of f(*x*) is $f(x) \geq 9$

iii One-to-one function as each value of *x* maps to a single value of *y*

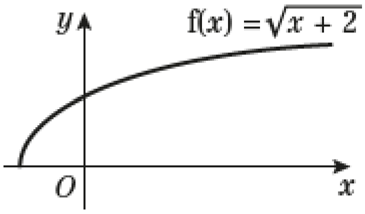
c i



ii Range of f(*x*) is $0 \leq f(x) \leq 2$

iii Many-to-one function as there are two values of *x* which map to a single value of *y*

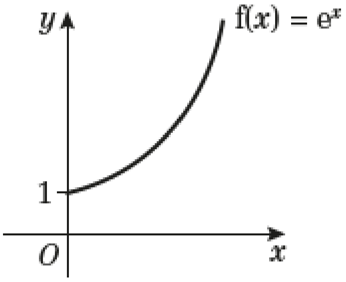
d i



ii Range of f(*x*) is $f(x) \geq 0$

iii One-to-one function as each value of *x* maps to a single value of *y*

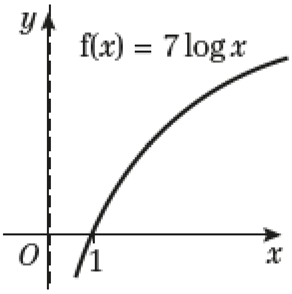
5 e i



ii Range of f(x) is f(x) ≥ 1

iii One-to-one function as each value of x maps to a single value of y

f i



ii Range is $f(x) \in \mathbb{R}$

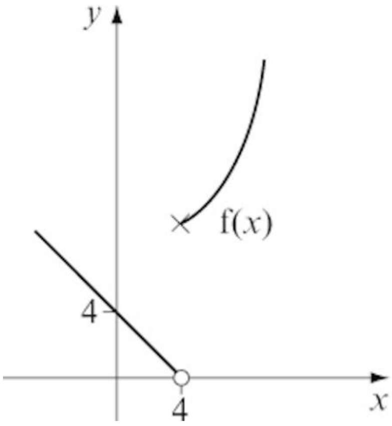
iii One-to-one function as each value of x maps to a single value of y

6 a Although g(x) is supposed to be defined on all real numbers, it does not map the element ‘4’ of the domain to any point in the range. Hence g(x) is not a function.

$f(4) = 25$, so for each $x \in \mathbb{R}$ there exists a y such that $f(x) = y$

Hence, $f(x)$ is a function.

b



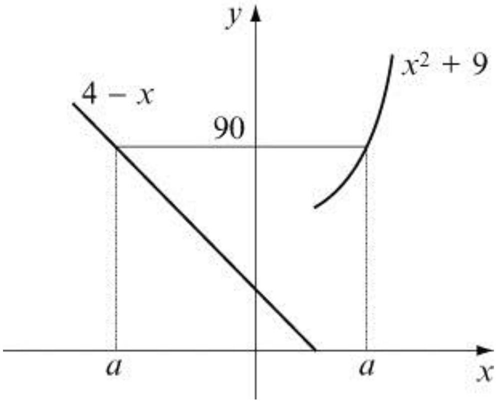
c i $f(3) = 4 - 3 = 1$

(Use $4 - x$ as $3 < 4$)

ii $f(10) = 10^2 + 9 = 109$

(Use $x^2 + 9$ as $10 > 4$)

d



The negative value of a is where

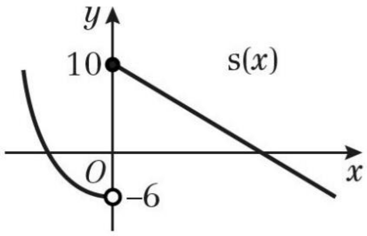
$4 - a = 90 \Rightarrow a = -86$

The positive value of a is where

$$a^2 + 9 = 90$$
$$a^2 = 81$$
$$a = \pm 9$$
$$a = 9$$

The values of a are -86 and 9

7 a



b There is no solution to
$10 - x = 43$ for $x \geq 0$

$s(a) = 43$ only when

$x^2 - 6 = 43$

$x^2 = 49$

$x = -7$

x cannot be 7, since

$s(x) = x^2 - 6$ for $x < 0$

c The negative solution is where

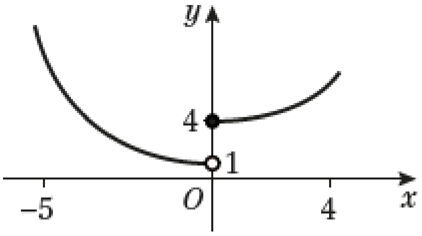
$$x^2 - 6 = x$$
$$x^2 - x - 6 = 0$$
$$(x - 3)(x + 2) = 0$$
$$x = 3 \text{ or } x = -2$$
$$\text{As } x < 0, x = -2$$

The positive solution is where

$$10 - x = x$$
$$2x = 10$$
$$x = 5$$

The solutions are $x = -2$ and $x = 5$

8 a



b $p(a) = 50$

The negative solution is where

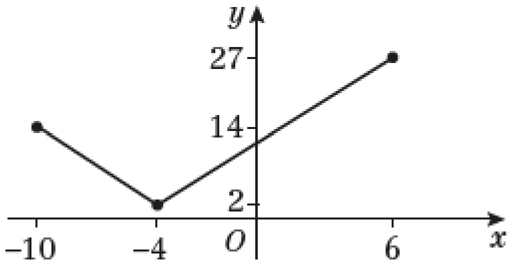
$$e^{-a} = 50$$
$$-a = \ln(50)$$
$$a = -3.91$$

The positive solution is where

$$a^3 + 4 = 50$$
$$a^3 = 46$$
$$a = 3.58$$

The solutions are
$a = -3.91$ and $a = 3.58$

9 a



b Range of $h(x)$ is $\{2 \leq h(x) \leq 27\}$

c $h(a) = 12$

One solution is for the function
$h(x) = -2x - 6$

$$\Rightarrow -2a - 6 = 12$$
$$\Rightarrow a = -9$$

The other solution is for the function

$$h(x) = \frac{5}{2}x + 12$$
$$\Rightarrow \frac{5}{2}a + 12 = 12$$
$$\Rightarrow a = 0$$

The solutions are $a = -9$ and $a = 0$

10 $\mathrm{g}(x)=cx+d$

$\mathrm{g}(3)=10 \Rightarrow c\times 3+d=10$

$\Rightarrow 3c+d=10 \quad (1)$

$\mathrm{g}(8)=12 \Rightarrow c\times 8+d=12$

$\Rightarrow 8c+d=12 \quad (2)$

$(2)-(1) \Rightarrow 5c=2$

$\Rightarrow c=\frac{2}{5}$

Substitute $c=\frac{2}{5}$ into (1):

$3\times\frac{2}{5}+d=10$

$\frac{6}{5}+d=10$

$d=\frac{44}{5}$

11 $\mathrm{f}(x)=ax^3+bx-5$

$\mathrm{f}(1)=-4 \Rightarrow a\times 1^3+b\times 1-5=-4$

$\Rightarrow a+b-5=-4$

$\Rightarrow a+b=1 \quad (1)$

$\mathrm{f}(2)=9 \Rightarrow a\times 2^3+b\times 2-5=9$

$\Rightarrow 8a+2b-5=9$

$\Rightarrow 8a+2b=14$

$\Rightarrow 4a+b=7 \quad (2)$

$(2)-(1) \Rightarrow 3a=6$

$\Rightarrow a=2$

Substitute $a=2$ in (1):

$2+b=1$

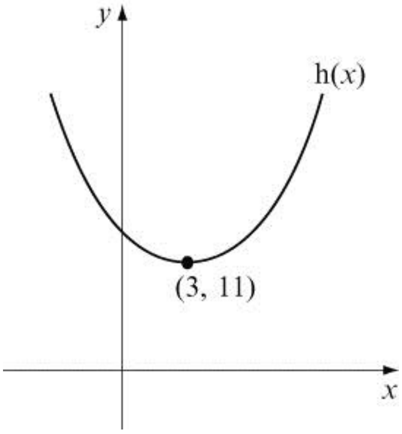
$b=-1$

12 $\mathrm{h}(x)=x^2-6x+20$

$=(x-3)^2-9+20$

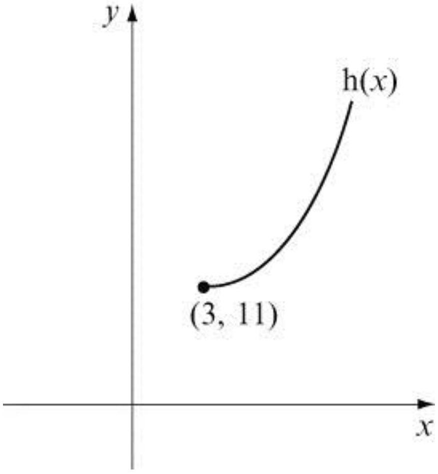
$=(x-3)^2+11$

This is a ∪–shaped quadratic with minimum point at (3, 11)



This is a many-to-one function.

For h(x) to be one-to-one, we must restrict domain to $x\geq 3$



Hence smallest value of a is $a=3$

Functions and graphs 2C

1 **a** $$\begin{aligned}\mathrm{pq}(-8) &= \mathrm{p}\left(\frac{-8}{4}\right)\\ &= \mathrm{p}(-2)\\ &= 1-3(-2)\\ &= 7\end{aligned}$$

b $$\begin{aligned}\mathrm{qr}(5) &= \mathrm{q}[(5-2)^2]\\ &= \mathrm{q}(9)\\ &= \frac{9}{4}\end{aligned}$$

c $$\begin{aligned}\mathrm{rq}(6) &= \mathrm{r}\left(\frac{6}{4}\right)\\ &= \mathrm{r}\left(\frac{3}{2}\right)\\ &= \left(\frac{3}{2}-2\right)^2\\ &= \frac{1}{4}\end{aligned}$$

d $$\begin{aligned}\mathrm{p}^2(-5) &= \mathrm{p}(1-3(-5))\\ &= \mathrm{p}(16)\\ &= 1-3(16)\\ &= -47\end{aligned}$$

e $$\begin{aligned}\mathrm{pqr}(8) &= \mathrm{pq}[(8-2)^2]\\ &= \mathrm{pq}(36)\\ &= \mathrm{p}\left(\frac{36}{4}\right)\\ &= \mathrm{p}(9)\\ &= 1-3(9)\\ &= -26\end{aligned}$$

2 **a** $$\begin{aligned}\mathrm{fg}(x) &= \mathrm{f}(x^2-4)\\ &= 4(x^2-4)+1\\ &= 4x^2-15, \quad x\in\mathbb{R}\end{aligned}$$

b $$\begin{aligned}\mathrm{gf}(x) &= \mathrm{g}(4x+1)\\ &= (4x+1)^2-4\\ &= 16x^2+8x-3, \quad x\in\mathbb{R}\end{aligned}$$

c $$\begin{aligned}\mathrm{gh}(x) &= \mathrm{g}\left(\frac{1}{x}\right)\\ &= \left(\frac{1}{x}\right)^2-4\\ &= \frac{1}{x^2}-4, \quad x\in\mathbb{R}, x\neq 0\end{aligned}$$

d $$\begin{aligned}\mathrm{fh}(x) &= \mathrm{f}\left(\frac{1}{x}\right)\\ &= 4\times\left(\frac{1}{x}\right)+1\\ &= \frac{4}{x}+1, \quad x\in\mathbb{R}, x\neq 0\end{aligned}$$

e $$\begin{aligned}\mathrm{f}^2(x) &= \mathrm{ff}(x)\\ &= \mathrm{f}(4x+1)\\ &= 4(4x+1)+1\\ &= 16x+5, \quad x\in\mathbb{R}\end{aligned}$$

3 **a** $\text{fg}(x) = \text{f}(x^2)$

$= 3x^2 - 2, \quad x \in \mathbb{R}$

b $\text{gf}(x) = \text{g}(3x-2)$

$= (3x-2)^2$

When $\text{fg}(x) = \text{gf}(x)$ then

$$3x^2 - 2 = (3x-2)^2$$
$$3x^2 - 2 = 9x^2 - 12x + 4$$
$$0 = 6x^2 - 12x + 6$$
$$0 = x^2 - 2x + 1$$
$$0 = (x-1)^2$$

Hence $x = 1$

4 **a** $\text{qp}(x) = \text{q}\left(\dfrac{1}{x-2}\right)$

$= 3 \times \left(\dfrac{1}{x-2}\right) + 4$

$= \dfrac{3}{x-2} + \dfrac{4(x-2)}{x-2}$

$= \dfrac{4x-5}{x-2}, \quad x \in \mathbb{R}, x \neq 2$

b If $\text{qp}(m) = 16$ then

$$3\left(\frac{1}{m-2}\right) + 4 = 16$$
$$\frac{3}{m-2} = 12$$
$$3 = 12(m-2)$$
$$\frac{3}{12} = m-2$$
$$\frac{1}{4} = m-2$$
$$m = \frac{9}{4}$$

5 **a** $\text{fg}(6) = \text{f}\left(\dfrac{3(6)-2}{2}\right)$

$= \text{f}(8)$

$= |9 - 4(8)|$

$= |-23|$

$= 23$

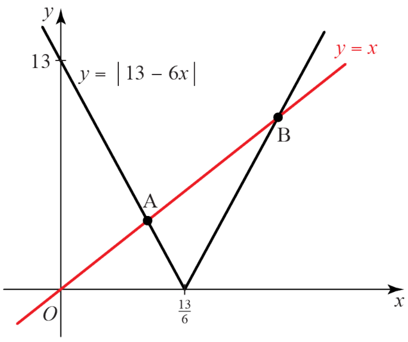
b $\text{fg}(x) = \text{f}\left(\dfrac{3x-2}{2}\right)$

$= \left|9 - 4\left(\dfrac{3x-2}{2}\right)\right|$

$= |9 - 6x + 4|$

$= |13 - 6x|$

Now $\text{fg}(x) = x$ when $|13 - 6x| = x$



At A: $13 - 6x = x$

$13 = 7x$

$x = \dfrac{13}{7}$

At B: $-(13 - 6x) = x$

$5x = 13$

$x = \dfrac{13}{5}$

The solutions are

$x = \dfrac{13}{7}$ and $x = \dfrac{13}{5}$

6 **a** $$f^2(x) = f\left(\frac{1}{x+1}\right)$$

$$= \left(\frac{1}{\left(\frac{1}{x+1}\right)+1}\right)$$

$$= \left(\frac{1}{\left(\frac{1+x+1}{x+1}\right)}\right)$$

$$= \left(\frac{x+1}{x+2}\right), \quad x \neq -1, x \neq -2$$

b $$f^3(x) = f\left(\frac{x+1}{x+2}\right)$$

$$= \left(\frac{1}{\left(\frac{x+1}{x+2}\right)+1}\right)$$

$$= \left(\frac{1}{\left(\frac{x+1+x+2}{x+2}\right)}\right)$$

$$= \left(\frac{x+2}{2x+3}\right), x \neq -1, x \neq -2, x \neq -\tfrac{3}{2}$$

7 **a** $$st(x) = s(x+3)$$
$$= 2^{x+3}, \quad x \in \mathbb{R}$$

b $$ts(x) = t(2^x)$$
$$= 2^x + 3, \quad x \in \mathbb{R}$$

c $$2^{x+3} = 2^x + 3$$
$$2^{x+3} - 2^x = 3$$
$$2^x \times 2^3 - 2^x = 3$$
$$2^x(8-1) = 3$$
$$2^x = \frac{3}{7}$$
$$x \ln 2 = \ln\left(\frac{3}{7}\right)$$
$$x = \frac{\ln\left(\frac{3}{7}\right)}{\ln 2}$$

8 **a** $$gf(x) = g(e^{5x})$$
$$= 4\ln(e^{5x})$$
$$= 4(5x)$$
$$= 20x, \ x \in \mathbb{R}$$

b $$fg(x) = f(4\ln x)$$
$$= e^{5(4\ln x)}$$
$$= e^{\ln x^{20}}$$
$$= x^{20}, \ x \in \mathbb{R}, x > 0$$

9 **a** $$qp(x) = q(\ln(x+3))$$
$$= e^{3(\ln(x+3))} - 1$$
$$= e^{\ln(x+3)^3} - 1$$
$$= (x+3)^3 - 1, \ x \in \mathbb{R}, x > -3$$

Since $x > -3$, so $qp(x) > -1$

b $$qp(7) = (7+3)^3 - 1$$
$$= 999$$

c $$qp(x) = (x+3)^3 - 1 = 124$$
$$(x+3)^3 = 125$$
$$x + 3 = 5$$
$$x = 2$$

10 $t^2(x) = t(5 - 2x)$
$= 5 - 2(5 - 2x)$
$= 5 - 10 + 4x$
$= -5 + 4x$

$$t^2(x) - (t(x))^2 = 0$$
$$-5 + 4x - (5 - 2x)^2 = 0$$
$$-5 + 4x - 25 + 20x - 4x^2 = 0$$
$$-4x^2 + 24x - 30 = 0$$
$$2x^2 - 12x + 15 = 0$$

Using the formula:

$$x = \frac{12 \pm \sqrt{(-12)^2 - 4\times 2\times 15}}{2\times 2}$$
$$= \frac{12 \pm \sqrt{24}}{4}$$
$$= \frac{12 \pm 2\sqrt{6}}{4}$$
$$= 3 \pm \frac{\sqrt{6}}{2}$$

11 a Range of g is $-8 \leq x \leq 12$

b From the graph,

$$g(x) = -\frac{1}{2}x + 12 \text{ for } 0 \leq x \leq 14$$
and $g(0) = 12$

So $gg(0) = g(12)$
$$= -\frac{1}{2}(12) + 12$$
$$= 6$$

c $$gh(7) = g\left(\frac{2(7)-5}{10-7}\right)$$
$$= g(3)$$
$$= -\frac{1}{2}(3) + 12$$
$$= 10.5$$

Functions and graphs 2D

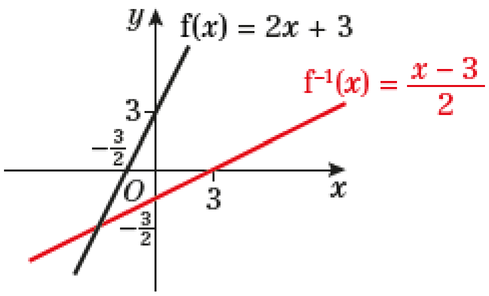
1 a i $y \in \mathbb{R}$

ii Let $y = \text{f}(x)$

$$y = 2x + 3$$
$$x = \frac{y-3}{2}$$
$$\text{f}^{-1}(x) = \frac{x-3}{2}$$

iii The domain of $\text{f}^{-1}(x)$ is $x \in \mathbb{R}$
The range of $\text{f}^{-1}(x)$ is $y \in \mathbb{R}$

iv



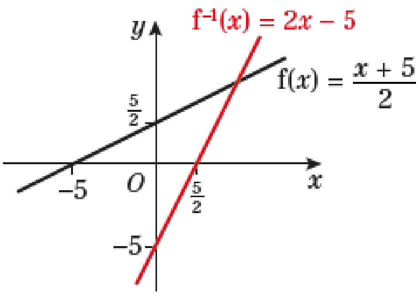
b i $y \in \mathbb{R}$

ii Let $y = \text{f}(x)$

$$y = \frac{x+5}{2}$$
$$x = 2y - 5$$
$$\text{f}^{-1}(x) = 2x - 5$$

iii The domain of $\text{f}^{-1}(x)$ is $x \in \mathbb{R}$
The range of $\text{f}^{-1}(x)$ is $y \in \mathbb{R}$

iv



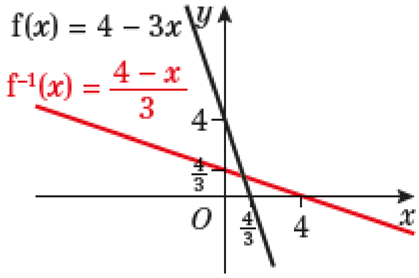
c i $y \in \mathbb{R}$

ii Let $y = \text{f}(x)$

$$y = 4 - 3x$$
$$x = \frac{4-y}{3}$$
$$\text{f}^{-1}(x) = \frac{4-x}{3}$$

iii The domain of $\text{f}^{-1}(x)$ is $x \in \mathbb{R}$
The range of $\text{f}^{-1}(x)$ is $y \in \mathbb{R}$

iv



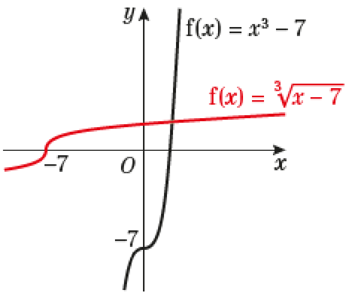
d i $y \in \mathbb{R}$

ii Let $y = \text{f}(x)$

$$y = x^3 - 7$$
$$x = \sqrt[3]{y+7}$$
$$\text{f}^{-1}(x) = \sqrt[3]{x+7}$$

iii The domain of $\text{f}^{-1}(x)$ is $x \in \mathbb{R}$
The range of $\text{f}^{-1}(x)$ is $y \in \mathbb{R}$

iv



2 a Range of f is $\text{f}(x) \in \mathbb{R}$
Let $y = \text{f}(x)$

$$y = 10 - x$$
$$x = 10 - y$$
$$\text{f}^{-1}(x) = 10 - x, \ \{x \in \mathbb{R}\}$$

2 b Range of f is $\text{f}(x) \in \mathbb{R}$

Let $y = \text{g}(x)$

$$y = \frac{x}{5}$$

$$x = 5y$$

$\text{g}^{-1}(x) = 5x, \ \{x \in \mathbb{R}\}$

c Range of f is $\text{f}(x) \neq 0$

Let $y = \text{h}(x)$

$$y = \frac{3}{x}$$

$$x = \frac{3}{y}$$

$\text{h}^{-1}(x) = \frac{3}{x}, \ \{x \neq 0\}$

d Range of f is $\text{f}(x) \in \mathbb{R}$

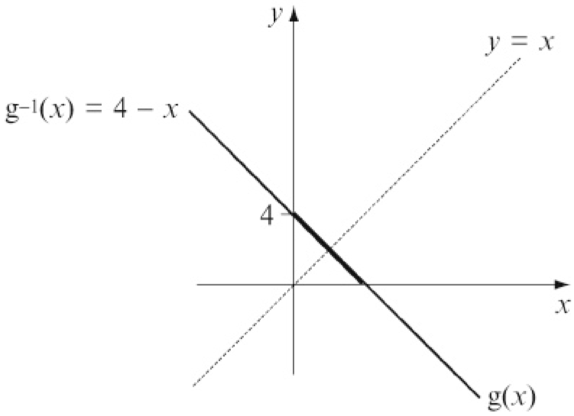
Let $y = \text{k}(x)$

$$y = x - 8$$

$$x = y + 8$$

$\text{k}^{-1}(x) = y + 8, \ \{x \in \mathbb{R}\}$

3



$\text{g} : x \mapsto 4 - x, \ \{x \in \mathbb{R}, \ x > 0\}$

g has range $\{\text{g}(x) \in \mathbb{R}, \ \text{g}(x) < 4\}$

The inverse function is $\text{g}^{-1}(x) = 4 - x$

Now $\{\text{Range g}\} = \{\text{Domain g}^{-1}\}$

and $\{\text{Domain g}\} = \{\text{Range g}\}$

Hence, $\text{g}^{-1}(x) = 4 - x, \ \{x \in \mathbb{R}, x < 4\}$

Although $\text{g}(x)$ and $\text{g}^{-1}(x)$ have identical equations, their domains and hence ranges are different, and so are not identical.

4 a i Maximum value of g when $x = \frac{1}{3}$

Hence $\left\{\text{g}(x) \in \mathbb{R}, \ 0 < \text{g}(x) \leq \frac{1}{3}\right\}$

ii $\text{g}^{-1}(x) = \frac{1}{x}$

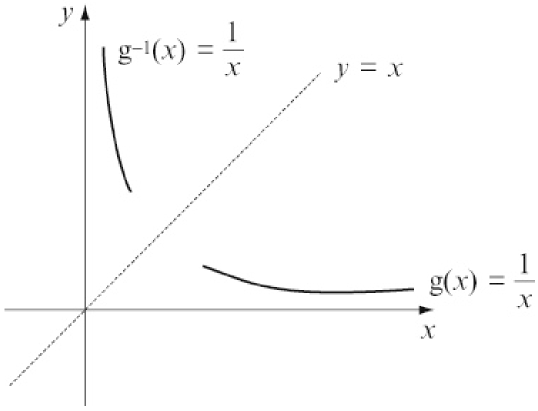
iii Domain g^{-1} = Range g

$\Rightarrow$ Domain $\text{g}^{-1} : \left\{x \in \mathbb{R}, 0 < x \leq \frac{1}{3}\right\}$

Range g^{-1} = Domain g

$\Rightarrow$ Range $\text{g}^{-1}(x) : \left\{\text{g}^{-1}(x) \in \mathbb{R}, \ \text{g}^{-1}(x) \geq 3\right\}$

iv



4 b i Minimum value of $g(x) = -1$

when $x = 0$

Hence $\{g(x) \in \mathbb{R},\ g(x) \geq -1\}$

ii Letting $y = 2x - 1 \Rightarrow x = \dfrac{y+1}{2}$

Hence $g^{-1}(x) = \dfrac{x+1}{2}$

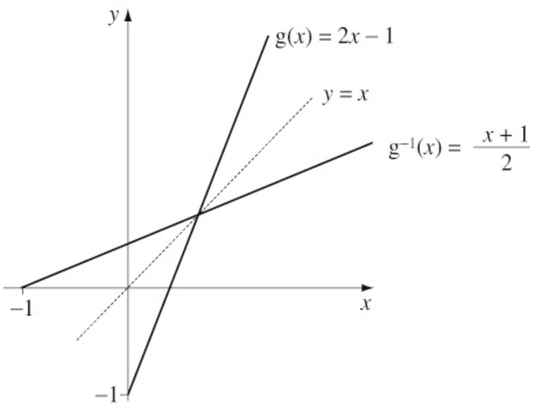
iii Domain g^{-1} = Range g

$\Rightarrow$ Domain $g^{-1}: \{x \in \mathbb{R}, x \geq -1\}$

Range g^{-1} = Domain g

$\Rightarrow$ Range $g^{-1}(x): \left\{\begin{array}{l} g^{-1}(x) \in \mathbb{R}, \\ \qquad g^{-1}(x) \geq 0 \end{array}\right\}$

iv



c i $g(x) \to +\infty$ as $x \to 2$

Hence $\{g(x) \in \mathbb{R},\ g(x) > 0\}$

ii Letting $y = \dfrac{3}{x-2} \Rightarrow x = \dfrac{2y+3}{y}$

Hence $g^{-1}(x) = \dfrac{2x+3}{x}$

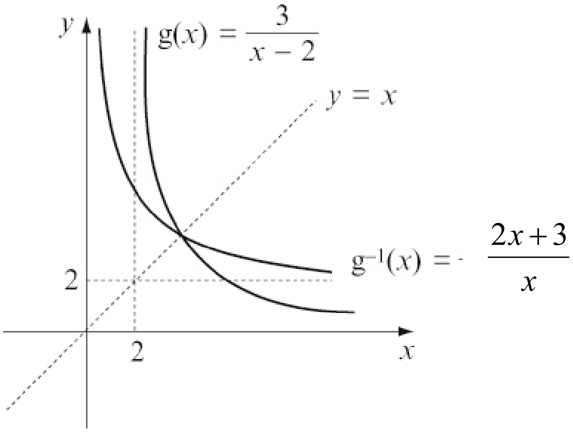
iii Domain g^{-1} = Range g

$\Rightarrow$ Domain $g^{-1}: \{x \in \mathbb{R}, x > 0\}$

Range g^{-1} = Domain g

$\Rightarrow$ Range $g^{-1}(x): \{g^{-1}(x) \in \mathbb{R},\ g^{-1}(x) > 2\}$

iv



d i Minimum value of $g(x) = 2$

when $x = 7$

Hence $\{g(x) \in \mathbb{R},\ g(x) \geq 2\}$

ii Letting $y = \sqrt{x-3} \Rightarrow x = y^2 + 3$

Hence $g^{-1}(x) = x^2 + 3$

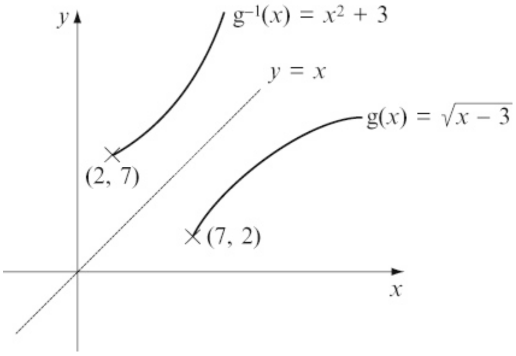
iii Domain g^{-1} = Range g

$\Rightarrow$ Domain $g^{-1}: \{x \in \mathbb{R}, x \geq 2\}$

Range g^{-1} = Domain g

$\Rightarrow$ Range $g^{-1}(x): \{g^{-1}(x) \in \mathbb{R},\ g^{-1}(x) \geq 7\}$

iv



4 e i $2^2+2=6$

Hence $\{g(x)\in\mathbb{R},\ g(x)>6\}$

ii Letting $y=x^2+2$

$y-2=x^2$

$x=\sqrt{y-2}$

Hence $g^{-1}(x)=\sqrt{x-2}$

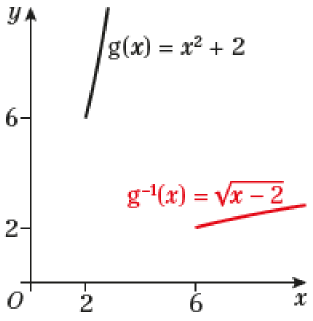
iii Domain g^{-1} = Range g

$\Rightarrow$ Domain $g^{-1}:\{x\in\mathbb{R},\ x>6\}$

Range g^{-1} = Domain g

$\Rightarrow$ Range $g^{-1}(x):\left\{g^{-1}(x)\in\mathbb{R},\ g^{-1}(x)>2\right\}$

iv



f i Minimum value of $g(x)=0$

when $x=2$

Hence $\{g(x)\in\mathbb{R},\ g(x)\geq 0\}$

ii Letting $y=x^3-8\Rightarrow x=\sqrt[3]{y+8}$

Hence $g^{-1}(x)=\sqrt[3]{x+8}$

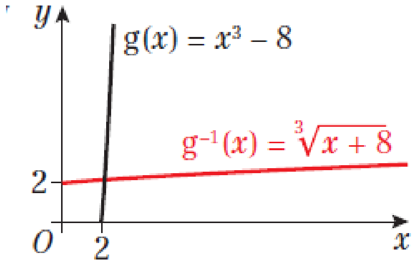
iii Domain g^{-1} = Range g

$\Rightarrow$ Domain $g^{-1}:\{x\in\mathbb{R},\ x\geq 0\}$

Range g^{-1} = Domain g

$\Rightarrow$ Range $g^{-1}(x):\left\{g^{-1}(x)\in\mathbb{R},\ g^{-1}(x)\geq 2\right\}$

iv



5 $t(x)=x^2-6x+5,\ \{x\in\mathbb{R},\ x\geq 5\}$

Let $y=x^2-6x+5$

$y=(x-3)^2-9+5$ (completing the square)

$y=(x-3)^2-4$

This has a minimum point at $(3,-4)$

For the domain $x\geq 5$, $t(x)$ is a one-to-one function so we can find an inverse function.

Make y the subject:

$y=(x-3)^2-4$

$y+4=(x-3)^2$

$\sqrt{y+4}=x-3$

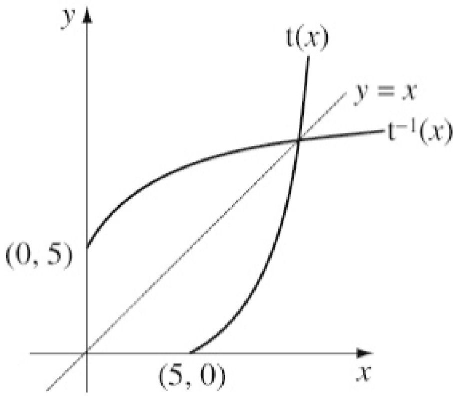
$\sqrt{y+4}+3=x$

5 (continued)

Domain t^{-1} = Range t

$\Rightarrow$ Domain $g^{-1}: \{x \in \mathbb{R}, x \geq 0\}$

Hence, $t^{-1}(x) = \sqrt{x+4}+3, \ \{x \in \mathbb{R}, x \geq 0\}$



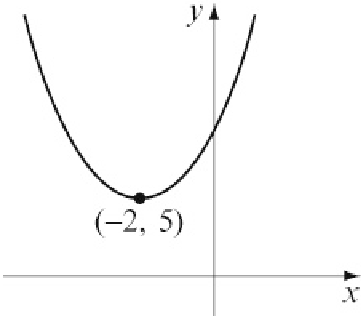
6 a $m(x) = x^2+4x+9, \ \{x \in \mathbb{R}, \ x > a\}$

Let $y = x^2+4x+9$

$$y = (x+2)^2 - 4 + 9$$

$$y = (x+2)^2 + 5$$

This has a minimum value of $(-2, 5)$



For $m(x)$ to have an inverse it must be one-to-one. Hence the least value of a is -2

b Changing the subject of the formula:

$$y = (x+2)^2 + 5$$

$$y - 5 = (x+2)^2$$

$$\sqrt{y-5} = x+2$$

$$\sqrt{y-5} - 2 = x$$

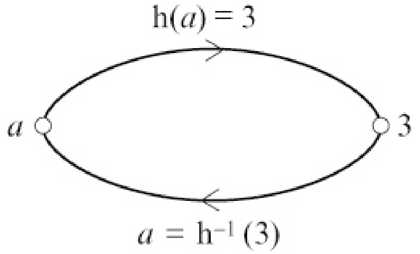
Hence $m^{-1}(x) = \sqrt{x-5} - 2$

c Domain of $m^{-1}(x)$: $\{x \in \mathbb{R}, \ x > 5\}$

7 a As $x \to 2$, $h(x) \to \frac{5}{0}$

and hence $h(x) \to \infty$

b To find $h^{-1}(3)$ we can find what element of the domain gets mapped to 3



Suppose $h(a) = 3$ for some a such that $a \neq 2$

Then $\frac{2a+1}{a-2} = 3$

$$2a + 1 = 3a - 6$$

$$7 = a$$

So $h^{-1}(3) = 7$

7 c Let $y = \dfrac{2x+1}{x-2}$ and find x as a function of y

$$y(x-2) = 2x+1$$
$$yx - 2y = 2x+1$$
$$yx - 2x = 2y+1$$
$$x(y-2) = 2y+1$$
$$x = \frac{2y+1}{y-2}$$

So $\mathrm{h}^{-1}(x) = \dfrac{2x+1}{x-2}, \quad \{x \in \mathbb{R},\ x \neq 2\}$

d If an element b is mapped to itself, then $\mathrm{h}(b) = b$

$$\frac{2b+1}{b-2} = b$$
$$2b+1 = b(b-2)$$
$$2b+1 = b^2 - 2b$$
$$0 = b^2 - 4b - 1$$
$$b = \frac{4 \pm \sqrt{16+4}}{2} = \frac{4 \pm \sqrt{20}}{2}$$
$$= \frac{4 \pm 2\sqrt{5}}{2} = 2 \pm \sqrt{5}$$

The elements $2+\sqrt{5}$ and $2-\sqrt{5}$ get mapped to themselves by the function.

8 a $\mathrm{nm}(x) = \mathrm{n}(2x+3)$

$$= \frac{2x+3-3}{2}$$
$$= x$$

b $\mathrm{mn}(x) = \mathrm{m}\left(\dfrac{x-3}{2}\right)$

$$= 2\left(\frac{x-3}{2}\right) + 3$$
$$= x$$

The functions m(x) and n(x) are the inverse of each other as $\mathrm{mn}(x) = \mathrm{nm}(x) = x$.

9 $\mathrm{st}(x) = \mathrm{s}\left(\dfrac{3-x}{x}\right)$

$$= \frac{3}{\left(\frac{3-x}{x}+1\right)}$$
$$= \frac{3}{\left(\frac{3-x+x}{x}\right)}$$
$$= x$$

$\mathrm{st}(x) = \mathrm{t}\left(\dfrac{3}{x+1}\right)$

$$= \frac{\left(3 - \frac{3}{x+1}\right)}{\left(\frac{3}{x+1}\right)}$$
$$= \frac{\left(\frac{3x+3-3}{x+1}\right)}{\left(\frac{3}{x+1}\right)}$$
$$= x$$

The functions s(x) and t(x) are the inverse of each other as $\mathrm{st}(x) = \mathrm{ts}(x) = x$

10 a Let $y = 2x^2 - 3$

The domain of $\mathrm{f}^{-1}(x)$ is the range of $\mathrm{f}(x)$.

$\mathrm{f}(x) = 2x^2 - 3,\ \{x \in \mathbb{R},\ x < 0\}$ has range $\mathrm{f}(x) > -3$

Letting $y = 2x^2 - 3 \Rightarrow x = \pm\sqrt{\dfrac{x+3}{2}}$

We need to consider the domain of $\mathrm{f}(x)$ to determine if *either*

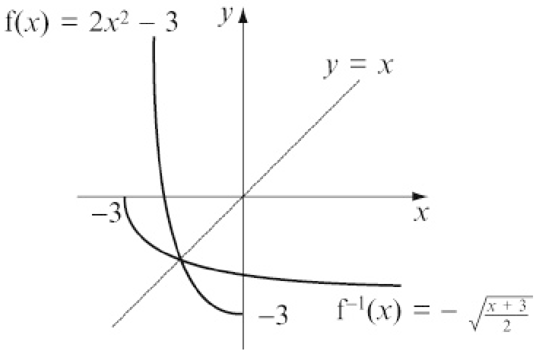
$$\mathrm{f}^{-1}(x) = +\sqrt{\frac{x+3}{2}} \quad or \quad \mathrm{f}^{-1}(x) = -\sqrt{\frac{x+3}{2}}$$

$\mathrm{f}(x) = 2x^2 - 3$ has domain $\{x \in \mathbb{R},\ x < 0\}$

10 a (continued)

Hence $f^{-1}(x)$ must be the negative square root

$$f^{-1}(x) = -\sqrt{\frac{x+3}{2}},\ \{x \in \mathbb{R},\ x > -3\}$$



b If $f(a) = f^{-1}(a)$ then a is negative (see graph).

Solve $f(a) = a$

$$2a^2 - 3 = a$$

$$2a^2 - a - 3 = 0$$

$$(2a-3)(a+1) = 0$$

$$a = \frac{3}{2}, -1$$

Therefore $a = -1$

11 a Range of f(x) is $f(x) > -5$

b Let $y = f(x)$

$y = e^x - 5$

$e^x = y + 5$

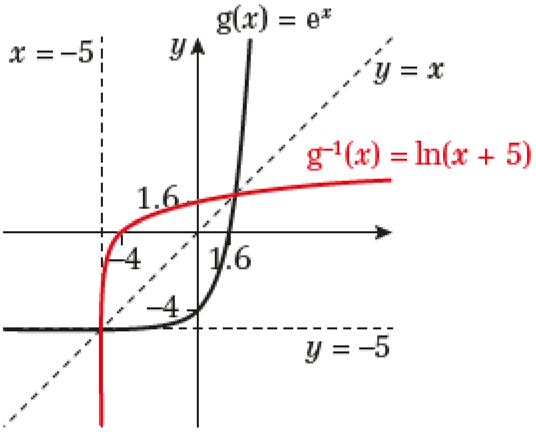
$x = \ln(y + 5)$

$f^{-1}(x) = \ln(x + 5)$

Range of f(x) is $f(x) > -5$,

so domain of $f^{-1}(x)$ is $\{x \in \mathbb{R}, x > -5\}$

c



d Let $y = g(x)$

$y = \ln(x - 4)$

$e^y = x - 4$

$x = e^y + 4$

$g^{-1}(x) = e^x + 4$

Range of g(x) is $g(x) \in \mathbb{R}$,

so domain of $g^{-1}(x)$ is $\{x \in \mathbb{R}\}$

e $g^{-1}(x) = 11$

$e^x + 4 = 11$

$e^x = 7$

$x = \ln 7$

$x = 1.95$

12 a

$$f(x) = \frac{3(x+2)}{x^2+x-20} - \frac{2}{x-4}$$

$$= \frac{3(x+2)}{(x+5)(x-4)} - \frac{2}{x-4}$$

$$= \frac{3(x+2)}{(x+5)(x-4)} - \frac{2(x+5)}{(x+5)(x-4)}$$

$$= \frac{3x+6-2x-10}{(x+5)(x-4)}$$

$$= \frac{x-4}{(x+5)(x-4)}$$

$$= \frac{1}{x+5},\ x > 4$$

12 b The range of f is

$$\left\{ f(x) \in \mathbb{R},\ f(x) < \frac{1}{9} \right\}$$

c Let $y = f(x)$

$$y = \frac{1}{x+5}$$

$$yx + 5y = 1$$

$$yx = 1 - 5y$$

$$x = \frac{1-5y}{y}$$

$$x = \frac{1}{y} - 5$$

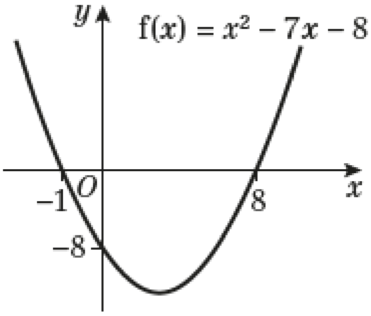
$$f^{-1}(x) = \frac{1}{x} - 5$$

The domain of $f^{-1}(x)$ is

$$\left\{ x \in \mathbb{R}, x < \frac{1}{9} \text{ and } x \neq 0 \right\}$$

Functions and graphs 2E

1 a



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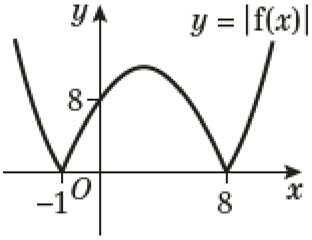

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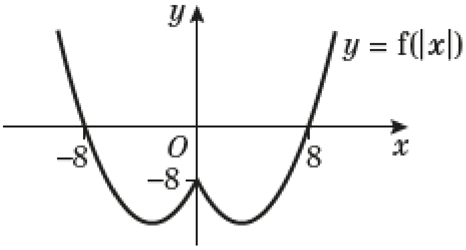
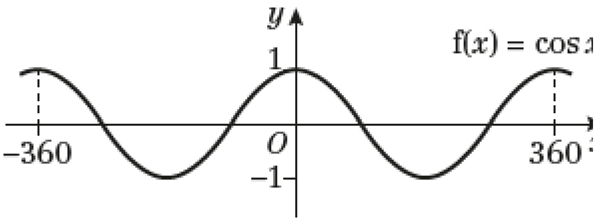


2 a



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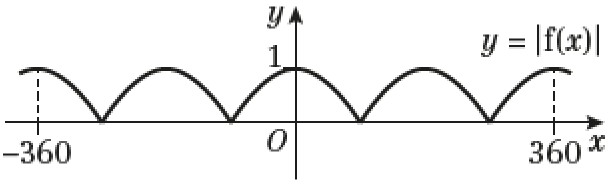

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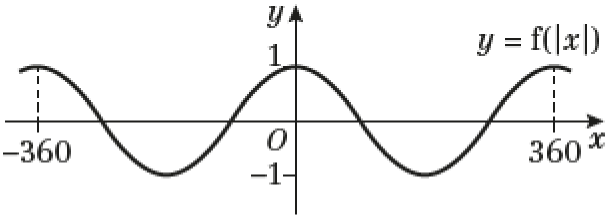
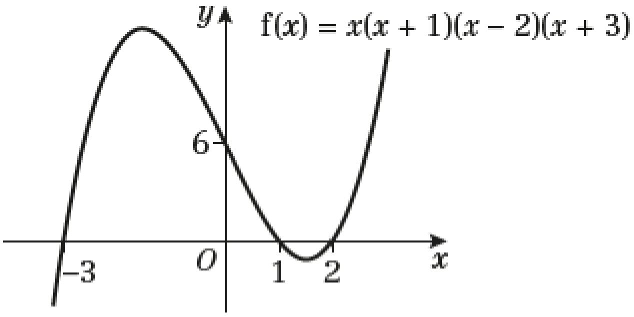


3 a



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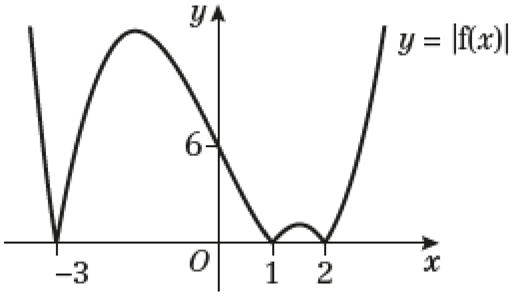
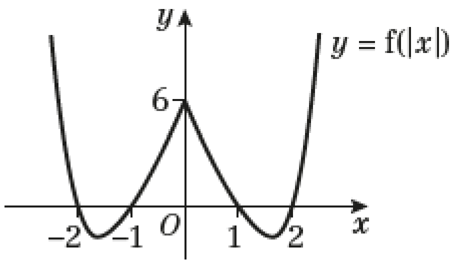
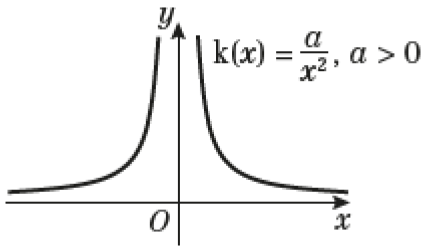


3 c

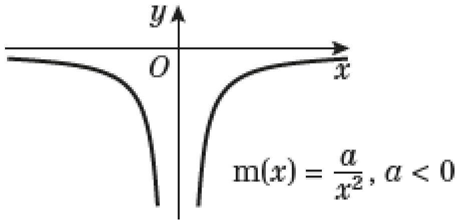


4 a



b There is no need to sketch $y=|\mathrm{k}(x)|$ and $y=\mathrm{k}(|x|)$ as these graphs would match the original graph.

c



d i $|\mathrm{k}(x)|=|\mathrm{m}(x)|$ is true:

$$|\mathrm{k}(x)|=\left|\frac{a}{x^2}\right|=\left|\frac{-a}{x^2}\right|=|\mathrm{m}(x)|$$

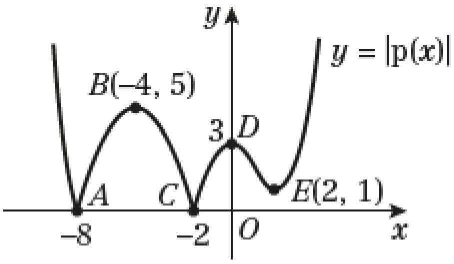
ii $\mathrm{k}(|x|)=\mathrm{m}(|x|)$ is false:

$$\mathrm{k}(|x|)=\frac{a}{|x|^2}\neq\frac{-a}{|x|^2}=\mathrm{m}(|x|)$$

iii $m(x)=\mathrm{m}(|x|)$ is true:

$$\mathrm{m}(x)=\frac{-a}{|x|^2}=\frac{-a}{|x|^2}=\mathrm{m}(|x|)$$

5 a



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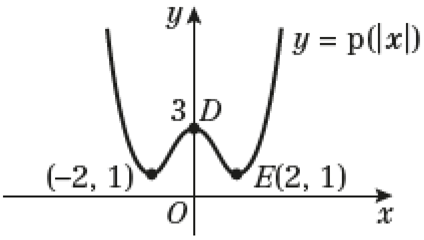
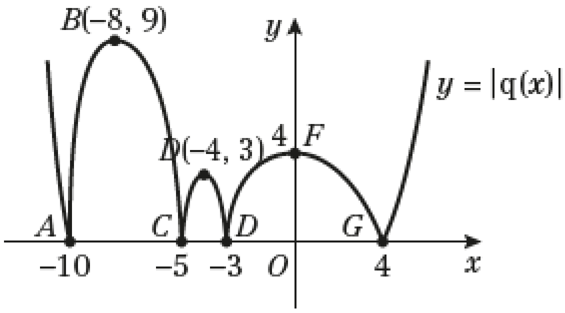


6 a



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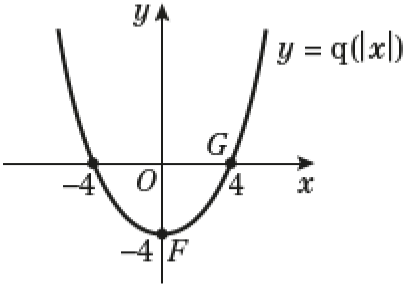
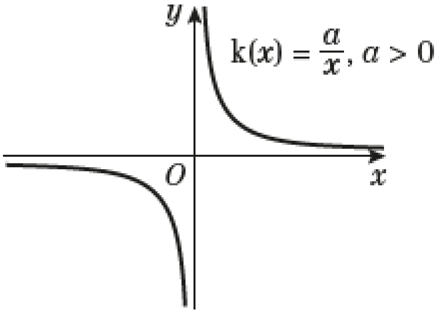


7 a



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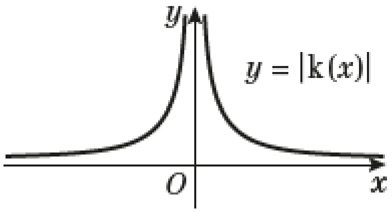

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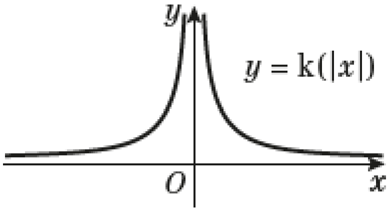
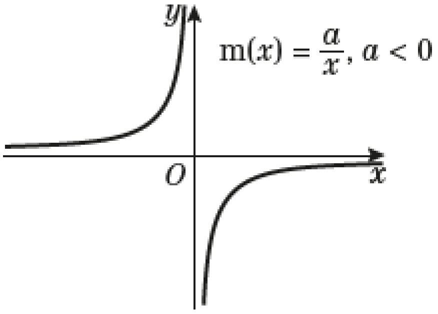


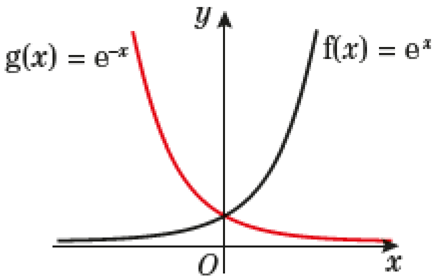
8 a



b $y = |\mathrm{m}(x)|$ and $y = \mathrm{m}(|x|)$ are reflections of each other in the x-axis.

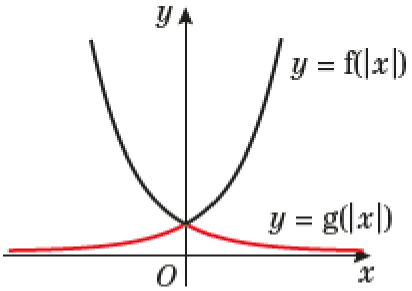
$|\mathrm{m}(x)| = -\mathrm{m}(|x|)$

9 a

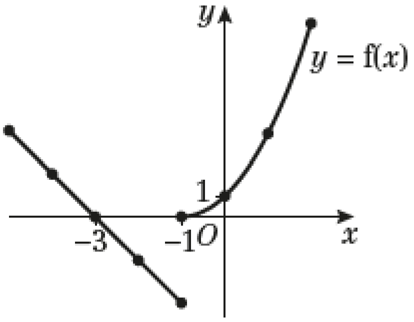


b The graphs of $y = |\mathrm{f}(x)|$ and $y = |\mathrm{g}(x)|$ are the same as the original graph.

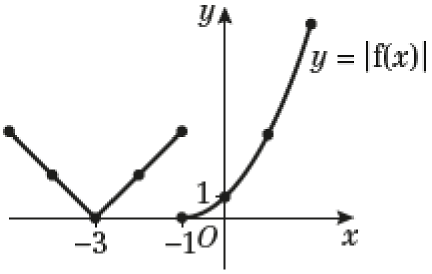
c



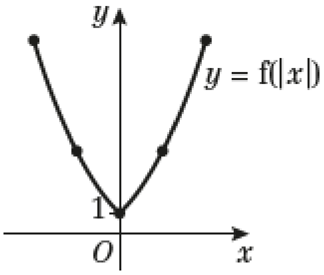
10 a



10 b



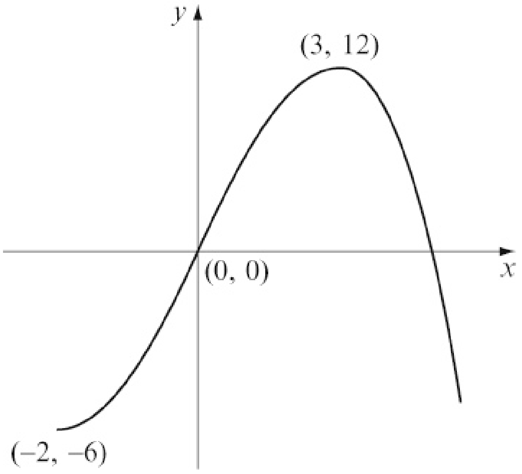
c



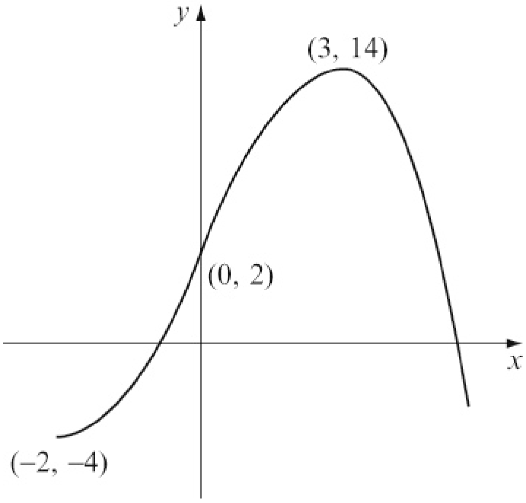
Functions and graphs 2F

1 **a** $y = 3\text{f}(x)$

Vertical stretch, scale factor 3.

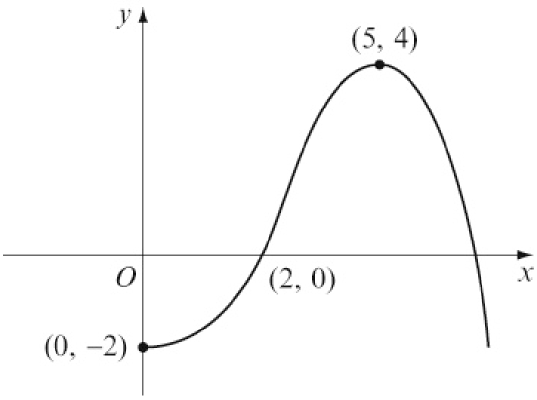


$y = 3\text{f}(x) + 2$. Vertical translation of +2.



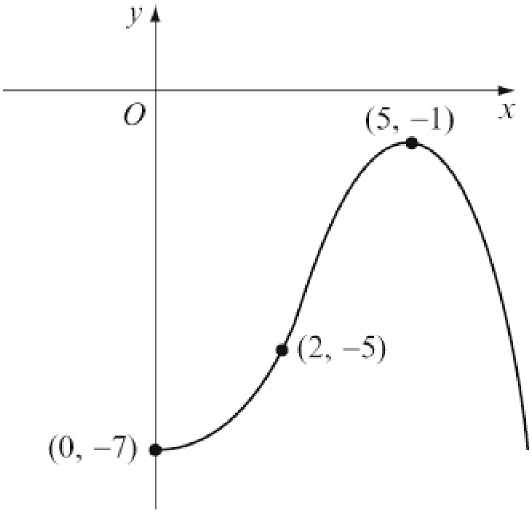
b $y = \text{f}(x-2)$.

Horizontal translation of $+2$.



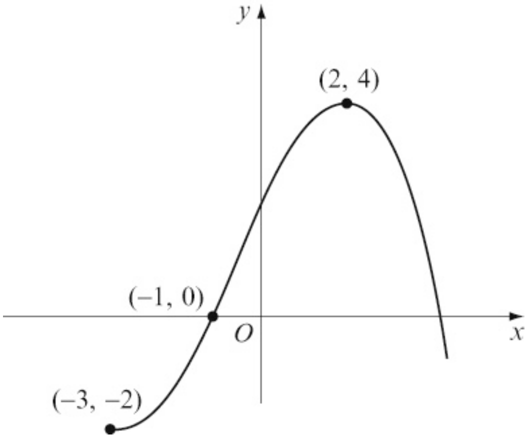
$y = \text{f}(x-2) - 5$.

Vertical translation of -5.



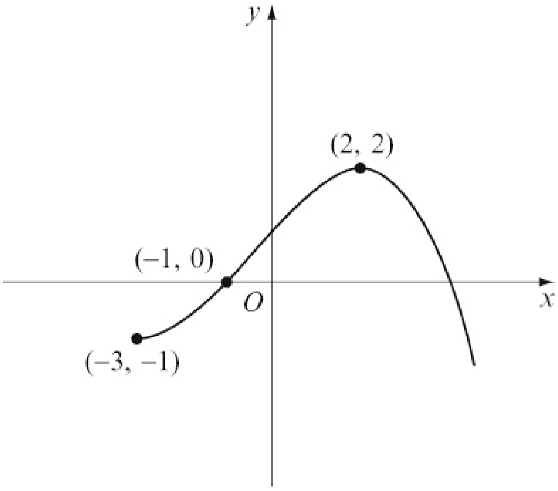
1 **c** $y = \mathrm{f}(x+1)$

Horizontal translation of -1.



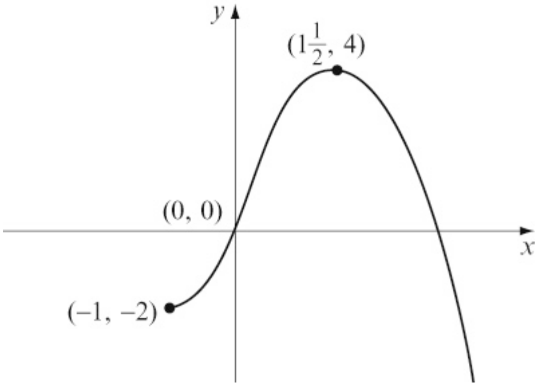
$y = \frac{1}{2}\mathrm{f}(x+1)$

Vertical stretch, scale factor $\frac{1}{2}$



d $y = \mathrm{f}(2x)$

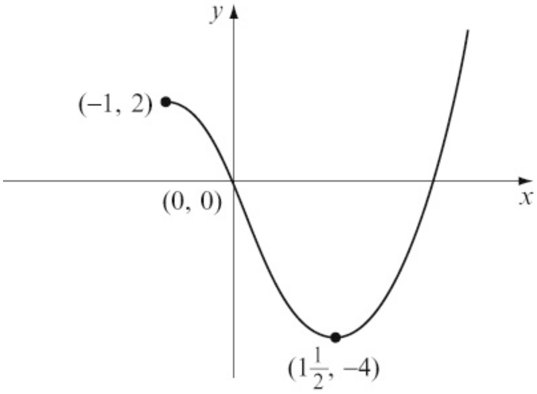
Horizontal stretch, scale factor $\frac{1}{2}$



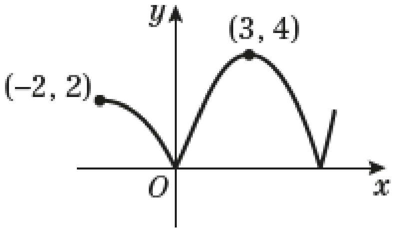
$y = -\mathrm{f}(2x)$

Reflection in the x-axis.

(or Vertical stretch, scale factor -1).



e $y = |\mathrm{f}(x)|$. Reflect, in the x-axis, the parts of the graph that lie below the x-axis.

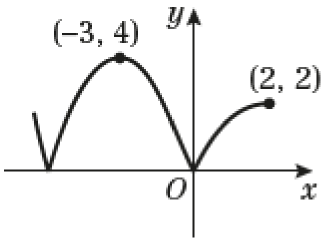


1 f $y = f(-x)$.

Reflection in the y-axis.

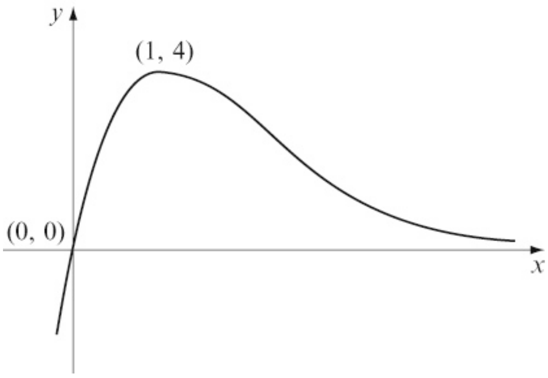
$y = |f(-x)|$.

Reflect, in the x-axis, the parts of the graph that lie below the x-axis.



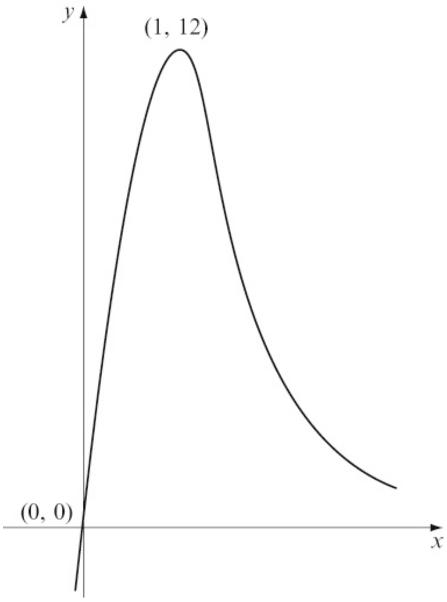
2 a $y = f(x-2)$

Horizontal translation of $+2$



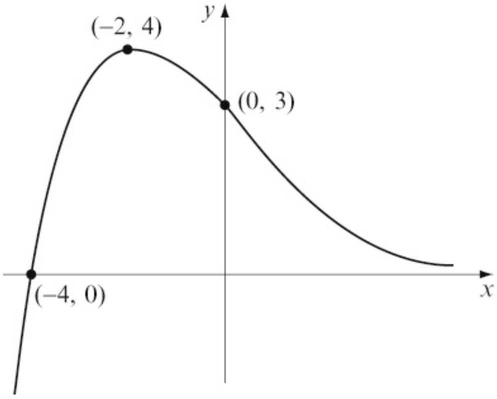
$y = 3f(x-2)$

Vertical stretch, scale factor 3.



b $y = f\left(\frac{1}{2}x\right)$

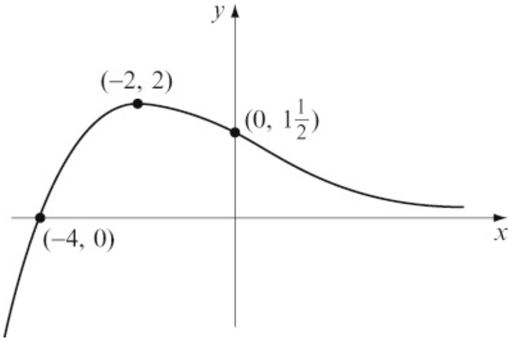
Horizontal stretch, scale factor 2.



2 b (continued)

$y = \frac{1}{2}\mathrm{f}\left(\frac{1}{2}x\right)$

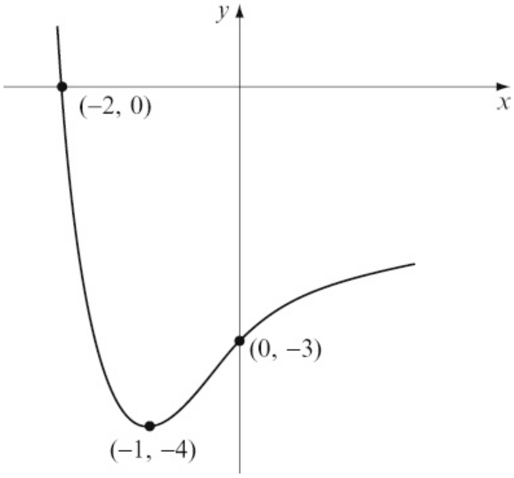
Vertical stretch, scale factor $\frac{1}{2}$



c $y = -\mathrm{f}(x)$

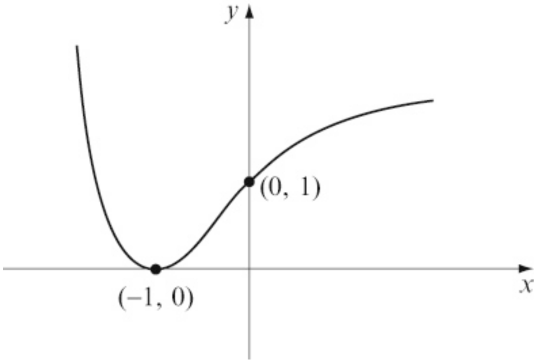
Reflection in the x-axis.

(Or vertical stretch, scale factor -1).



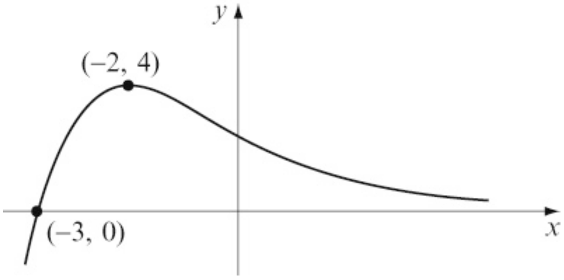
$y = -\mathrm{f}(x) + 4$

Vertical translation of $+4$.



d $y = \mathrm{f}(x+1)$

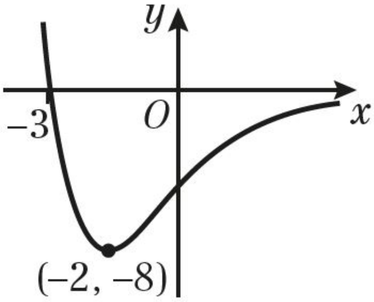
Horizontal translation of -1.



$y = -2\mathrm{f}(x+1)$

Reflection in the x-axis,

and vertical stretch, scale factor 2.



2 e $y = \mathrm{f}(|x|)$ can be written

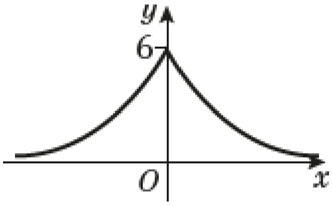
$$y = \begin{cases} \mathrm{f}(x), & x \geq 0 \\ \mathrm{f}(-x), & x < 0 \end{cases}$$

$y = \mathrm{f}(-x)$ is a reflection of $y = \mathrm{f}(x)$ in the y-axis.

Hence, $y = \mathrm{f}(|x|)$ is the following:

$y = 2\mathrm{f}(|x|)$

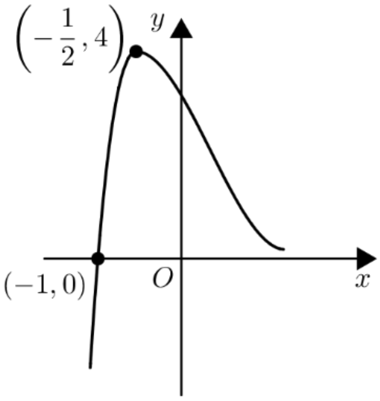
Vertical stretch, scale factor 2.



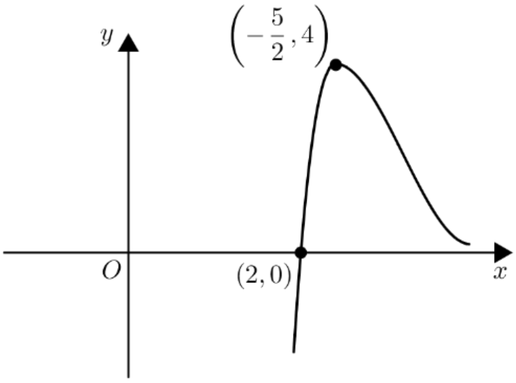
2 f $y = \mathrm{f}(2x-6)$ can be written as

$y = \mathrm{f}(2(x-3))$

$y = \mathrm{f}(2x)$: Horizontal stretch, scale factor $\frac{1}{2}$

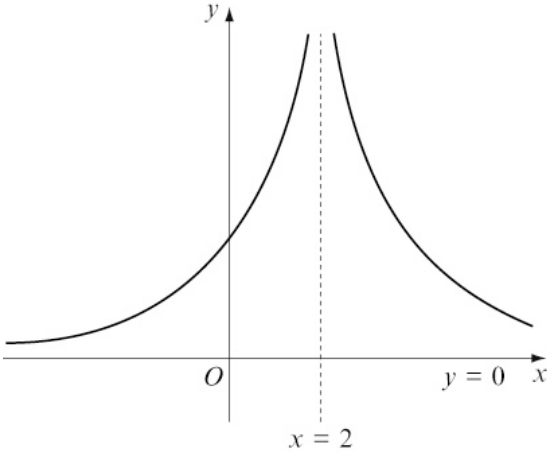


$y = \mathrm{f}(2(x-3))$: Horizontal translation of $+3$



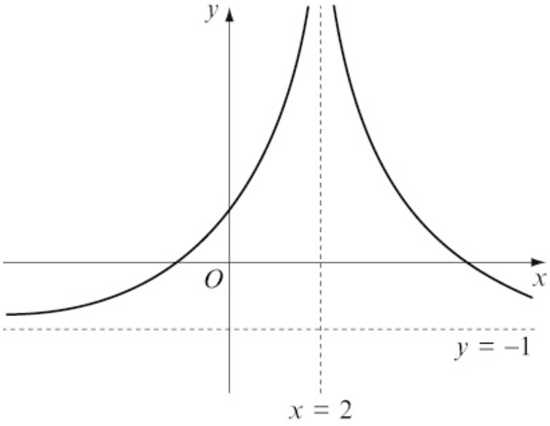
3 a $y = 3\mathrm{f}(x)$

Vertical stretch, scale factor 3.



$y = 3\mathrm{f}(x) - 1$

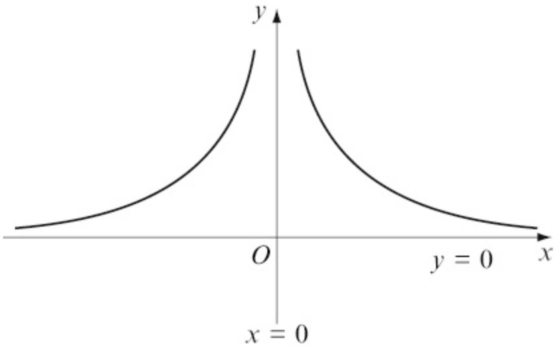
Vertical translation of -1.



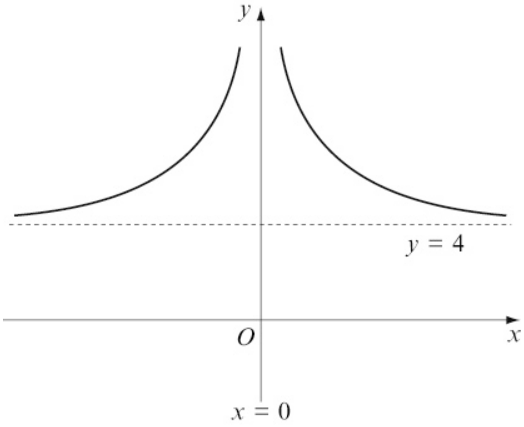
Asymptotes: $x = 2$, $y = -1$

A: (0, 2)

3 **b** $y = \mathrm{f}(x+2)$
Horizontal translation of -2.



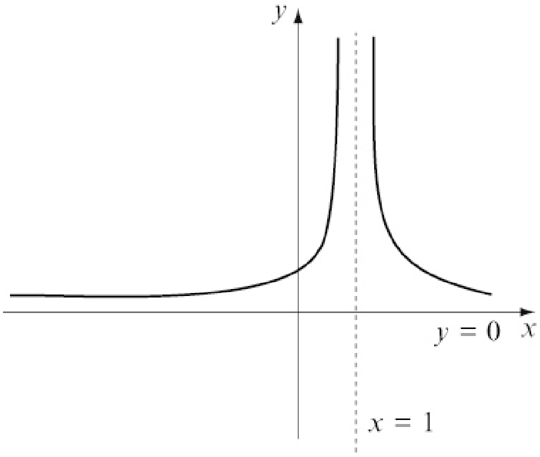
$y = \mathrm{f}(x+2)+4$
Vertical translation of $+4$.



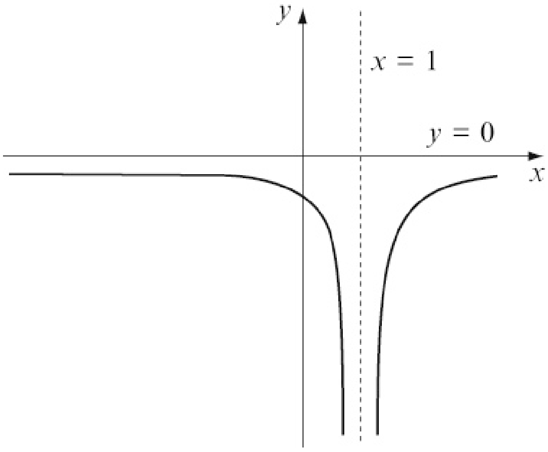
Asymptotes: $x = 0$, $y = 4$
A: (–2, 5)

c $y = \mathrm{f}(2x)$

Horizontal stretch, scale factor $\frac{1}{2}$



$y = -\mathrm{f}(2x)$. Reflection in the x-axis.



Asymptotes: $x = 1$, $y = 0$
A: (0, –1)

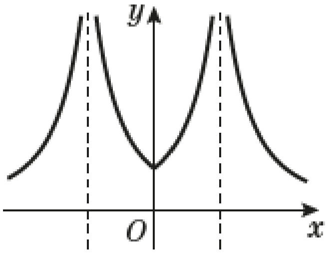
3 **d** $y = \mathrm{f}(|x|)$ can be written

$$y = \begin{cases} \mathrm{f}(x), & x \geq 0 \\ \mathrm{f}(-x), & x < 0 \end{cases}$$

$y = \mathrm{f}(-x)$ is a reflection of

$y = \mathrm{f}(x)$ in the y-axis.

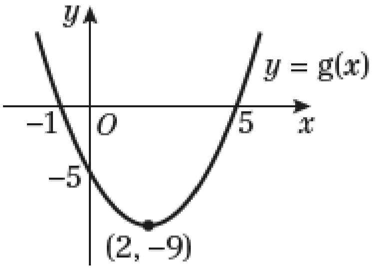
Hence, $y = \mathrm{f}(|x|)$ is the following:



Asymptotes are $x = -2$, $x = 2$ and $y = 0$.

A: (0, 1)

4 **a**



b **i** $(2 + 4, -9 \times 2) = (6, -18)$

ii $(2 \times \frac{1}{2}, -9) = (1, -9)$

iii $(2, -9 \times -1) = (2, 9)$

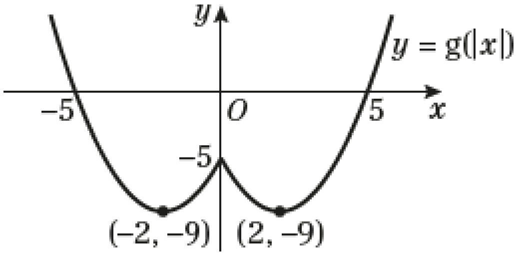
c $y = \mathrm{g}(|x|)$ can be written

$$y = \begin{cases} \mathrm{g}(x) = (x-2)^2 - 9, & x \geq 0 \\ \mathrm{g}(-x) = (x+2)^2 - 9, & x < 0 \end{cases}$$

$y = \mathrm{g}(-x)$ is a reflection of

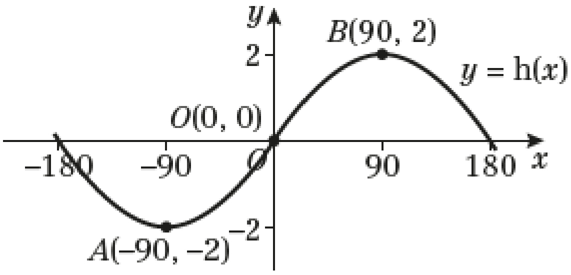
$y = \mathrm{g}(x)$ in the y-axis.

Hence, $y = \mathrm{g}(|x|)$ is the following:



5 **a** $y = 2\sin x$ is a vertical stretch of

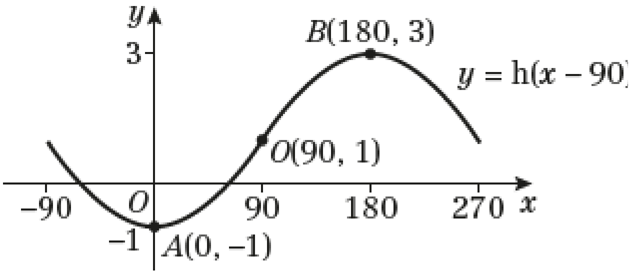
$y = \sin x$ by a scale factor 2.



b minimum $A(-90°, -2)$ and maximum $B(90°, 2)$

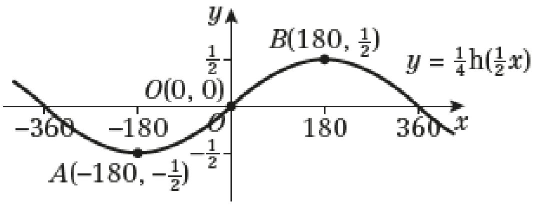
5 **c** **i** $\mathrm{h}(x-90)$ is a horizontal translation of +90°

$\mathrm{h}(x-90)+1$ is a vertical translation of $+1$.



ii $\mathrm{h}\left(\frac{1}{2}x\right)$ is a horizontal stretch scale factor 2

$\frac{1}{4}\mathrm{h}\left(\frac{1}{2}x\right)$ is a vertical stretch scale factor $\frac{1}{4}$

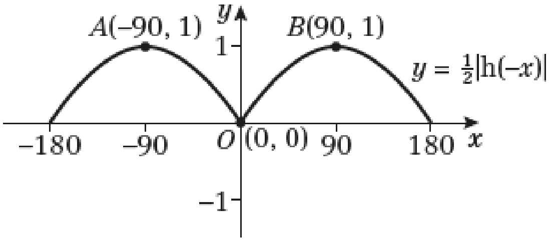


iii $\mathrm{h}(-x)$ is a reflection in the y-axis

$|\mathrm{h}(-x)|$ causes the part of the graph below the x-axis to be reflected in the x-axis.

$\frac{1}{2}|\mathrm{h}(-x)|$ is a vertical stretch scale factor

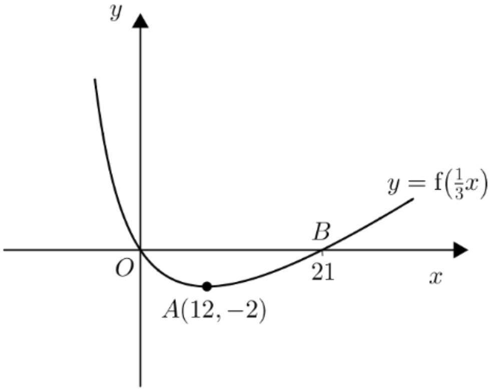
$\frac{1}{2}$



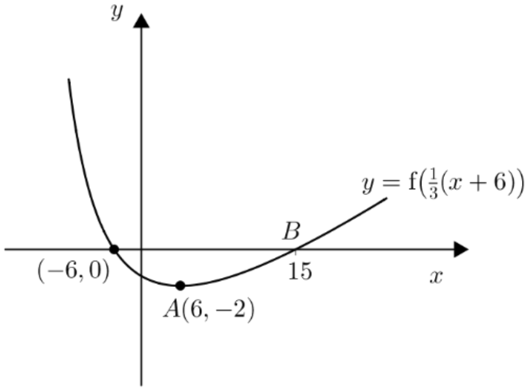
6 $y=\mathrm{f}\left(\frac{1}{3}x+2\right)$ can be written as

$y=\mathrm{f}\left(\frac{1}{3}(x+6)\right)$

$y=\mathrm{f}\left(\frac{1}{3}x\right)$: Horizontal stretch, scale factor 3



$y=\mathrm{f}\left(\frac{1}{3}(x+6)\right)$**:** Horizontal translation of -6



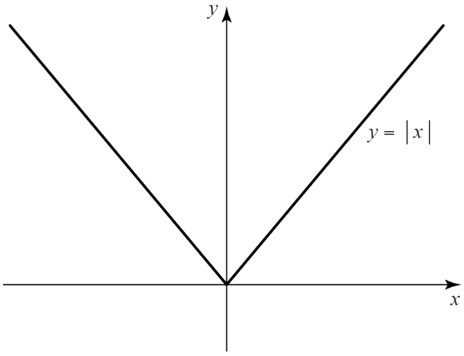
So O is transformed to $(-6,0)$

A is transformed to $(6,-2)$

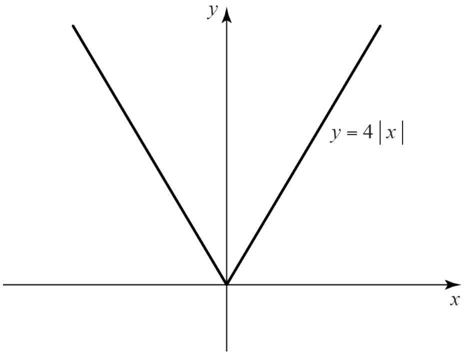
B is transformed to $(15,0)$

Functions and graphs 2G

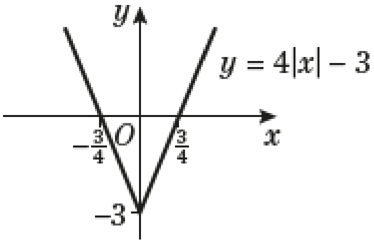
1 a Start with $y=|x|$



$y=4|x|$ is a vertical stretch by scale factor 4

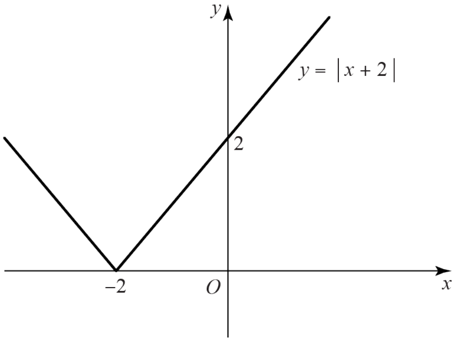


$y=4|x|-3$ is a horizontal translation by –3



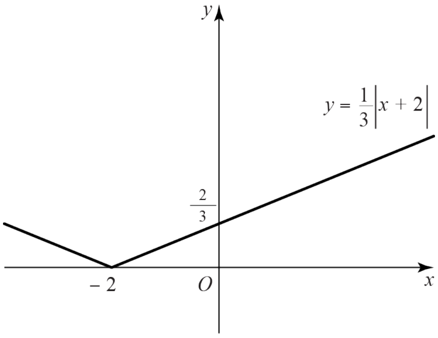
The range is $\mathrm{f}(x)\geq -3$

b Start with $y=|x|$

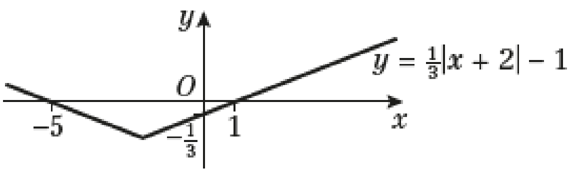


$y=|x+2|$ is a horizontal translation by –2

$y=\frac{1}{3}|x+2|$ is a vertical stretch by scale factor $\frac{1}{3}$

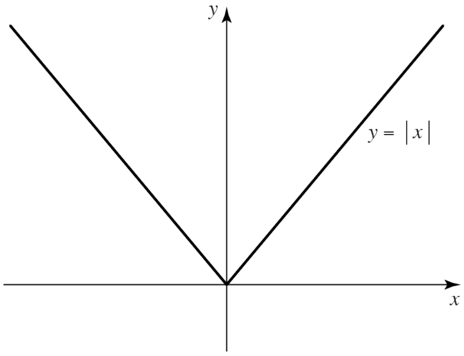


$y=\frac{1}{3}|x+2|-1$ is a vertical translation by –1

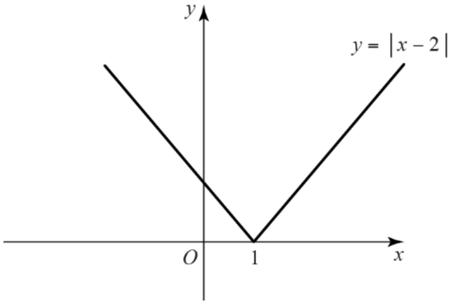


The range is $\mathrm{f}(x)\geq -1$

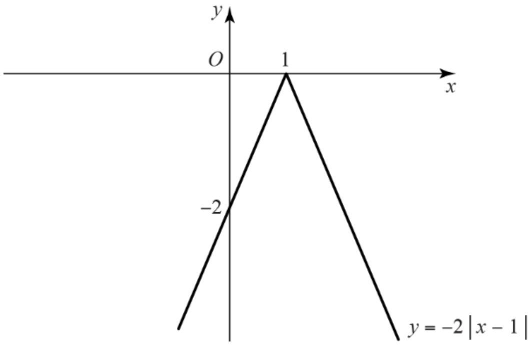
1 **c** Start with $y = |x|$



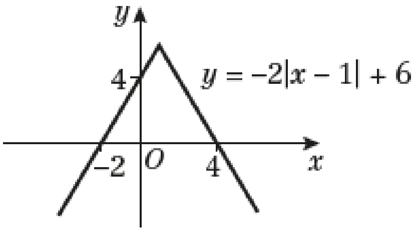
$y = |x-1|$ is a horizontal translation by +1



$y = -2|x-1|$ is a vertical stretch by scale factor –2

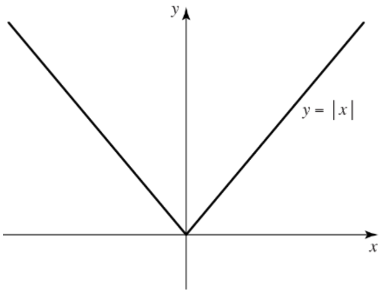


$y = -2|x-1| + 6$ is a vertical translation by +6

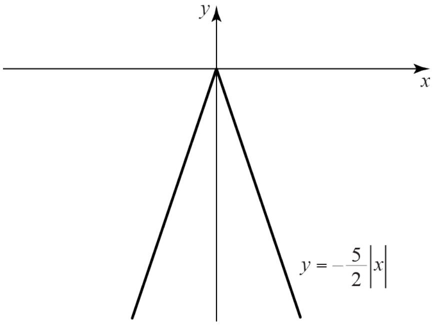


The range is $\mathrm{f}(x) \leq 6$

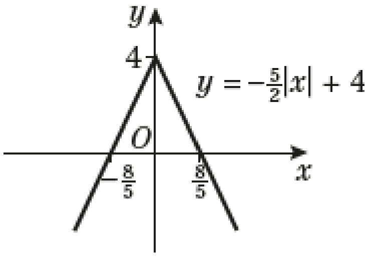
d Start with $y = |x|$



$y = -\frac{5}{2}|x|$ is a vertical stretch by scale factor $-\frac{5}{2}$



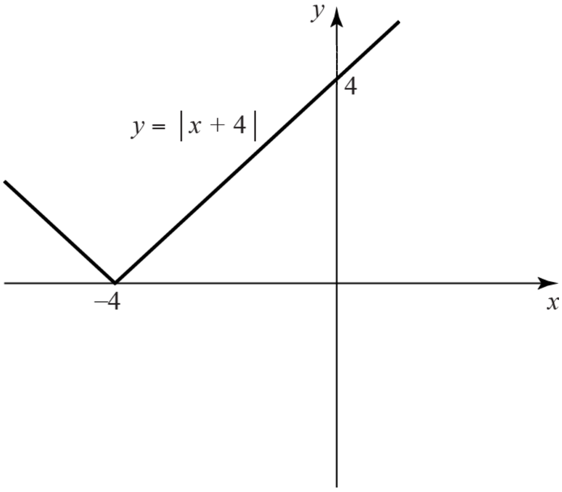
$y = -\frac{5}{2}|x| + 4$ is a horizontal translation by –3



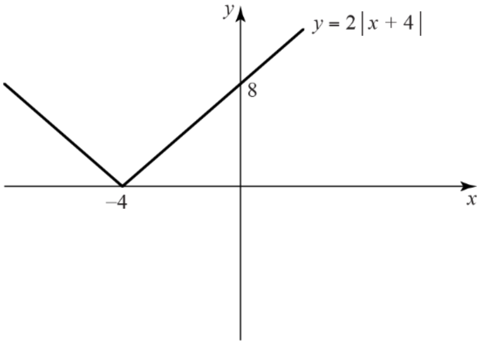
The range is $\mathrm{f}(x) \leq 4$

2 **a** Start with $y=|x|$

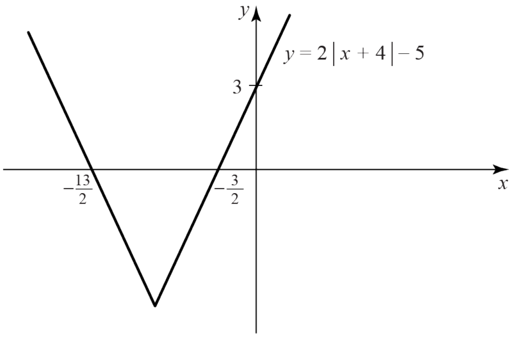
$y=|x+4|$ is a horizontal translation of –4



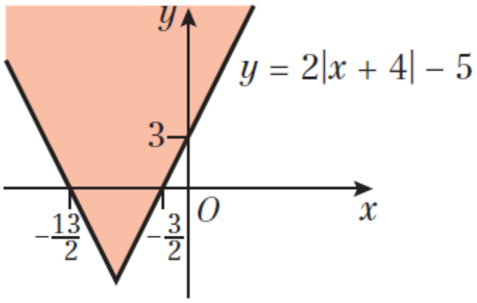
$y=2|x+4|$ is a vertical stretch scale factor 2



$y=2|x+4|-5$ is a vertical translation of –5



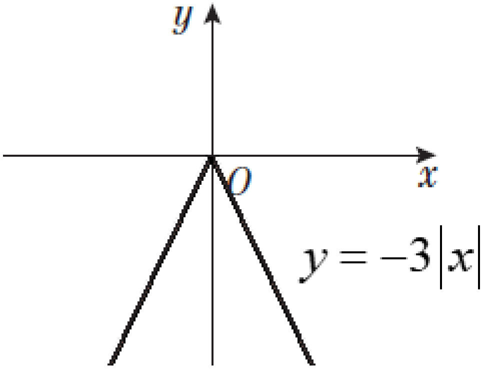
b The region where $y \geq \mathrm{p}(x)$ is the region which lies on and above the line $y=2|x+4|-5$



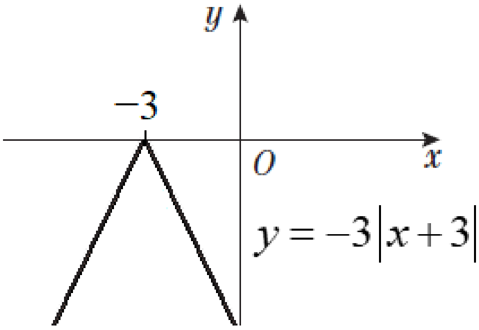
3 **a** $\mathrm{q}(x)=6-|3x+9|=-3|x+3|+6$

Start with $y=|x|$

$y=-3|x|$ is a vertical stretch scale factor –3

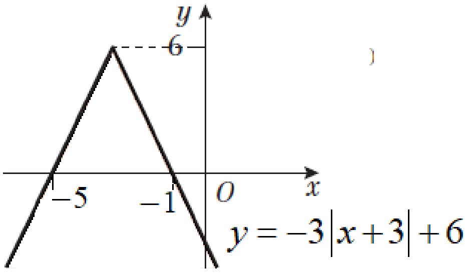


$y=-3|x+3|$ is a horizontal translation of –3

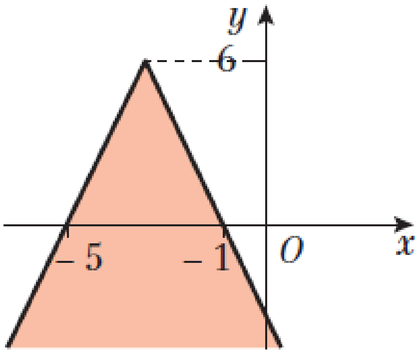


3 a (continued)

$y = -3|x+3|+6$ is a vertical translation of +6

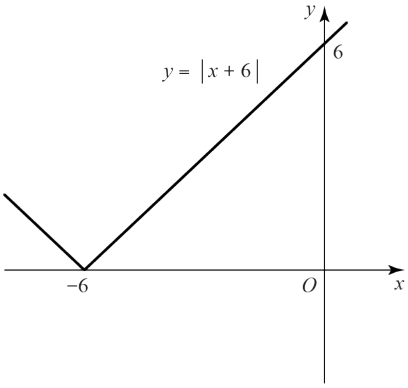


3 b The region where $y < \mathrm{q}(x)$ is the region which lies below the line $y = -3|x+3|+6$

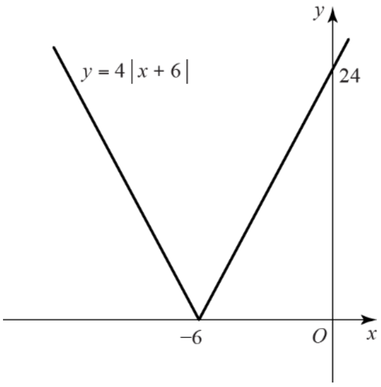


4 a Start with $y = |x|$

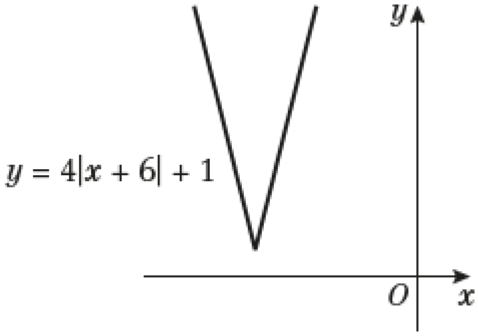
$y = |x+6|$ is a horizontal translation of –6



$y = 4|x+6|$ is a vertical stretch scale factor 4



$y = 4|x+6|+1$ is a vertical translation of +1



4 b The range is $f(x) \geq 1$

c At one point of intersection:

$$-4(x+6)+1=-\frac{1}{2}x+1$$
$$-4x-23=-\frac{1}{2}x+1$$
$$-8x-46=-x+2$$
$$-48=7x$$
$$x=-\frac{48}{7}$$

At other point of intersection:

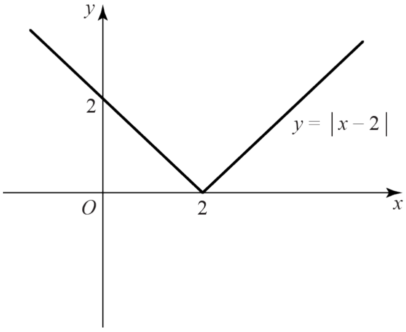
$$4(x+6)+1=-\frac{1}{2}x+1$$
$$4x+25=-\frac{1}{2}x+1$$
$$8x+50=-x+2$$
$$9x=-48$$
$$x=-\frac{16}{3}$$

So the solutions are

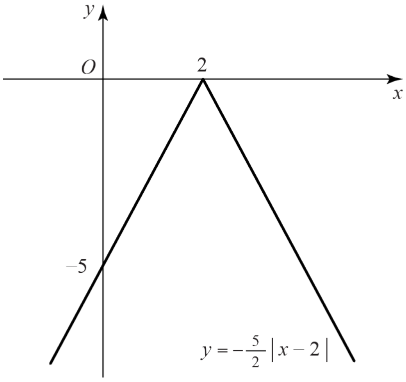
$x=-\frac{48}{7}$ and $x=-\frac{16}{3}$

5 a Start with $y=|x|$

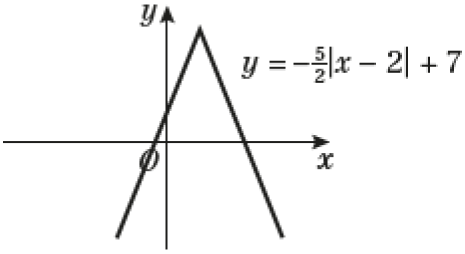
$y=|x-2|$ is a horizontal translation of +2



$y=-\frac{5}{2}|x-2|$ is a vertical stretch scale factor $-\frac{5}{2}$



$y=-\frac{5}{2}|x-2|+7$ is a vertical translation of +7



b The range is $g(x) \leq 7$

5 **c** At one point of intersection:

$$-\frac{5}{2}(x-2)+7=x+1$$
$$-\frac{5}{2}x+12=x+1$$
$$-5x+24=2x+2$$
$$22=7x$$
$$x=\frac{22}{7}$$

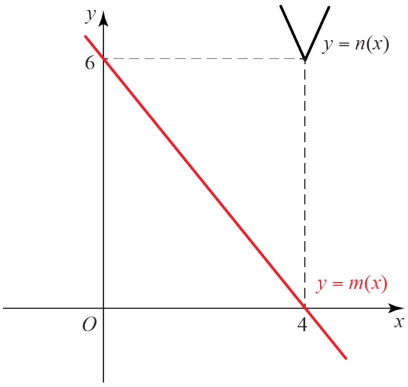
At other point of intersection:

$$\frac{5}{2}(x-2)+7=x+1$$
$$\frac{5}{2}x+2=x+1$$
$$5x+4=2x+2$$
$$3x=-2$$
$$x=-\frac{2}{3}$$

So the solutions are

$x=-\frac{2}{3}$ and $x=\frac{22}{7}$

6 For the equation $\mathrm{m}(x) = \mathrm{n}(x)$ to have no real roots, it must be the case that $y = \mathrm{m}(x)$ and $y = \mathrm{n}(x)$ do not intersect.



The least value of

$y=\mathrm{n}(x)=3|x-4|+6$ is

$y = 6$ when $x = 4$

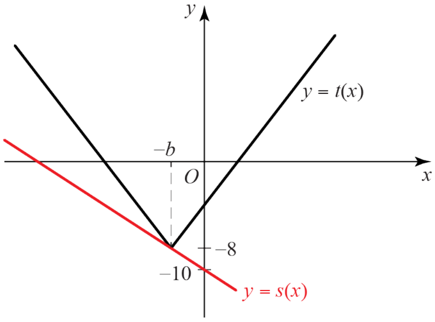
Hence, we need $\mathrm{m}(4) < 6$ to avoid intersection

So $-2(4) + k < 6$

$-8 + k < 6$

$k < 14$

7 For the equation s(x) = t(x) to have exactly one real root, it must be the case that y = s(x) and y = t(x) intersect at the minimum point of t(x).



The least value of

$$y = \mathrm{t}(x) = 2|x+b| - 8 \text{ is}$$

$$y = -8 \text{ when } x = -b$$

Hence, we need s(–b) = –8 to ensure one intersection

$$\Rightarrow -8 = -10 - (-b)$$

$$b = 2$$

8 a The range is h(x) ≥ -7

b h(x) is many-to-one, therefore $\mathrm{h}^{-1}(x)$ would be one-to-many, and so would not be a function.

c At one point of intersection:

$$-\frac{2}{3}(x-1) - 7 = -6$$

$$2x - 2 + 21 = 18$$

$$2x = -1$$

$$x = -\frac{1}{2}$$

At other point of intersection:

$$\frac{2}{3}(x-1) - 7 = -6$$

$$2x - 2 - 21 = -18$$

$$2x = 5$$

$$x = \frac{5}{2}$$

So the solutions are

$$x = -\frac{1}{2} \text{ and } x = \frac{5}{2}$$

h(x) < –6 between the two points of intersection, so the solution to the inequality h(x) < −6 is

$$-\frac{1}{2} < x < \frac{5}{2}$$

d Since $\mathrm{h}(x) \geq -7$ and $\mathrm{h}(1) = -7$,

then for the equation $\mathrm{h}(x) = \frac{2}{3}x + k$

to have no solutions, we require

$$\frac{2}{3}(1) + k < -7$$

$$\Rightarrow k < -\frac{23}{3}$$

9 a We can write h as

$$\mathrm{h}(x)=\begin{cases}a+2(x+3), & x\le -3\\ a-2(x+3), & x\ge -3\end{cases}$$

The line which has gradient –2 and passes through (0, 4) is $y=-2x+4$

So, for $x\ge -3$

$$-2(x+3)+a=-2x+4$$
$$-2x-6+a=-2x+4$$
$$a=10$$

b At P, h(x) = 10 (from part **a**)

So $10 = 10 - 2(x + 3)$

$$-2x-6=0$$
$$x=-3$$

At Q, h(x) = 0

So $0 = 10 - 2(x + 3)$

$$4-2x=0$$
$$x=2$$

$P(-3, 10)$ and $Q(2, 0)$

c $\mathrm{h}(x) = \frac{1}{3}x+6$

At one point of intersection:

$$10-2(x+3)=\frac{1}{3}x+6$$
$$4-2x=\frac{1}{3}x+6$$
$$12-6x=x+18$$
$$7x=-6$$
$$x=-\frac{6}{7}$$

At other point of intersection:

$$10+2(x+3)=\frac{1}{3}x+6$$
$$16+2x=\frac{1}{3}x+6$$
$$48+6x=x+18$$
$$5x=-30$$
$$x=-6$$

So the solutions are

$x=-6$ and $x=-\frac{6}{7}$

10 a The range of m(x) is $\mathrm{m}(x)\le 7$

b $\mathrm{m}(x) = \frac{3}{5}x+2$

At one point of intersection:

$$-4(x+3)+7=\frac{3}{5}x+2$$
$$-4x-5=\frac{3}{5}x+2$$
$$-20x-25=3x+10$$
$$-23x=35$$
$$x=-\frac{35}{23}$$

At other point of intersection:

$$4(x+3)+7=\frac{3}{5}x+2$$
$$4x+19=\frac{3}{5}x+2$$
$$20x+95=3x+10$$
$$17x=-85$$
$$x=-5$$

So the solutions are $x=-5$ and $x=-\frac{35}{23}$

c For two distinct roots, there are two points of intersection, so m(x) < 7. Therefore, $k < 7$.

Challenge

1 a At A:

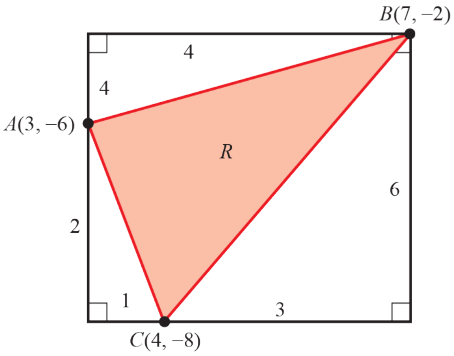
$$-2(x-4)-8=x-9$$
$$-2x=x-9$$
$$-3x=-9$$
$$x=3$$
$$y=3-9=-6$$

At B:

$$2(x-4)-8=x-9$$
$$2x-16=x-9$$
$$x=7$$
$$y=7-9=-2$$

$A(3, -6)$ and $B(7, -2)$

b Taking the shaded triangle R and enclosing it in a rectangle looks like:



$$R=(4\times6)-\left(\frac{1}{2}\times4\times4\right)-\left(\frac{1}{2}\times6\times3\right)-\left(\frac{1}{2}\times2\times1\right)$$
$$R=24-8-9-1$$
$$R=6 \text{ units}^2$$

2 At the first point of intersection:

$$x-3+10=-2(x-3)+2$$
$$x+7=-2x+8$$
$$3x=1$$
$$x=\frac{1}{3}$$

At the other point of intersection:

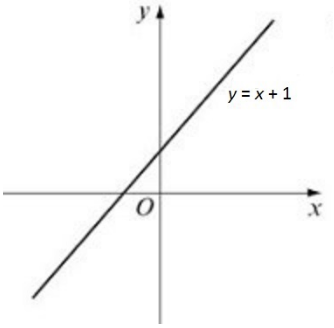
$$-(x-3)+10=2(x-3)+2$$
$$-x+13=2x-4$$
$$-3x=-17$$
$$x=\frac{17}{3}$$

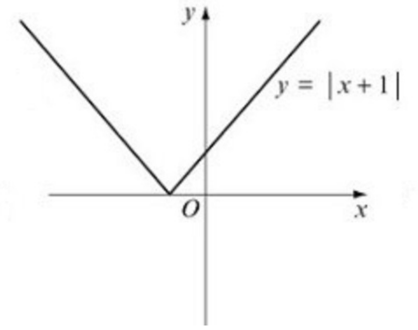
Maximum point of $f(x)$ is
$f(x)=10$ when $x=3$, so at $(3, 10)$
Minimum point of $g(x)$ is
$g(x)=2$ when $x=3$, so at $(3, 2)$

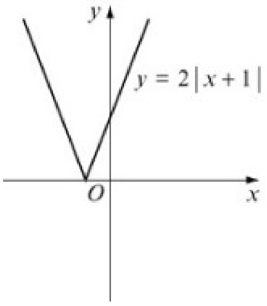
$$\text{Area of a kite} = \frac{1}{2}\times \text{width} \times \text{height}$$
$$=\frac{1}{2}\times\left(\frac{17}{3}-\frac{1}{3}\right)\times(10-2)$$
$$=\frac{1}{2}\times\frac{16}{3}\times 8$$
$$=\frac{64}{3} \text{ units}^2$$

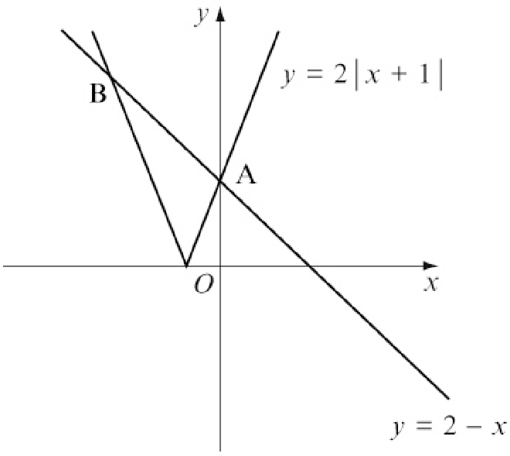
Functions and graphs Mixed exercise 2

1 a









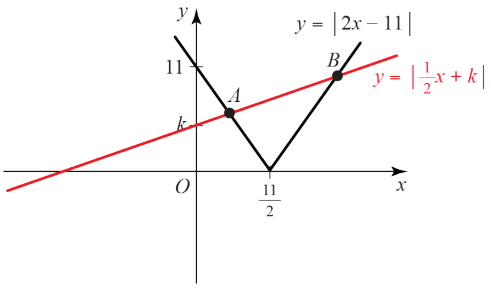
b Intersection point A:

$$2(x+1)=2-x$$
$$2x+2=2-x$$
$$3x=0$$
$$x=0$$

Intersection point B is on the reflected part of the modulus graph.

$$-2(x+1)=2-x$$
$$-2x-2=2-x$$
$$-x=4$$
$$x=-4$$

2



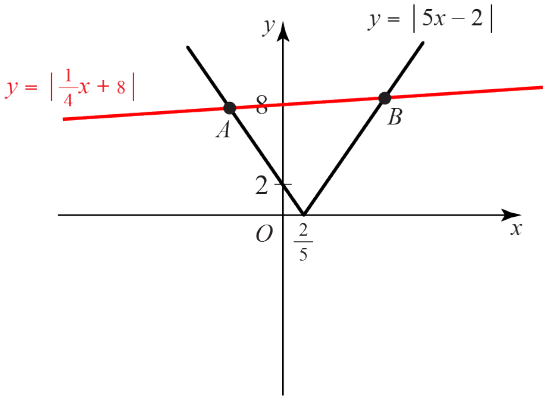
Minimum value of $y=|2x-11|$ is

$$y=0 \text{ at } x=\frac{11}{2}$$

For two distinct solutions to $|2x-11|=\frac{1}{2}x-k$, we must have

$$\frac{1}{2}x-k>0 \text{ at } x=\frac{11}{2}$$
$$\frac{1}{2}\times\frac{11}{2}+k>0$$
$$k>-\frac{11}{4}$$

3



At A:

$$-(5x-2) = -\frac{1}{4}x + 8$$
$$-20x + 8 = -x + 32$$
$$-19x = 24$$
$$x = -\frac{24}{19}$$

At B:

$$5x - 2 = -\frac{1}{4}x + 8$$
$$20x - 8 = -x + 32$$
$$21x = 40$$
$$x = \frac{40}{21}$$

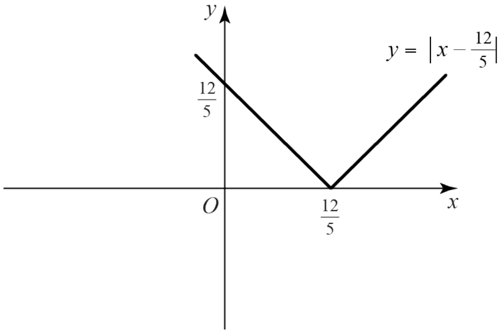
So the solution are

$x = -\frac{24}{19}$ and $x = \frac{40}{21}$

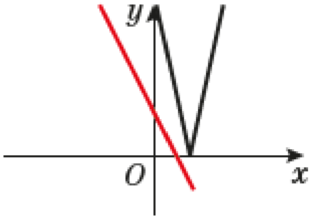
4 a $y = |12 - 5x| = 5\left|-\left(x - \frac{12}{5}\right)\right|$

Start with $y = |x|$

$y = \left|x - \frac{12}{5}\right|$ is a horizontal translation of $+\frac{12}{5}$

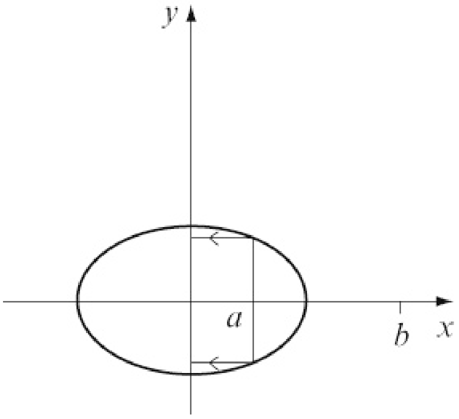


$y = 5\left|x - \frac{12}{5}\right|$ is a vertical stretch, scale factor 5



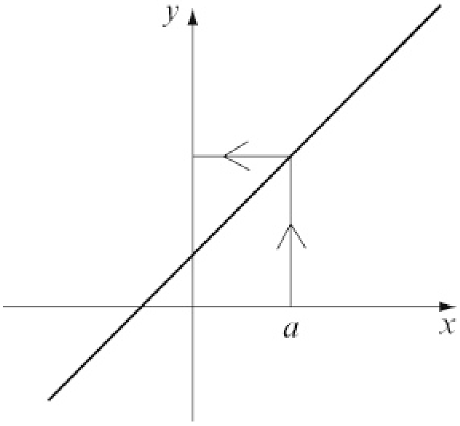
b The graphs do not intersect, so there are no solutions.

5 a i One-to-many.
ii Not a function.

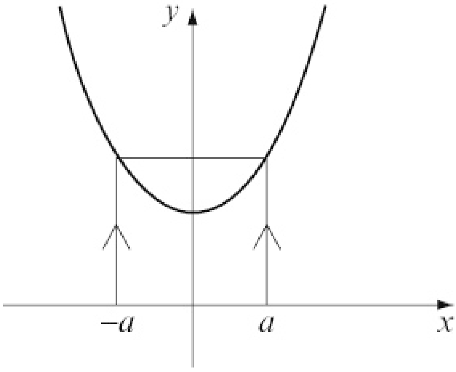


x value a gets mapped to two values of y.
x value b gets mapped to no values.

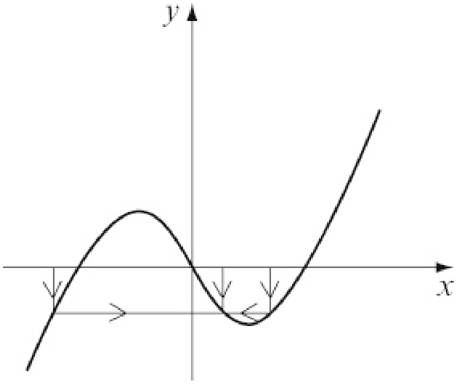
b i One-to-one.
ii Is a function.



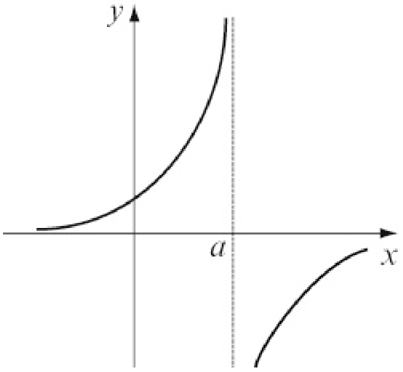
c i Many-to-one.
ii Is a function.



d i Many-to-one.
ii Is a function.

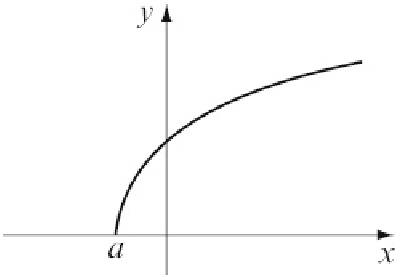


5 e i One-to-one.
ii Not a function.



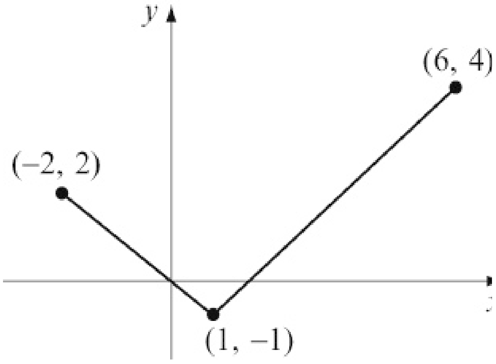
x value a doesn't get mapped to any value of y. It could be redefined as a function if the domain is said to exclude point a.

f i One-to-one.
ii Not a function for this domain.



x values less than a don't get mapped anywhere. Again, we could define the domain to be $x \leqslant a$ and then it would be a function.

6 a



For $x \leqslant 1$, $f(x) = -x$

This is a straight line of gradient -1.

At point $x = 1$, its y-coordinate is -1.

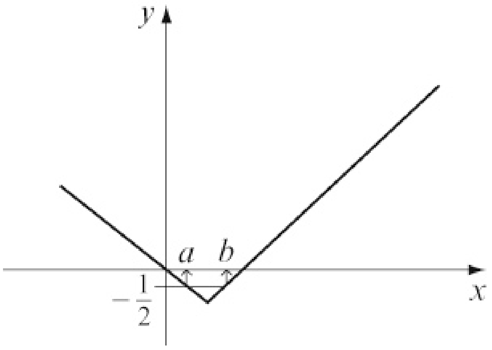
For $x > 1$, $f(x) = x - 2$

This is a straight line of gradient $+1$.

At point $x = 1$, its y-coordinate is also -1.

Hence, the graph is said to be continuous.

b There are two values x in the range $-2 \leqslant x \leqslant 6$ for which $f(x) = -\frac{1}{2}$



Point a is where

$$-x = -\frac{1}{2} \Rightarrow x = \frac{1}{2}$$

Point b is where

$$x - 2 = -\frac{1}{2} \Rightarrow x = 1\frac{1}{2}$$

Hence, the values of x for which $f(x) = -\frac{1}{2}$ are $x = \frac{1}{2}$ and $x = 1\frac{1}{2}$

7 a $pq(x) = p(2x + 1)$
$= (2x + 1)^2 + 3(2x + 1) - 4$
$= 4x^2 + 4x + 1 + 6x + 3 - 4$
$= 4x^2 + 10x,\ x \in \mathbb{R}$

b $qq(x) = q(2x + 1)$
$= 2(2x + 1) + 1$
$= 4x + 3$

$pq(x) = qq(x)$ gives
$4x^2 + 10x = 4x + 3$
$4x^2 + 6x - 3 = 0$

Using the formula:

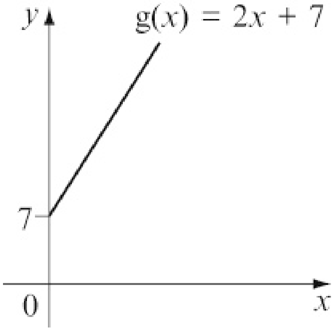
$$x = \frac{-6 \pm \sqrt{6^2 - 4 \times 4 \times (-3)}}{2 \times 4}$$

$$x = \frac{-6 \pm \sqrt{84}}{8}$$

$$x = \frac{-6 \pm 2\sqrt{21}}{8}$$

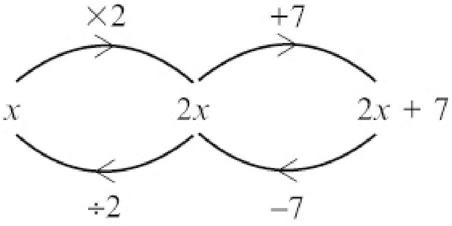
$$x = \frac{-3 \pm \sqrt{21}}{4}$$

8 a $y = 2x + 7$ is a straight line with gradient 2 and y-intercept 7



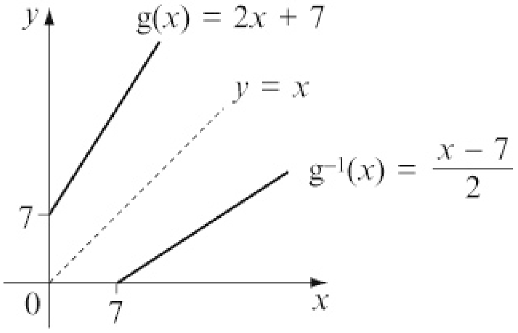
For $x \geqslant 0$, the range is $g(x) \geqslant 7$

b The range is $g^{-1}(x) \geqslant 0$.
To find the equation of the inverse function, you can use a flow chart.



$g^{-1}(x)=\dfrac{x-7}{2}$ and has domain $x \geqslant 7$

8 c



$g^{-1}(x)$ is the reflection of $g(x)$ in the line $y = x$.

9 a To find $f^{-1}(x)$, you can change the subject of the formula.

$$\text{Let } y=\frac{2x+3}{x-1}$$
$$y(x-1)=2x+3$$
$$yx-y=2x+3$$
$$yx-2x=y+3$$
$$x(y-2)=y+3$$
$$x=\frac{y+3}{y-2}$$

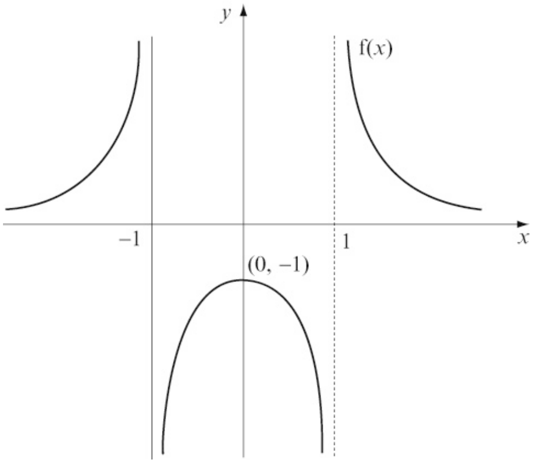
Therefore $f^{-1}(x)=\dfrac{x+3}{x-2}, x\in\mathbb{R}, x>2$

b Domain $f(x)=$ Range $f^{-1}(x)$

$\therefore$ Range $f^{-1}(x)=\{y\in\mathbb{R}, y>1\}$

10 a $$f(x)=\frac{x}{x^2-1}-\frac{1}{x+1}$$
$$=\frac{x}{(x+1)(x-1)}-\frac{1}{(x+1)}$$
$$=\frac{x}{(x+1)(x-1)}-\frac{x-1}{(x+1)(x-1)}$$
$$=\frac{x-(x-1)}{(x+1)(x-1)}$$
$$=\frac{1}{(x+1)(x-1)}$$

b Consider the graph of $y=\dfrac{1}{(x-1)(x+1)}$ for $x\in\mathbb{R}$:



For $x>1$, $f(x)>0$

c $$gf(x)=g\left(\frac{1}{(x-1)(x+1)}\right)$$
$$=\frac{2}{\left(\frac{1}{(x-1)(x+1)}\right)}$$
$$=2\times\frac{(x-1)(x+1)}{1}$$
$$=2(x-1)(x+1)$$

$$gf(x)=70\Rightarrow 2(x-1)(x+1)=70$$
$$(x-1)(x+1)=35$$
$$x^2-1=35$$
$$x^2=36$$
$$x=\pm6$$

11 a $\mathrm{f}(7) = 4(7-2)$

$= 4\times 5$

$= 20$

$\mathrm{g}(3) = 3^3+1$

$= 27+1$

$= 28$

$\mathrm{h}(-2) = 3^{-2}$

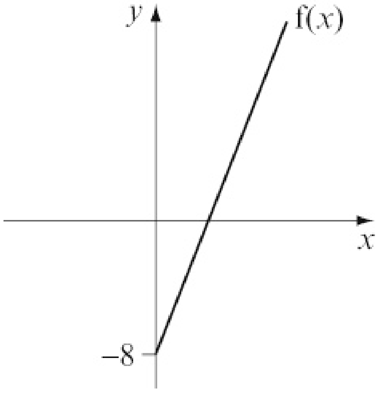
$= \dfrac{1}{3^2}$

$= \dfrac{1}{9}$

b $\mathrm{f}(x) = 4(x-2) = 4x-8$

This is a straight line with gradient 4 and intercept −8.

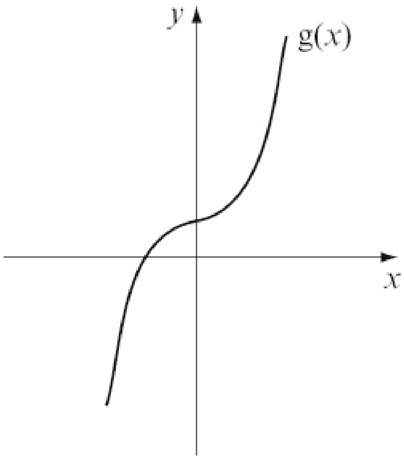
The domain tells us that $x \geqslant 0$, so the graph of $y = \mathrm{f}(x)$ is:



The range of $\mathrm{f}(x)$ is

$\mathrm{f}(x) \in \mathbb{R}, \mathrm{f}(x) \geqslant -8$

$g(x) = x^3+1$



The range of $\mathrm{g}(x)$ is $\mathrm{g}(x) \in \mathbb{R}$

c Let $y = x^3+1$

(change the subject of the formula)

$y-1 = x^3$

$\sqrt[3]{y-1} = x$

Hence $g^{-1}(x) = \sqrt[3]{x-1}\ \{x \in \mathbb{R}\}$

d $\mathrm{fg}(x) = \mathrm{f}(x^3+1)$

$= 4(x^3+1-2)$

$= 4(x^3-1), x \in \mathbb{R}, x \geq -1$

e First find $\mathrm{gh}(x)$:

$\mathrm{gh}(x) = \mathrm{g}(3^x)$

$= (3^x)^3+1$

$= 3^{3x}+1$

$\mathrm{gh}(a) = 244$

$3^{3a}+1 = 244$

$3^{3a} = 243$

$3^{3a} = 3^5$

$3a = 5$

$a = \dfrac{5}{3}$

f First find $\mathrm{f}^{-1}(x)$

Let $y = 4(x-2)$

(changing the subject of the formula)

$\dfrac{y}{4} = x-2$

$\dfrac{y}{4}+2 = x$

Hence $\mathrm{f}^{-1}(x) = \dfrac{x}{4}+2$

$\mathrm{f}^{-1}(x) = -\dfrac{1}{2}$

$\dfrac{x}{4}+2 = -\dfrac{1}{2}$

$x = 4\left(-\dfrac{1}{2}-2\right) = -10$

12 a f^{-1} exists when f is one-to-one.

Now $f(x) = x^2 + 6x - 4$
Completing the square:
$f(x) = (x + 3)^2 - 13$
The minimum value is
$f(x) = -13$ when $x + 3 = 0$

$\Rightarrow x = -3$

Hence, f is one-to-one when $x > -3$
So least value of a is $a = -3$

b Let $y = f(x)$

$$y = x^2 + 6x - 4$$
$$y = (x + 3)^2 - 13$$
$$y + 13 = (x + 3)^2$$
$$x + 3 = \sqrt{y + 13}$$
$$x = \sqrt{y + 13} - 3$$

So $f^{-1} : x \mapsto \sqrt{x + 13} - 3$

For $a = 0$, Range $f(x)$ is $y > -4$
So Domain $f^{-1}(x)$ is $x > -4$

13 a $f : x \mapsto 4x - 1$
Let $y = 4x - 1$ and change the subject of the formula.

$$\Rightarrow y + 1 = 4x$$
$$\Rightarrow \quad x = \frac{y+1}{4}$$

Hence $f^{-1} : x \mapsto \frac{x+1}{4}, \quad x \in \mathbb{R}$

b $gf(x) = g(4x - 1)$

$$= \frac{3}{2(4x-1)-1}$$
$$= \frac{3}{8x-3}$$

Hence $gf : x \mapsto \frac{3}{8x-3}$
$gf(x)$ is undefined when $8x - 3 = 0$

That is, at $x = \frac{3}{8}$

$\therefore$ Domain $gf(x) = \left\{x \in \mathbb{R}, x \neq \frac{3}{8}\right\}$

c If $2f(x) = g(x)$

$$2 \times (4x - 1) = \frac{3}{2x-1}$$
$$8x - 2 = \frac{3}{2x-1}$$
$$(8x - 2)(2x - 1) = 3$$
$$16x^2 - 12x + 2 = 3$$
$$16x^2 - 12x - 1 = 0$$

Use $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
with $a = 16$, $b = -12$ and $c = -1$.

Then $x = \frac{12 \pm \sqrt{144 + 64}}{32}$

$$= \frac{12 \pm \sqrt{208}}{32}$$
$$= 0.826, -0.076$$

Values of x are -0.076 and 0.826

14 a Let $y = \frac{x}{x-2}$

$$y(x - 2) = x$$
$$yx - 2y = x \quad \text{(rearrange)}$$
$$yx - x = 2y$$
$$x(y - 1) = 2y$$
$$x = \frac{2y}{y-1}$$
$$f^{-1}(x) = \frac{2x}{x-1}, \quad x \neq 1$$

b The range of $f^{-1}(x)$ is the domain of $f(x)$:
$\{f^{-1}(x) \in \mathbb{R},\ f^{-1}(x) \neq 2\}$

c $gf(1.5) = g\left(\frac{1.5}{1.5-2}\right)$

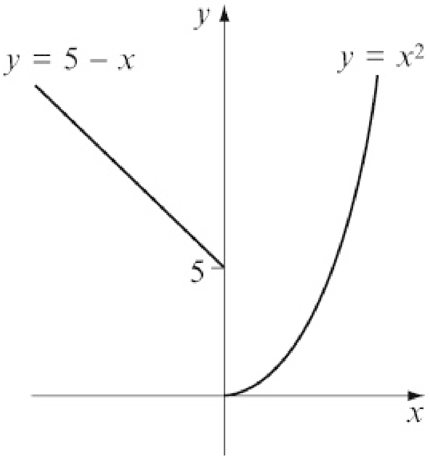
$$= g\left(\frac{1.5}{-0.5}\right)$$
$$= g(-3)$$
$$= \frac{3}{-3}$$
$$= -1$$

14 d If $g(x) = f(x) + 4$

$$\frac{3}{x} = \frac{x}{x-2} + 4$$
$$3(x-2) = x^2 + 4x(x-2)$$
$$3x - 6 = x^2 + 4x^2 - 8x$$
$$0 = 5x^2 - 11x + 6$$
$$0 = (5x-6)(x-1)$$
$$\Rightarrow x = \frac{6}{5}, 1$$

15 $y = 5 - x$ is a straight line with gradient -1 passing through 5 on the y axis.

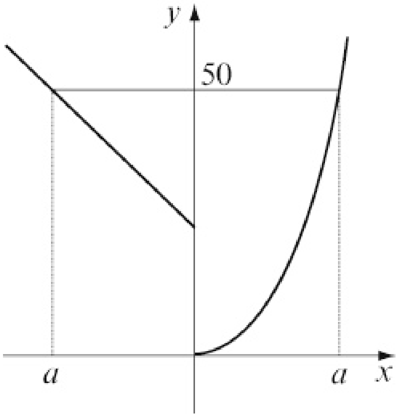
$y = x^2$ is a $\cup$-shaped quadratic passing through $(0, 0)$



a $n(-3) = 5 - (-3)$
$= 5 + 3$
$= 8$

$n(3) = 32$
$= 9$

b From the diagram, you can see there are two values of x for which $n(x) = 50$

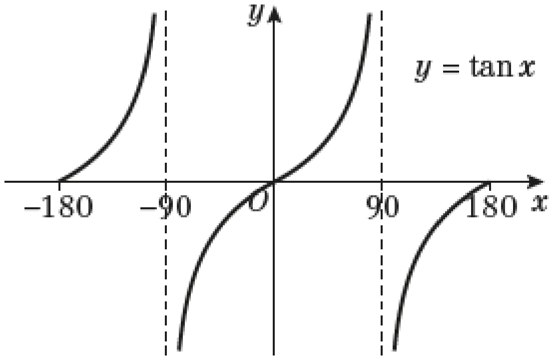


The negative value of x is where
$5 - x = 50$
$x = 5 - 50$
$x = -45$

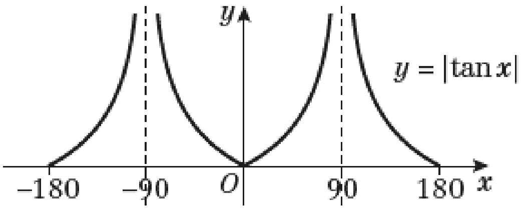
The positive value of x is where
$x^2 = 50$
$x = \sqrt{50}$
$x = 5\sqrt{2}$

The values of x such that $n(x) = 50$ are -45 and $+5\sqrt{2}$

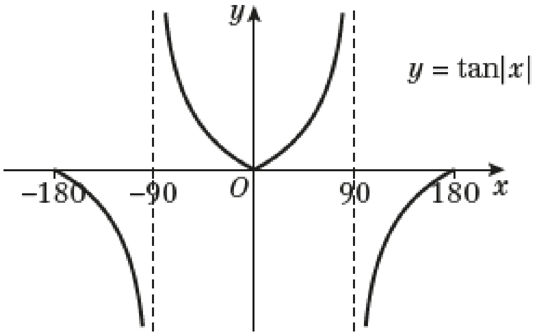
16 a



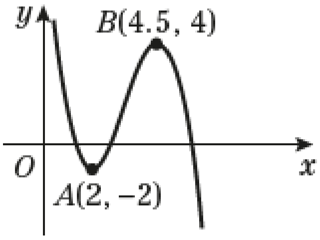
16 b $y = |\tan(x)|$ reflects the negative parts of $\tan x$ in the x axis.



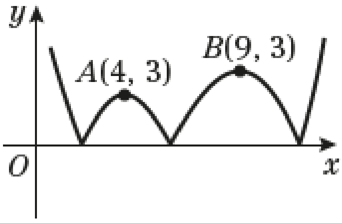
c $y = \tan(|x|)$ reflects $\tan x$ in the y-axis.



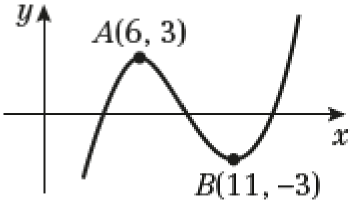
17 a



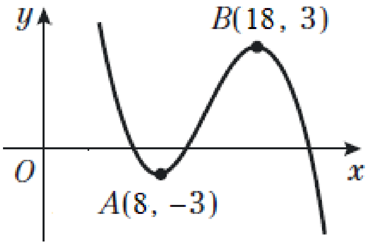
b



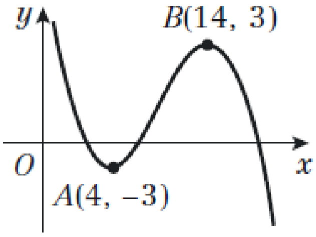
c



d $y = \mathrm{f}(\frac{1}{2}x + 2)$ can be written as
$y = \mathrm{f}(\frac{1}{2}(x + 4))$
$y = \mathrm{f}(\frac{1}{2}x)$
Horizontal stretch, scale factor 2.



$y = \mathrm{f}(\frac{1}{2}(x + 4))$
Horizontal translation of −4

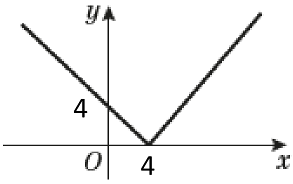


18 a $\mathrm{g}(x) \geqslant 0$

b $\mathrm{gf}(x) = \mathrm{g}(4 - x)$
$= 3(4 - x)^2$
$= 3x^2 - 24x + 48$

$\mathrm{gf}(x) = 48$
$3x^2 - 24x + 48 = 48$
$3x^2 - 24x = 0$
$3x(x - 8) = 0$
$x = 0$ or $x = 8$

18 c

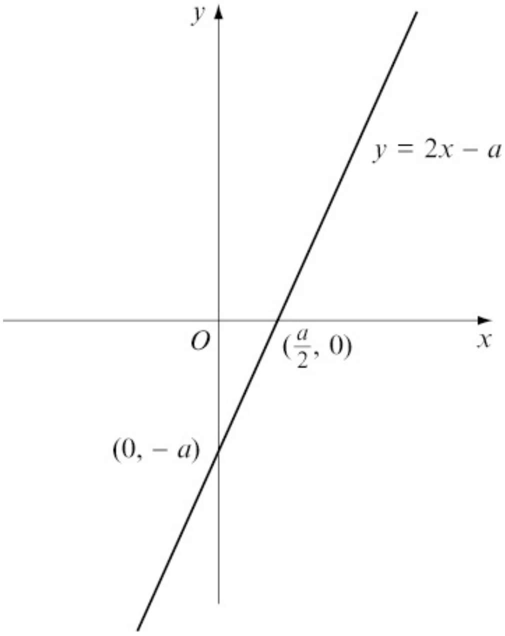


$|\mathrm{f}(x)| = 2$ when $|4-x| = 2$, so

$4-x=2 \Rightarrow x=2$

or $-(4-x)=2 \Rightarrow x=6$

19 a

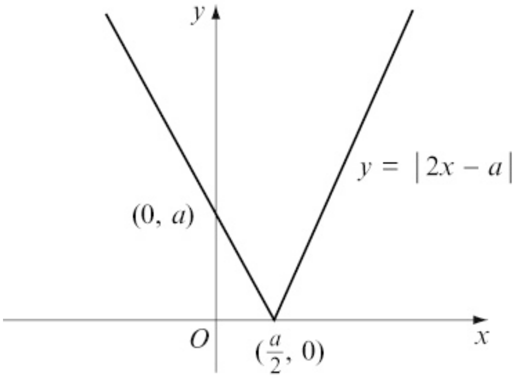


For $y=|2x-a|$:

When $x=0$, $y=|-a|=a$ $\quad (0, a)$

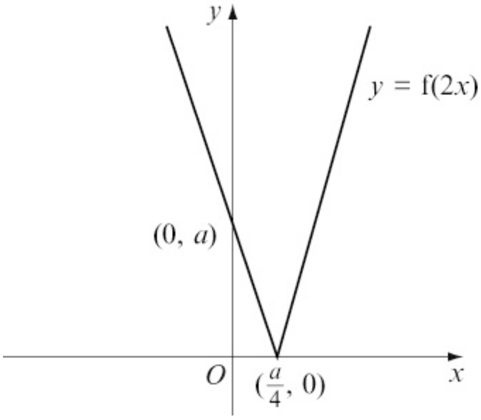
When $y=0$, $2x-a=0$

$\Rightarrow x=\frac{a}{2}$ $\quad \left(\frac{a}{2}, 0\right)$



19 b $y=\mathrm{f}(2x)$

Horizontal stretch, scale factor $\frac{1}{2}$



c $|2x-a|=\frac{1}{2}x$

Either $(2x-a)=\frac{1}{2}x$

$\Rightarrow a=\frac{3}{2}x$

Given that $x=4$,

$a=\frac{3\times 4}{2}=6$

Or

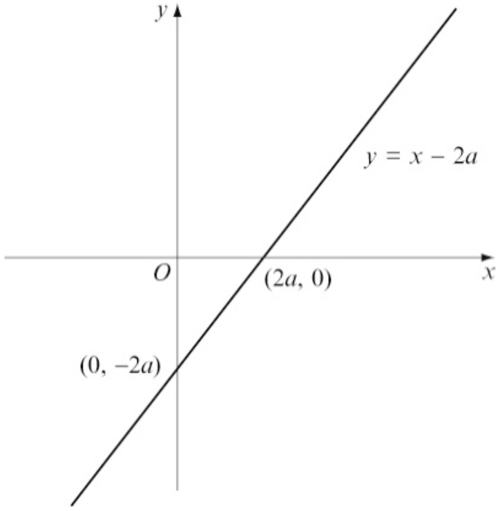
$-(2x-a)=\frac{1}{2}x$

$\Rightarrow a=\frac{5}{2}x$

Given that $x=4$,

$a=\frac{5\times 4}{2}=10$

20 a

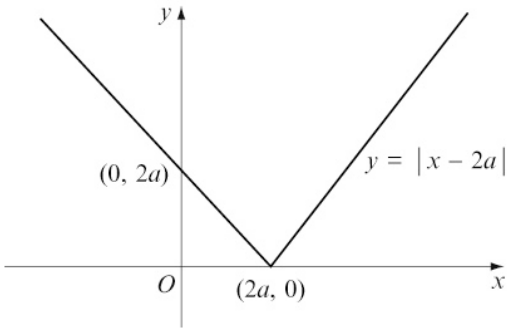


For $y = |x - 2a|$:

When $x = 0$, $y = |-2a| = 2a$ $(0, 2a)$

When $y = 0$, $x - 2a = 0$

$\Rightarrow x = 2a$ $(2a, 0)$



b $|x - 2a| = \frac{1}{3}x$

Either $(x - 2a) = \frac{1}{3}x$

$\Rightarrow x - \frac{1}{3}x = 2a$

$\Rightarrow \frac{2}{3}x = 2a$

$\Rightarrow x = 3a$

or $-(x - 2a) = \frac{1}{3}x$

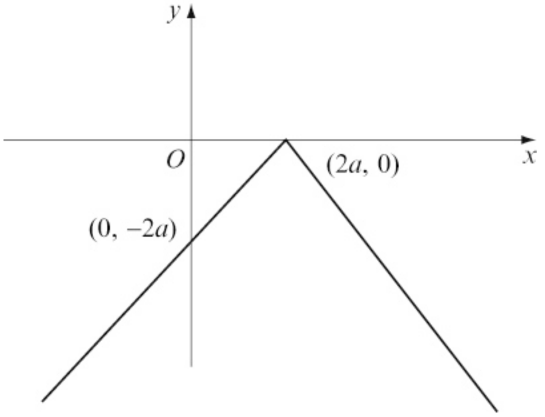
$\Rightarrow -x + 2a = \frac{1}{3}x$

$\Rightarrow \frac{4}{3}x = 2a$

$\Rightarrow x = \frac{3}{2}a$

c $y = -|x - 2a|$

Reflect $y = |x - 2a|$ in the x-axis



$y = a - |x - 2a|$ Vertical translation by $+a$

For $y = a - |x - 2a|$:

When $x = 0$,

$y = a - |-2a|$

$= a - 2a$

$= -a$ $(0, -a)$

When $y = 0$,

$a - |x - 2a| = 0$

$|x - 2a| = a$

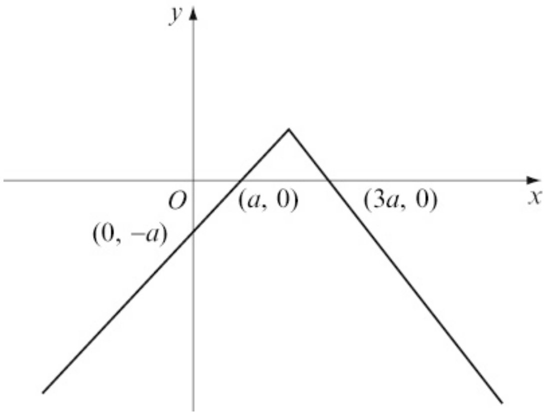
Either $x - 2a = a$

$\Rightarrow x = 3a$ $(3a, 0)$

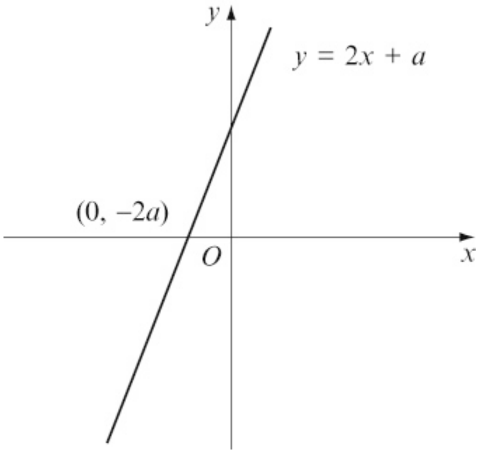
or $-(x - 2a) = a$

$\Rightarrow -x + 2a = a$

$\Rightarrow x = a$ $(a, 0)$



21 a & b

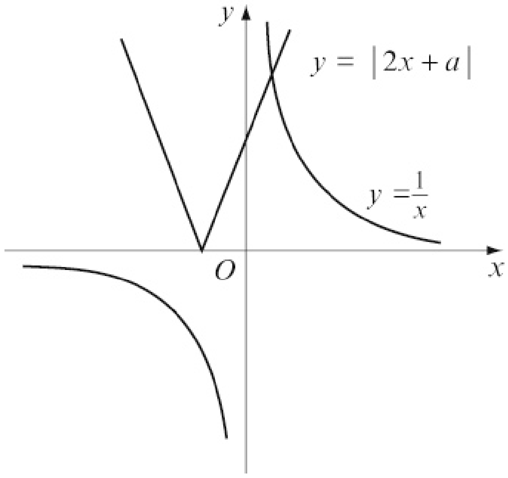


For $y = |2x + a|$:

When $x = 0,\ y = |a| = a \qquad (0,\ a)$

When $y = 0,\ 2x + a = 0$

$$\Rightarrow x = -\frac{a}{2} \qquad \left(-\frac{a}{2},\ 0\right)$$



c Intersection of graphs in **b** gives solutions to the equation:

$$|2x + a| = \frac{1}{x}$$
$$x|x + a| = 1$$
$$x|2x + a| - 1 = 0$$

The graphs intersect once only, so $x|2x + a| - 1 = 0$ has only one solution.

d The intersection point is on the non-reflected part of the modulus graph, so here $|2x - a| = 2x - a$

$$x(2x + a) - 1 = 0$$
$$2x^2 + ax - 1 = 0$$
$$x = \frac{-a \pm \sqrt{a^2 + 8}}{4}$$

As shown on the graph,
x is positive at intersection,

so $x = \dfrac{-a + \sqrt{a^2 + 8}}{4}$

22 a $f(x) = x^2 - 7x + 5\ln x + 8$

$$f'(x) = 2x - 7 + \frac{5}{x}$$

At stationary points, $f'(x) = 0$:

$$2x - 7 + \frac{5}{x} = 0$$
$$2x^2 - 7x + 5 = 0$$
$$(2x - 5)(x - 1) = 0$$
$$x = \frac{5}{2},\ x = 1$$

Point A: $x = 1$,

$$f(x) = 1 - 7 + 5\ln 1 + 8$$
$$= 2$$

A is $(1,\ 2)$

Point B: $x = \frac{5}{2}$,

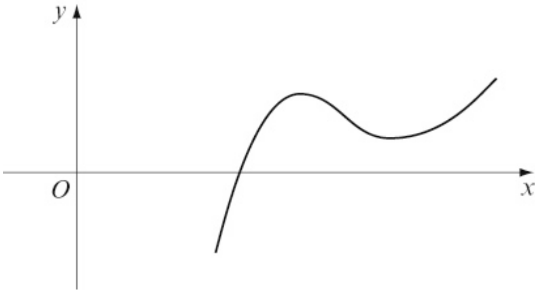
$$f(x) = \frac{25}{4} - \frac{35}{2} + 5\ln\frac{5}{2} + 8$$
$$= 5\ln\frac{5}{2} - \frac{13}{4}$$

B is $\left(\frac{5}{2}, 5\ln\frac{5}{2} - \frac{13}{4}\right)$

22 b $y = \mathrm{f}(x-2)$

Horizontal translation of $+2$.

Graph looks like:

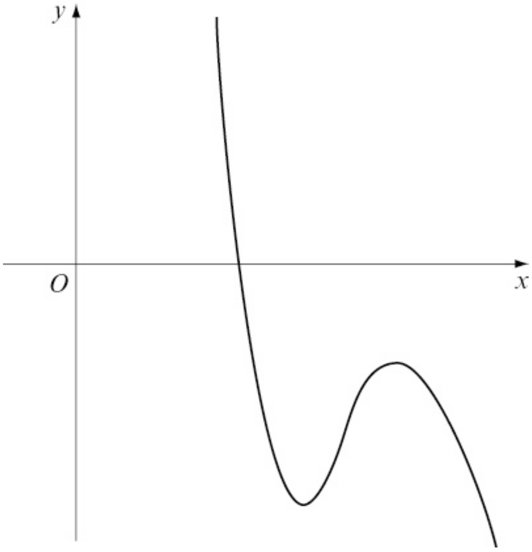


$y = -3f(x-2)$

Reflection in the x-axis, and

vertical stretch, scale factor 3.

Graph looks like:



c Using the transformations,

point $(X,\ Y)$

becomes $(X+2, -3Y)$

$(1,2) \rightarrow (3,-6)$

Minimum

$$\left(\frac{5}{2}, 5\ln\frac{5}{2} - \frac{13}{4}\right) \rightarrow$$

$$\left(\frac{9}{2}, \frac{39}{4} - 15\ln\frac{5}{2}\right)$$

Maximum

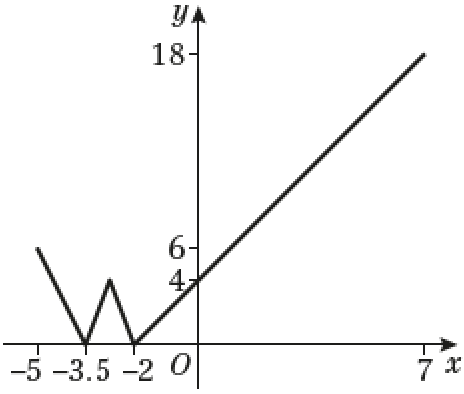
23 a The range of f(x) is $-2 \leqslant \mathrm{f}(x) \leqslant 18$

b $\mathrm{ff}(-3) = \mathrm{f}(-2)$

Using $\mathrm{f}(x) = 2x + 4$

$\mathrm{f}(-2) = 2 \times (-2) + 4 = 0$

c



23 d Look at each section of $\text{f}(x)$ separately.

$-5 \leqslant x \leqslant -3$:

$$\text{Gradient} = \frac{-2-6}{-3-(-5)} = -4$$

$$\therefore \text{f}(x) - (-2) = -4(x-(-3)) \Rightarrow \text{f}(x) = -4x - 14$$

So in this region, $\text{f}(x) = 2$ when $x = -4$

$\therefore \text{fg}(x) = 2$ has a corresponding solution if

$$\text{g}(x) = -4 \Rightarrow \text{g}(x) + 4 = x^2 - 7x + 14 = 0$$

Discriminant $(-7)^2 - 4(1)(14) = -7 < 0$

So no solution

$$-3 \leqslant x \leqslant 7\text{: Gradient} = \frac{18-(-2)}{7-(-3)} = 2$$

$$\therefore \text{f}(x) - (-2) = 2(x-(-3)) \Rightarrow \text{f}(x) = 2x + 4$$

So in this region, $\text{f}(x) = 2$ when $x = -1$

$\therefore \text{fg}(x) = 2$ has a corresponding solution if

$$\text{g}(x) = -1 \Rightarrow \text{g}(x) + 1 = x^2 - 7x + 11 = 0$$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(11)}}{2(1)} = \frac{7 \pm \sqrt{5}}{2}$$

$$\therefore x = \frac{7+\sqrt{5}}{2} \text{ or } x = \frac{7-\sqrt{5}}{2}$$

24 a The range of $\text{p}(x)$ is $\text{p}(x) \leqslant 10$

b $\text{p}(x)$ is many-to-one, therefore the inverse is one-to-many, which is not a function.

c At first point of intersection:

$2(x + 4) + 10 = -4$

$2x + 18 = -4$

$x = -11$

At the other point of intersection:

$-2(x + 4) + 10 = -4$

$-2x + 2 = -4$

$x = 3$

$-11 < x < 3$

d For no solutions, $\text{p}(x) > 10$ at $x = -4$

$$\text{So } -\frac{1}{2}x + k > 10 \text{ at } x = -4$$

$$-\frac{1}{2}(-4) + k > 10$$

$$2 + k > 10$$

$$k > 8$$

25 a Completing the square

$$\begin{aligned} 3x^2 - 12x + 20 &= 3(x^2 - 4x) + 20 \\ &= 3((x-2)^2 - 4) + 20 \\ &= 3(x-2)^2 - 12 + 20 \\ &= 3(x-2)^2 + 8 \end{aligned}$$

b $$\text{g}(x) = \frac{1}{3x^2 - 12x + 20} = \frac{1}{3(x-2)^2 + 8}$$

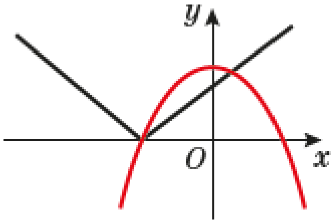
The maximum value of $\text{g}(x)$ is $\frac{1}{8}$ (when $x = 2$)

As x approaches infinity, $\text{g}(x)$ approaches 0

Therefore the range is $0 < \text{g}(x) \leq \frac{1}{8}$

Challenge

a



b $y = (a + x)(a - x)$

When $y = 0$, $x = -a$ or $x = a$

When $x = 0$, $y = a^2$

$(-a, 0)$, $(a, 0)$, $(0, a^2)$

c When $x = 4$, $y = a^2 - x^2$

$= a^2 - 16$

and $y = x + a$

$= 4 + a$

$a^2 - 16 = 4 + a$

$a^2 - a - 20 = 0$

$(a - 5)(a + 4) = 0$

As $a > 1$, $a = 5$

Sequences and series 3A

1 a i $u_n = 5n + 2$

$n = 1 \rightarrow \quad u_1 = 5(1) + 2 = 7$
$n = 2 \rightarrow \quad u_2 = 5(2) + 2 = 12$
$n = 3 \rightarrow \quad u_3 = 5(3) + 2 = 17$
$n = 4 \rightarrow \quad u_4 = 5(4) + 2 = 22$

ii $a = 7$ and $d = 5$

b i $u_n = 9 - 2n$

$n = 1 \rightarrow \quad u_1 = 9 - 2(1) = 7$
$n = 2 \rightarrow \quad u_2 = 9 - 2(2) = 5$
$n = 3 \rightarrow \quad u_3 = 9 - 2(3) = 3$
$n = 4 \rightarrow \quad u_4 = 9 - 2(4) = 1$

ii $a = 7$ and $d = -2$

c i $u_n = 7 + 0.5n$

$n = 1 \rightarrow \quad u_1 = 7 + 0.5(1) = 7.5$
$n = 2 \rightarrow \quad u_2 = 7 + 0.5(2) = 8$
$n = 3 \rightarrow \quad u_3 = 7 + 0.5(3) = 8.5$
$n = 4 \rightarrow \quad u_4 = 7 + 0.5(4) = 9$

ii $a = 7.5$ and $d = 0.5$

d i $u_n = n - 10$

$n = 1 \rightarrow \quad u_1 = 1 - 10 = -9$
$n = 2 \rightarrow \quad u_2 = 2 - 10 = -8$
$n = 3 \rightarrow \quad u_3 = 3 - 10 = -7$
$n = 4 \rightarrow \quad u_4 = 4 - 10 = -6$

ii $a = -9$ and $d = 1$

2 a $5 \xrightarrow[+2]{} 7 \xrightarrow[+2]{} 9 \xrightarrow[+2]{} 11$

$$\begin{aligned} 10\text{th term} &= 5 + 9 \times 2 = 5 + 18 = 23 \\ n\text{th term} &= 5 + (n - 1) \times 2 \\ &= 5 + 2n - 2 \\ &= 2n + 3 \end{aligned}$$

b $5 \xrightarrow[+3]{} 8 \xrightarrow[+3]{} 11 \xrightarrow[+3]{} 14$

$$\begin{aligned} 10\text{th term} &= 5 + 9 \times 3 = 5 + 27 = 32 \\ n\text{th term} &= 5 + (n - 1) \times 3 \\ &= 5 + 3n - 3 \\ &= 3n + 2 \end{aligned}$$

c $24 \xrightarrow[-3]{} 21 \xrightarrow[-3]{} 18 \xrightarrow[-3]{} 15$

$$\begin{aligned} 10\text{th term} &= 24 + 9 \times (-3) \\ &= 24 - 27 = -3 \\ n\text{th term} &= 24 + (n - 1) \times (-3) \\ &= 24 - 3n + 3 \\ &= 27 - 3n \end{aligned}$$

d $-1 \xrightarrow[+4]{} 3 \xrightarrow[+4]{} 7 \xrightarrow[+4]{} 11$

$$\begin{aligned} 10\text{th term} &= -1 + 9 \times 4 \\ &= -1 + 36 = 35 \\ n\text{th term} &= -1 + (n - 1) \times 4 \\ &= -1 + 4n - 4 \\ &= 4n - 5 \end{aligned}$$

e $x \xrightarrow[+x]{} 2x \xrightarrow[+x]{} 3x \xrightarrow[+x]{} 4x$

$$\begin{aligned} 10\text{th term} &= x + 9 \times x = 10x \\ n\text{th term} &= x + (n - 1)x = nx \end{aligned}$$

f $a \xrightarrow[+d]{} a + d \xrightarrow[+d]{} a + 2d \xrightarrow[+d]{} a + 3d$

$$\begin{aligned} 10\text{th term} &= a + 9d \\ n\text{th term} &= a + (n - 1)d \end{aligned}$$

3 **a** $3 \underset{+4}{\rightarrow} 7 \underset{+4}{\rightarrow} 11 \ldots 83 \underset{+4}{\rightarrow} 87$

$$\text{number of jumps} = \frac{87-3}{4} = 21$$

so number of terms = 21 + 1 = 22

3 **b** $5 \underset{+3}{\rightarrow} 8 \underset{+3}{\rightarrow} 11 \ldots 119 \underset{+3}{\rightarrow} 122$

$$\text{number of jumps} = \frac{122-5}{3} = 39$$

therefore number of terms = 40

c $90 \underset{-2}{\rightarrow} 88 \underset{-2}{\rightarrow} 86 \ldots 16 \underset{-2}{\rightarrow} 14$

$$\text{number of jumps} = \frac{90-14}{2} = 38$$

therefore number of terms = 39

d $4 \underset{+5}{\rightarrow} 9 \underset{+5}{\rightarrow} 14 \ldots 224 \underset{+5}{\rightarrow} 229$

$$\text{number of jumps} = \frac{229-4}{5} = 45$$

therefore number of terms = 46

e $x \underset{+2x}{\rightarrow} 3x \underset{+2x}{\rightarrow} 5x \ldots 35x$

$$\text{number of jumps} = \frac{35x-x}{2x} = 17$$

therefore number of terms = 18

f $a \underset{+d}{\rightarrow} a+d \underset{+d}{\rightarrow} a+2d \ldots a+(n-1)d$

$$\text{number of jumps} = \frac{a+(n-1)d-a}{d}$$

$$= \frac{(n-1)d}{d} = n-1$$

therefore number of terms $= n$

4 $u_1 = 14$ and $u_4 = 32$
$d = (32 - 14) \div 3$
$d = 6$

5 $u_n = pn + q$
$u_6 = 9$, so $6p + q = 9$ **(1)**
$u_9 = 11$, so $9p + q = 11$ **(2)**
(2) − **(1)** gives:
$3p = 2$
$p = \frac{2}{3}$
Substitute $p = \frac{2}{3}$ in **(1)**:
$6\left(\frac{2}{3}\right) + q = 9$
$q = 5$
Constants are $p = \frac{2}{3}$ and $q = 5$

6 $u_3 = 30$ and $u_9 = 9$
$d = (9 - 30) \div 6 = -3.5$
$u_{10} = 5.5$, $u_{11} = 2$, $u_{12} = -1.5$
The first negative term is −1.5

7 $u_{20} = 14$ and $u_{40} = -6$
$d = (-6 - 14) \div 20 = -1$
$u_{10} = 14 - 10(-1) = 24$

8 $u_1 = 5p$, $u_2 = 20$ and $u_3 = 3p$
$d = 20 - 5p$ and $d = 3p - 20$
$20 - 5p = 3p - 20$
$8p = 40$
$p = 5$
$d = 20 - 5 \times 5 = -5$
$u_{20} = 5 \times 5 - 5(20 - 1) = -70$

9 $u_1 = -8$, $u_2 = k^2$ and $u_3 = 17k$
$d = k^2 + 8$ and $d = 17k - k^2$
$k^2 + 8 = 17k - k^2$
$2k^2 - 17k + 8 = 0$
$(2k - 1)(k - 8) = 0$
$k = \frac{1}{2}$ or $k = 8$

10 $a = k^2$, $d = k$, $u_5 = 41$

$u_5 = k^2 + (5-1)k = 41$

$k^2 + 4k - 41 = 0$

Using the formula:

$$k = \frac{-4 \pm \sqrt{4^2 - 4\times(1)\times(-41)}}{2\times1}$$

$$k = \frac{-4 \pm \sqrt{180}}{2}$$

$$k = \frac{-4 \pm 6\sqrt{5}}{2}$$

$$k = -2 \pm 3\sqrt{5}$$

As $k > 0$, $k = -2 + 3\sqrt{5}$

Challenge

$u_n = \ln a + (n - 1)\ln b$

$u_3 = \ln 16$ and $u_7 = \ln 256$

$d = \ln b$

$d = \frac{1}{4}(\ln 256 - \ln 16)$

$\ln b = \frac{1}{4}(\ln 256 - \ln 16)$

$\ln b = \ln 256^{\frac{1}{4}} - \ln 16^{\frac{1}{4}}$

$\ln b = \ln 4 - \ln 2$

$\ln b = \ln\left(\frac{4}{2}\right)$

$\ln b = \ln 2$

$b = 2$

$u_3 = \ln 16$

$= \ln a + (3 - 1)\ln 2$

$= \ln a + \ln 2^2$

So $\ln 16 = \ln a + \ln 4 = \ln 4a$

$a = 4$, $b = 2$

Sequences and series 3B

1 **a** 3 + 7 + 11 + 14 + . . . (for 20 terms)

Substitute $a = 3$, $d = 4$, $n = 20$ into

$$S_n = \frac{n}{2}\left(2a+\left(n-1\right)d\right) = \frac{20}{2}\left(6+19\times4\right)$$
$$= 10\times82 = 820$$

b 2 + 6 + 10 + 14 + . . . (for 15 terms)

Substitute $a = 2$, $d = 4$, $n = 15$ into

$$S_n = \frac{n}{2}\left(2a+\left(n-1\right)d\right) = \frac{15}{2}\left(4+14\times4\right)$$
$$= \frac{15}{2}\times60 = 450$$

c 30 + 27 + 24 + 21 + . . . (40 terms)

Substitute $a = 30$, $d = -3$, $n = 40$ into

$$S_n = \frac{n}{2}\left(2a+\left(n-1\right)d\right)$$
$$= \frac{40}{2}\left(60+39\times\left(-3\right)\right)$$
$$= 20\times\left(-57\right) = -1140$$

d 5 + 1 + −3 + −7 + . . . (14 terms)

Substitute $a = 5$, $d = -4$, $n = 14$ into

$$S_n = \frac{n}{2}\left(2a+\left(n-1\right)d\right)$$
$$= \frac{14}{2}\left(10+13\times\left(-4\right)\right)$$
$$= 7\times\left(-42\right) = -294$$

e 5 + 7 + 9 + . . . + 75

Here, $a = 5$, $d = 2$ and $L = 75$.

Use $L = a + (n - 1)d$ to find n:

$$75 = 5+\left(n-1\right)\times2$$
$$70 = \left(n-1\right)\times2$$
$$35 = n-1$$
$$n = 36\ \left(36\ \text{terms}\right)$$

Substitute $a = 5$, $d = 2$, $n = 36$ and $L = 75$ into

$$S_n = \frac{n}{2}\left(a+L\right) = \frac{36}{2}\left(5+75\right)$$
$$= 18\times80 = 1440$$

f 4 + 7 + 10 + . . . + 91

Here, $a = 4$, $d = 3$ and $L = 91$.

Use $L = a + (n - 1)d$ to find n:

$$91 = 4+\left(n-1\right)\times3$$
$$87 = \left(n-1\right)\times3$$
$$29 = \left(n-1\right)$$
$$n = 30\ \left(30\ \text{terms}\right)$$

Substitute $a = 4$, $d = 3$, $L = 91$ and $n = 30$ into

$$S_n = \frac{n}{2}\left(a+L\right) = \frac{30}{2}\left(4+91\right)$$
$$= 15\times95 = 1425$$

1 **g** $34 + 29 + 24 + 19 + \ldots + -111$

Here, $a = 34$, $d = -5$ and $L = -111$.

Use $L = a + (n - 1)d$ to find n:

$$\begin{aligned}-111 &= 34 + (n-1)\times(-5)\\ -145 &= (n-1)\times(-5)\\ 29 &= (n-1)\\ n &= 30\ (30\,\text{terms})\end{aligned}$$

Substitute $a = 34$, $d = -5$, $L = -111$ and $n = 30$ into

$$\begin{aligned}S_n &= \frac{n}{2}(a+L) = \frac{30}{2}(34+(-111))\\ &= 15\times(-77) = -1155\end{aligned}$$

h $(x + 1) + (2x + 1) + (3x + 1) + \ldots + (21x + 1)$

Here, $a = x + 1$, $d = x$ and $L = 21x + 1$.

Use $L = a + (n - 1)d$ to find n:

$$\begin{aligned}21x+1 &= x+1+(n-1)\times x\\ 20x &= (n-1)\times x\\ 20 &= (n-1)\\ n &= 21(21\,\text{terms})\end{aligned}$$

Substitute $a = x + 1$, $d = x$, $L = 21x + 1$ and $n = 21$ into

$$\begin{aligned}S_n &= \frac{n}{2}(a+L) = \frac{21}{2}(x+1+21x+1)\\ &= \frac{21}{2}\times(22x+2) = 21(11x+1)\\ &= 231x+21\end{aligned}$$

2 **a** $5 + 8 + 11 + 14 + \ldots = 670$

Substitute $a = 5$, $d = 3$, $S_n = 670$ into

$$\begin{aligned}S_n &= \frac{n}{2}(2a+(n-1)d)\\ 670 &= \frac{n}{2}(10+(n-1)\times 3)\\ 670 &= \frac{n}{2}(3n+7)\\ 1340 &= n(3n+7)\\ 0 &= 3n^2+7n-1340\\ 0 &= (n-20)(3n+67)\\ n &= 20 \text{ or } -\frac{67}{3}\end{aligned}$$

Number of terms is 20.

b $3 + 8 + 13 + 18 + \ldots = 1575$

Substitute $a = 3$, $d = 5$, $S_n = 1575$ into

$$\begin{aligned}S_n &= \frac{n}{2}(2a+(n-1)d)\\ 1575 &= \frac{n}{2}(6+(n-1)\times 5)\\ 1575 &= \frac{n}{2}(5n+1)\\ 3150 &= n(5n+1)\\ 0 &= 5n^2+n-3150\\ 0 &= (5n+126)(n-25)\\ n &= -\frac{126}{5}, 25\end{aligned}$$

Number of terms is 25.

2 **c** $64 + 62 + 60 + \ldots = 0$

Substitute $a = 64$, $d = -2$ and $S_n = 0$ into

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

$$0 = \frac{n}{2}\left(128 + (n-1)\times(-2)\right)$$

$$0 = \frac{n}{2}(130 - 2n)$$

$$0 = n(65 - n)$$

$$n = 0 \text{ or } 65$$

Number of terms is 65.

d $34 + 30 + 26 + 22 + \ldots = 112$

Substitute $a = 34$, $d = -4$ and $S_n = 112$ into

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

$$112 = \frac{n}{2}\left(68 + (n-1)\times(-4)\right)$$

$$112 = \frac{n}{2}(72 - 4n)$$

$$112 = n(36 - 2n)$$

$$2n^2 - 36n + 112 = 0$$

$$n^2 - 18n + 56 = 0$$

$$(n-4)(n-14) = 0$$

$$n = 4 \text{ or } 14$$

Number of terms is 4 or 14

3 $S = \underbrace{2 + 4 + 6 + 8 + \ldots}_{\text{50 terms}}$

This is an arithmetic series with $a = 2$, $d = 2$ and $n = 50$.

Use $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$

So $S = \frac{50}{2}(4 + 49\times 2)$

$= 25\times 102 = 2550$

4 $7 + 12 + 17 + 22 + 27 + \ldots > 1000$

Using $S_n = \frac{n}{2}(2a + (n-1)d)$

$$1000 = \frac{n}{2}(2\times 7 + (n-1)5)$$

$$2000 = n(14 + 5n - 5)$$

$$2000 = n(5n + 9)$$

$$5n^2 + 9n - 2000 = 0$$

$$n = \frac{-9 \pm \sqrt{9^2 - 4\times 5\times(-2000)}}{2\times 5}$$

$$n = \frac{-9 \pm \sqrt{40081}}{10}$$

$n = 19.12\ldots$ or $n = -20.92\ldots$

So 20 terms are needed.

5 Let common difference = d.

Substitute $a = 4$, $n = 20$, and $S_{20} = -15$ into

$$S_n = \frac{n}{2}\left(2a + \left(n-1\right)d\right)$$
$$-15 = \frac{20}{2}\left(8 + \left(20-1\right)d\right)$$
$$-15 = 10\left(8 + 19d\right)$$
$$-1.5 = 8 + 19d$$
$$19d = -9.5$$
$$d = -0.5$$

The common difference is −0.5.

Use nth term = $a + (n - 1)d$ to find

$$\begin{aligned}20\text{th term} &= a + 19d\\ &= 4 + 19 \times (-0.5)\\ &= 4 - 9.5 = -5.5\end{aligned}$$

20th term is −5.5.

6 Let the first term be a and the common difference d.

Sum of first three terms is 12, so

$$a + (a + d) + (a + 2d) = 12$$
$$3a + 3d = 12$$

$$a + d = 4 \qquad \textbf{(1)}$$

20th term is −32, so

$$a + 19d = -32 \qquad \textbf{(2)}$$

Equation (**2**) − Equation (**1**):

$$18d = -36$$
$$d = -2$$

Substitute $d = -2$ into Equation (**1**):

$$a + \left(-2\right) = 4$$
$$a = 6$$

Therefore, first term is 6 and common difference is −2.

7
$$S_{50} = 1 + 2 + 3 + \ldots + 48 + 49 + 50 \qquad \textbf{(1)}$$
$$S_{50} = 50 + 49 + 48 + \ldots + 3 + 2 + 1 \qquad \textbf{(2)}$$

Adding (**1**) and (**2**):

$$2 \times S_{50} = 50 \times 51$$
$$S_{50} = \frac{50 \times 51}{2}$$
$$= 1275$$

8 Sum required $= \underbrace{1 + 2 + 3 + \ldots + 2n}_{}$

Arithmetic series with $a = 1$, $d = 1$ and $n = 2n$.

$$\text{Use } S_n = \frac{n}{2}\left(2a+\left(n-1\right)d\right)$$
$$= \frac{2n}{2}\left(2\times1+\left(2n-1\right)\times1\right)$$
$$= \frac{\cancel{2}n}{\cancel{2}}\left(2n+1\right)$$
$$= n\left(2n+1\right)$$

9 Required sum $= \underbrace{1 + 3 + 5 + 7 + \ldots}_{n\,\text{terms}}$

This is an arithmetic series with $a = 1$, $d = 2$ and $n = n$.

$$\text{Use } S_n = \frac{n}{2}\left(2a+\left(n-1\right)d\right)$$
$$= \frac{n}{2}\left(2\times1+\left(n-1\right)\times2\right)$$
$$= \frac{n}{2}\left(2+2n-2\right)$$
$$= \frac{n\times\cancel{2}n}{\cancel{2}}$$
$$= n\times n$$
$$= n^2$$

10 a $u_5 = 33$, so $a + 4d = 33$ **(1)**
$u_{10} = 68$, so $a + 9d = 68$ **(2)**
(2) − **(1)** gives:
$5d = 35$
$d = 7$
$a = 5$

$$2225 = \frac{n}{2}(2\times5+(n-1)7)$$
$$4450 = n(7n+3)$$
$$7n^2+3n-4450=0$$

10 b
$$n = \frac{-3\pm\sqrt{3^2-4\times7\times(-4450)}}{2\times7}$$
$$n = \frac{-3\pm\sqrt{124\,609}}{14}$$
$$n = \frac{-3\pm353}{14}$$
$n = 25$ or -25.42
So $n = 25$

11 a $u_n = a + (n - 1)d$
$303 = k + 1 + (n - 1)(k + 2)$
$303 = k + 1 + nk + 2n - k - 2$
$303 = nk + 2n - 1$
$304 = n(k + 2)$
$$n = \frac{304}{k+2}$$

b
$$S_n = \frac{\left(\frac{304}{k+2}\right)}{2}(k+1+303)$$
$$S_n = \frac{152}{k+2}(k+304)$$
$$S_n = \frac{152k+46\,208}{k+2}$$

c
$$2568 = \frac{152k+46\,208}{k+2}$$
$$2568(k+2) = 152k+46\,208$$
$2416k = 41\,072$
$k = 17$

12 a
$$S_n = \frac{33}{2}(3+99)$$
$= 1683$

b i $4p + (n - 1)4p = 400$
$4pn = 400$
$$n = \frac{100}{p}$$

12 b ii $$S_n = \frac{\left(\frac{100}{p}\right)}{2}(4p + 400)$$

$$S_n = \frac{50}{p}(4p + 400)$$

$$S_n = 200 + \frac{20\,000}{p}$$

c $u_{80} = 3p + 2 + (80 - 1)(2p + 1)$
$= 3p + 2 + 158p + 79$
$= 161p + 81$

13 a $u_n = a + (n - 1)d$
$= 6 + (n - 1)5$
$= 6 + 5n - 5$
$= 5n + 1$

b $u_{10} = 5 \times 10 + 1 = 51$

$$S_{10} = \frac{10}{2}(6 + 51)$$

$= 5 \times 57$
$= 285$

c $$S_k = \frac{k}{2}(2 \times 6 + (k - 1)5)$$

$$= \frac{k}{2}(12 + 5k - 5)$$

$$= \frac{k}{2}(5k + 7)$$

$$\frac{k}{2}(5k + 7) \leqslant 1029$$

$5k^2 + 7k \leqslant 2058$

$5k^2 + 7k - 2058 \leqslant 0$

$(5k - 98)(k + 21) \leqslant 0$

d As $k > 0$, $5k - 98 = 0$, $k = 19.6$
So $k = 19$

Challenge

$u_n = \ln 9 + (n - 1)\ln 3$
$a = \ln 9$, $d = \ln 3$

$$S_n = \frac{n}{2}(2\ln 9 + (n - 1)\ln 3)$$

$$= \frac{n}{2}(\ln 81 - \ln 3 + n\ln 3)$$

$$= \frac{n}{2}(\ln 27 + n\ln 3)$$

$$= \frac{n}{2}(\ln 3^3 + \ln 3^n)$$

$$= \frac{n}{2}(\ln 3^{n+3})$$

$$= \frac{1}{2}(\ln 3^{n^2+3n})$$

Therefore, $a = \dfrac{1}{2}$

Sequences and series 3C

1 **a** $1 \xrightarrow[\times 2]{} 2 \xrightarrow[\times 2]{} 4 \xrightarrow[\times 2]{} 8 \xrightarrow[\times 2]{} 16 \xrightarrow[\times 2]{} 32$

Geometric, $r = 2$

b $2 \xrightarrow[+3]{} 5 \xrightarrow[+3]{} 8 \xrightarrow[+3]{} 11 \xrightarrow[+3]{} 14$

Not geometric
(this is an arithmetic sequence)

c $40 \xrightarrow[-4]{} 36 \xrightarrow[-4]{} 32 \xrightarrow[-4]{} 28$

Not geometric (arithmetic)

d $2 \xrightarrow[\times 3]{} 6 \xrightarrow[\times 3]{} 18 \xrightarrow[\times 3]{} 54$

Geometric, $r = 3$

e $10 \xrightarrow[\times\frac{1}{2}]{} 5 \xrightarrow[\times\frac{1}{2}]{} 2.5 \xrightarrow[\times\frac{1}{2}]{} 1.25$

Geometric, $r = \frac{1}{2}$

f $5 \xrightarrow[\times(-1)]{} -5 \xrightarrow[\times(-1)]{} 5 \xrightarrow[\times(-1)]{} -5$

Geometric, $r = -1$

g $3 \xrightarrow[\times 1]{} 3 \xrightarrow[\times 1]{} 3 \xrightarrow[\times 1]{} 3 \xrightarrow[\times 1]{} 3$

Geometric, $r = 1$

h $4 \xrightarrow[\times\left(-\frac{1}{4}\right)]{} -1 \xrightarrow[\times\left(-\frac{1}{4}\right)]{} 0.25 \xrightarrow[\times\left(-\frac{1}{4}\right)]{} -0.0625$

Geometric, $r = -\frac{1}{4}$

2 **a** $5 \xrightarrow[\times 3]{} 15 \xrightarrow[\times 3]{} 45 \xrightarrow[\times 3]{} 135 \xrightarrow[\times 3]{} 405 \xrightarrow[\times 3]{} 1215$

b $4 \xrightarrow[\times(-2)]{} -8 \xrightarrow[\times(-2)]{} 16 \xrightarrow[\times(-2)]{} -32 \xrightarrow[\times(-2)]{} 64 \xrightarrow[\times(-2)]{} -128$

c $60 \xrightarrow[\times\frac{1}{2}]{} 30 \xrightarrow[\times\frac{1}{2}]{} 15 \xrightarrow[\times\frac{1}{2}]{} 7.5 \xrightarrow[\times\frac{1}{2}]{} 3.75 \xrightarrow[\times\frac{1}{2}]{} 1.875$

d $1 \xrightarrow[\times\frac{1}{4}]{} \frac{1}{4} \xrightarrow[\times\frac{1}{4}]{} \frac{1}{16} \xrightarrow[\times\frac{1}{4}]{} \frac{1}{64} \xrightarrow[\times\frac{1}{4}]{} \frac{1}{256} \xrightarrow[\times\frac{1}{4}]{} \frac{1}{1024}$

e $1 \xrightarrow[\times p]{} p \xrightarrow[\times p]{} p^2 \xrightarrow[\times p]{} p^3 \xrightarrow[\times p]{} p^4 \xrightarrow[\times p]{} p^5$

f

$$x \xrightarrow[\times(-2x)]{} -2x^2 \xrightarrow[\times(-2x)]{} 4x^3 \xrightarrow[\times(-2x)]{} -8x^4 \xrightarrow[\times(-2x)]{} 16x^5 \xrightarrow[\times(-2x)]{} -32x^6$$

3 **a** $3 \quad x \quad 9$

$$\text{Common ratio} = \frac{\text{term}\,2}{\text{term}\,1} \text{ or } \frac{\text{term}\,3}{\text{term}\,2}\ \frac{x}{3} \text{ or } \frac{9}{x}$$

Therefore,

$$\frac{x}{3} = \frac{9}{x} \quad (\text{cross multiply})$$

$$x^2 = 27$$

$$x = \sqrt{27}$$

$$x = \sqrt{9 \times 3}$$

$$x = 3\sqrt{3}$$

3 **b** Term 4 = term 3 × r

Term 3 = 9 and

$$r = \frac{\text{term}\,2}{\text{term}\,1} = \frac{3\sqrt{3}}{3} = \sqrt{3}$$

So term $4 = 9\sqrt{3}$

4 **a** 2, 6, 18, 54, …
6th term = 2 × 3^5
= 2 × 243
= 486
nth term = 2 × 3^{n-1}

b 100, 50, 25, 12.5, …

$$\text{6th term} = 100 \times \left(\frac{1}{2}\right)^5$$

$$= 100 \times \frac{1}{32}$$

$$= \frac{25}{8}$$

$$n\text{th term} = 100 \times \left(\frac{1}{2}\right)^{n-1}$$

c 1, −2, 4, −8, ...
6th term = 1 × $(-2)^5$
= 1 × −32
= −32
nth term = $(-2)^{n-1}$

d 1, 1.1, 1.21, 1.331, ...
6th term = 1 × $(1.1)^5$
= 1 × 1.61051
= 1.61051
nth term = $(1.1)^{n-1}$

5 nth term = 2 × 5^n
1st term = 2 × 5^1 = 10
5th term = 2 × 5^5 = 6250

6 Let the first term be a and the common ratio = r
6th term is 32
$\Rightarrow ar^{6-1} = 32$
$\Rightarrow ar^5 = 32$ **(1)**

3rd term is 4
$\Rightarrow ar^{3-1} = 4$
$\Rightarrow ar^2 = 4$ **(2)**

(1) ÷ **(2)**:

$$\frac{\cancel{a}r^5}{\cancel{a}r^2} = \frac{32}{4}$$

$$r^3 = 8$$

$$r = 2$$

Common ratio is 2.

Substitute r = 2 into equation **(2)**
$a \times 2^2 = 4$
$a \times 4 = 4$
$a = 1$

First term is 1.

7 First term is 4.

$\Rightarrow a = 4$ **(1)**

Third term is 1 ⇒ $ar^{3-1} = 1$

$\Rightarrow ar^2 = 1$ **(2)**

Substitute a = 4 into **(2)**

$$4r^2 = 1$$

$$r^2 = \frac{1}{4}$$

$$r = \pm\frac{1}{2}$$

The sixth term = $ar^{6-1} = ar^5$

7 (continued)

If $r = \frac{1}{2}$ then sixth term $= 4 \times \left(\frac{1}{2}\right)^5 = \frac{1}{8}$

If $r = -\frac{1}{2}$ then sixth term $= 4 \times \left(-\frac{1}{2}\right)^5$

$= -\frac{1}{8}$

Possible values for sixth term: $\frac{1}{8}, -\frac{1}{8}$.

8 a $\frac{u_2}{u_1} = \frac{u_3}{u_2}$

$\frac{2x}{8-x} = \frac{x^2}{2x}$

$4x^2 = 8x^2 - x^3$

$x^3 - 4x^2 = 0$

b $x^2(x - 4) = 0$

$x = 0$ or 4

As $x > 0$, $x = 4$

$a = 4$, $r = 2$

20th term $= ar^{19}$

$= 4 \times 2^{19}$

$= 4 \times 524\,288$

$= 2\,097\,152$

c If 4096 in the sequence then,

for some n, $ar^{n-1} = 4096$

$4 \times 2^{n-1} = 4096$

$2^{n-1} = 1024$

$n - 1 = 10$

$n = 11$

Yes, 4096 is in the sequence as n is an integer.

9 a $a = 200$, $r = p$

$u_6 = 200p^5 = 40$

$p^5 = \frac{1}{5}$

$\log p^5 = \log \frac{1}{5}$

$5\log p = \log 1 - \log 5$

$5\log p + \log 5 = 0$

b $\log p = \frac{-\log 5}{5}$

$p = 10^{\frac{-\log 5}{5}}$

$p = 0.725$

10 $a = 4$, $u_4 = 108 = 4r^3$

$r^3 = 27$

$r = 3$

We want kth term > 500 000

So $4 \times 3^{k-1} > 500\,000$

$3^{k-1} > 125\,000$

$\log 3^{k-1} > \log 125\,000$

$(k - 1)\log 3 > \log 125\,000$

$k - 1 > \frac{\log 125\,000}{\log 3}$

$k - 1 > 10.68$

$k > 11.68$

So $k = 12$

11 $a = 9$, $r = 4$

$u_n = 9 \times 4^{n-1} = 383\,616$

$4^{n-1} = 42\,624$

$\log 4^{n-1} = \log 42\,624$

$(n - 1) \log 4 = \log 42\,624$

$n - 1 = \frac{\log 42\,624}{\log 4}$

$n - 1 = 7.69$

$n = 8.69$

n is not an integer so 383 616 is not in the sequence.

12 $a = 3$, $r = -4$

3, −12, 48, −192, 768, −3072, 12 288, −49 152

So 49 152 is not in the sequence, but −49 152 is.

13 $3 \underset{\times 4}{\rightarrow} 12 \underset{\times 4}{\rightarrow} 48 \ldots$

This is a geometric series with $a = 3$
and $r = 4$.
If a term exceeds 1 000 000 then

$ar^{n-1} > 1\,000\,000$

Substitute $a = 3$, $r = 4$:

$$\begin{aligned}3 \times 4^{n-1} &> 1000\,000\\ 4^{n-1} &> \frac{1000\,000}{3}\\ \log 4^{n-1} &> \log\left(\frac{1000\,000}{3}\right)\\ (n-1)\log 4 &> \log\left(\frac{1000\,000}{3}\right)\\ n-1 &> \frac{\log\left(\frac{1000\,000}{3}\right)}{\log 4}\\ n-1 &> 9.173\ldots\\ n &> 10.173\ldots\\ \text{So } n &= 11\end{aligned}$$

Term is $3 \times 4^{10} = 3\,145\,728$

Sequences and series 3D

1 **a** $1+2+4+8+\ldots$ (8 terms)

In this series $a=1$, $r=2$, $n=8$.

As $|r|>1$ use $S_n=\dfrac{a(r^n-1)}{r-1}$.

$$S_8=\frac{a(r^8-1)}{r-1}=\frac{1\times(2^8-1)}{2-1}=256-1=255$$

b $32+16+8+\ldots$ (10 terms)

In this series $a=32$, $r=\dfrac{1}{2}$, $n=10$.

As $|r|<1$ use $S_n=\dfrac{a(1-r^n)}{1-r}$.

$$S_{10}=\frac{a(1-r^{10})}{1-r}$$

$$=\frac{32\left(1-\left(\frac{1}{2}\right)^{10}\right)}{1-\frac{1}{2}}=63.938\ (3\text{ d.p.})$$

c $a=\dfrac{2}{3}$, $r=\dfrac{2}{5}$, $n=8$

$$S_8=\frac{\frac{2}{3}\left(1-\left(\frac{2}{5}\right)^8\right)}{1-\frac{2}{5}}$$

$=1.110$

d $4-12+36-108+\ldots$ (6 terms)

In this series $a=4$, $r=-3$, $n=6$.

As $|r|>1$ use $S_n=\dfrac{a(r^n-1)}{r-1}$.

$$S_6=\frac{a(r^6-1)}{r-1}=\frac{4\left((-3)^6-1\right)}{-3-1}=-728$$

e $729-243+81-\ \ldots\ -\dfrac{1}{3}$

Here, $a=729$, $r=\dfrac{-243}{729}=-\dfrac{1}{3}$

and the nth term is $-\dfrac{1}{3}$.

Using nth term $=ar^{n-1}$

$$-\frac{1}{3}=729\times\left(-\frac{1}{3}\right)^{n-1}$$

$$-\frac{1}{2187}=\left(-\frac{1}{3}\right)^{n-1}$$

$$\left(-\frac{1}{3}\right)^7=\left(-\frac{1}{3}\right)^{n-1}$$

So $n-1\ =7$

$\Rightarrow n\ =8$

There are 8 terms in the series.

As $|r|<1$ use $S_n=\dfrac{a(1-r^n)}{1-r}$ with $a=729$, $r=-\dfrac{1}{3}$ and $n=8$.

$$S_8=\frac{729\left(1-\left(-\frac{1}{3}\right)^8\right)}{1-\left(-\frac{1}{3}\right)}=546\frac{2}{3}$$

1 f $a = -\frac{5}{2}, r = -\frac{1}{2}, n = 15$

$$S_{15} = \frac{-\frac{5}{2}\left(1-\left(-\frac{1}{2}\right)^{15}\right)}{1+\frac{1}{2}}$$

$= -1.667$

2 $a = 3, r = 0.4, n = 10$

$$S_{10} = \frac{3\left(1-0.4^{10}\right)}{1-0.4}$$

$= 4.9995$

3 $a = 5, r = \frac{2}{3}, n = 8$

$$S_8 = \frac{5\left(1-\left(\frac{2}{3}\right)^8\right)}{1-\frac{2}{3}}$$

$= 14.4147$

4 Let the common ratio be r.

The first three terms are 8, $8r$ and $8r^2$.

Given that the first three terms add up to 30.5,

$$\begin{aligned} 8+8r+8r^2 &= 30.5 \quad (\times 2) \\ 16+16r+16r^2 &= 61 \\ 16r^2+16r-45 &= 0 \\ (4r-5)(4r+9) &= 0 \\ r &= \frac{5}{4}, \frac{-9}{4} \end{aligned}$$

Possible values of r are $\frac{5}{4}$ and $\frac{-9}{4}$.

5 3 + 6 + 12 + 24 + … is a geometric series with $a = 3$, $r = 2$.

So $S_n = \frac{a(r^n-1)}{r-1} = \frac{3(2^n-1)}{2-1} = 3(2^n-1)$

$$\begin{aligned} \text{We want } S_n &> 1.5\,\text{million} \\ S_n &> 1500\,000 \\ 3(2^n-1) &> 1500\,000 \\ 2^n-1 &> 500\,000 \\ 2^n &> 500\,001 \\ \log 2^n &> \log 500\,001 \\ n \log 2 &> \log 500\,001 \\ n &> \frac{\log 500\,001}{\log 2} \\ n &> 18.9 \end{aligned}$$

Least value of n is 19.

6 5 + 4.5 + 4.05 + … is a geometric series with $a = 5$ and $r = \frac{4.5}{5} = 0.9$.

Using $S_n = \frac{a(1-r^n)}{1-r} = \frac{5(1-0.9^n)}{1-0.9}$

$= 50(1-0.9^n)$

$$\begin{aligned} \text{We want } S_n &> 45 \\ 50(1-0.9^n) &> 45 \\ (1-0.9^n) &> \frac{45}{50} \\ 1-0.9^n &> 0.9 \\ 0.9^n &< 0.1 \end{aligned}$$

6 (continued)

$$\log(0.9)^n < \log(0.1)$$
$$n\log(0.9) < \log(0.1)$$
$$n > \frac{\log(0.1)}{\log(0.9)}$$
$$n > 21.85$$
$$\text{So } n = 22$$

7 **a** $a = 25,\ r = \frac{3}{5},\ S_k > 61$

$$\frac{25\left(1-\left(\frac{3}{5}\right)^k\right)}{1-\frac{3}{5}} > 61$$
$$\frac{25\left(1-0.6^k\right)}{0.4} > 61$$
$$25(1 - 0.6^k) > 24.4$$
$$1 - 0.6^k > 0.976$$
$$0.6^k > 0.024$$
$$k\log(0.6) > \log(0.024)$$
$$k > \frac{\log(0.024)}{\log(0.6)}$$

b $k > 7.301$

$k = 8$

8

$$S_2 = \frac{a\left(1-r^2\right)}{1-r} = 4.48$$
$$a\left(1-r^2\right) = 4.48(1-r)$$
$$a = \frac{4.48(1-r)}{1-r^2}$$
$$S_4 = \frac{a\left(1-r^4\right)}{1-r} = 5.1968$$
$$a\left(1-r^4\right) = 5.1968(1-r)$$
$$a = \frac{5.1968(1-r)}{1-r^4}$$
$$\frac{4.48(1-r)}{1-r^2} = \frac{5.1968(1-r)}{1-r^4}$$
$$\frac{1}{1-r^2} = \frac{1.16}{1-r^4}$$
$$\frac{1}{1-r^2} = \frac{1.16}{(1-r^2)(1+r^2)}$$
$$1 = \frac{1.16}{(1+r^2)}$$
$$1 + r^2 = 1.16$$
$$r^2 = 0.16$$
$$r = \pm 0.4$$

9 $a = a,\ r = \sqrt{3}$

$$S_{10} = \frac{a\left(\sqrt{3}^{10} - 1\right)}{\sqrt{3}-1}$$
$$= \frac{a\left(243-1\right)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$$
$$= \frac{242a(\sqrt{3}+1)}{3-1}$$
$$= 121a(\sqrt{3}+1)$$

10 First series:

$a = a,\ r = 2$

$$S_4 = \frac{a\left(2^4-1\right)}{2-1}$$
$$S_4 = 15a$$

Second series:

$a = b,\ r = 3$

$$S_4 = \frac{b\left(3^4-1\right)}{3-1}$$
$$S_4 = 40b$$
$$15a = 40b$$
$$a = \frac{8}{3}b$$

11 a $\frac{2k+5}{k}=\frac{k}{k-6}$

$(2k+5)(k-6)=k^2$

$2k^2+7k-30=k^2$

$k^2+7k-30=0$

b $(k+3)(k-10)=0$

$k=-3$ or $k=10$

As $k>0$, $k=10$

c $r=\frac{10}{10-6}=\frac{5}{2}=2.5$

d $S_{10}=\frac{4\left(2.5^{10}-1\right)}{2.5-1}$

$= 25\,429$

Sequences and series 3E

1 a i $r = 0.1$ so the series is convergent as $|r| < 1$.

ii $S_\infty = \frac{1}{1-0.1} = \frac{10}{9}$

b $r = 2$ so the series is not convergent as $|r| \geqslant 1$.

c i $r = -0.5$ so the series is convergent as $|r| < 1$.

ii $S_\infty = \frac{10}{1+0.5} = \frac{20}{3} = 6\frac{2}{3}$

d This is an arithmetic series and so does not converge.

e $r = 1$ so the series is not convergent as $|r| \geqslant 1$.

f i $r = \frac{1}{3}$ so the series is convergent as $|r| < 1$.

ii $S_\infty = \frac{3}{1-\frac{1}{3}} = \frac{9}{2} = 4\frac{1}{2}$

g This is an arithmetic series and so does not converge.

h i $r = 0.9$ so the series is convergent as $|r| < 1$.

ii $S_\infty = \frac{9}{1-0.9} = 90$

2 $a = 10,\ S_\infty = 30$

$\frac{10}{1-r} = 30$

$10 = 30(1 - r)$

$30r = 20$

$r = \frac{2}{3}$

3 $a = -5,\ S_\infty = -3$

$\frac{-5}{1-r} = -3$

$-5 = -3(1 - r)$

$3r = -2$

$r = -\frac{2}{3}$

4 $S_\infty = 60,\ r = \frac{2}{3}$

$\frac{a}{1-\frac{2}{3}} = 60$

$\frac{a}{\frac{1}{3}} = 60$

$a = 20$

5 $S_\infty = 10,\ r = -\frac{1}{3}$

$\frac{a}{1+\frac{1}{3}} = 10$

$\frac{a}{\frac{4}{3}} = 10$

$a = \frac{40}{3} = 13\frac{1}{3}$

6 $$0.\dot{2}\dot{3}\ldots = \frac{23}{100} \underset{\times\frac{1}{100}}{\xrightarrow{}} + \frac{23}{10\,000} \underset{\times\frac{1}{100}}{\xrightarrow{}} + \frac{23}{1\,000\,000} + \ldots$$

This is an infinite geometric series:
$a = \frac{23}{100}$ and $r = \frac{1}{100}$.

Use $S_\infty = \frac{a}{1-r}$.

$$0.\dot{2}\dot{3}\ldots = \frac{\frac{23}{100}}{1-\frac{1}{100}} = \frac{\frac{23}{100}}{\frac{99}{100}}$$
$$= \frac{23}{100} \times \frac{100}{99} = \frac{23}{99}$$

7 $S_3 = 9,\ S_\infty = 8$

$$S_3 = \frac{a(1-r^3)}{1-r} = 9 \quad (\mathbf{1})$$
$$S_\infty = \frac{a}{1-r} = 8 \quad (\mathbf{2})$$
$8(1-r^3) = 9$ (substituting (**2**) into (**1**))
$$1 - r^3 = \frac{9}{8}$$
$$r^3 = -\frac{1}{8}$$
$$r = -\frac{1}{2}$$
$a = 8\left(1+\frac{1}{2}\right)$ (from (**2**))
$a = 12$

8 a $a = 1, r = -2x$
As the series is convergent, $|-2x| < 1$

If $x < 0$ then $2x < 1$, so $x < \frac{1}{2}$;

if $x > 0$ then $-2x < 1$, so $x > -\frac{1}{2}$

Hence, $-\frac{1}{2} < x < \frac{1}{2}$.

b $$S_\infty = \frac{1}{1+2x}$$

9 a $a = 2,\ S_\infty = 16 \times S_3$
$$S_3 = \frac{2(1-r^3)}{1-r}$$
$$16 \times \frac{2(1-r^3)}{1-r} = \frac{2}{1-r}$$
$$32(1-r^3) = 2$$
$$r^3 = \frac{15}{16}$$
$$r = 0.9787$$

b $u_4 = ar^3 = 2 \times 0.9787^3 = 1.875$

10 a $a = 30,\ S_\infty = 240$
$$\frac{30}{1-r} = 240$$
$$\frac{1}{8} = 1 - r$$
$$r = \frac{7}{8}$$

b $u_4 - u_5 = ar^3 - ar^4$
$$= 30\left(\frac{7}{8}\right)^3 - 30\left(\frac{7}{8}\right)^4$$
$$= 2.51$$

c $$S_4 = \frac{30\left(1-\left(\frac{7}{8}\right)^4\right)}{1-\frac{7}{8}}$$
$$= 99.3$$

10 d If $S_n = \dfrac{30\left(1-\left(\frac{7}{8}\right)^n\right)}{1-\frac{7}{8}} = 180$

$$\frac{30\left(1-\left(\frac{7}{8}\right)^n\right)}{\frac{1}{8}} = 180$$
$$1-\left(\frac{7}{8}\right)^n = 0.75$$
$$0.875^n = 0.25$$
$$n = \frac{\log 0.25}{\log 0.875}$$
$$n = 10.38$$
$$n = 11$$

11 a $ar = \dfrac{15}{8}$, $S_\infty = 8$

$$\frac{a}{1-r} = 8$$
$$a = 8(1-r)$$
$$a = \frac{15}{8r}$$
$$\frac{15}{8r} = 8(1-r)$$
$$15 = 64r - 64r^2$$
$$64r^2 - 64r + 15 = 0$$

b $(8r-3)(8r-5) = 0$

$r = \dfrac{3}{8}$ or $r = \dfrac{5}{8}$

c When $r = \dfrac{3}{8}$

$$a = 8\left(1-\frac{3}{8}\right) = 5$$

When $r = \dfrac{5}{8}$

$$a = 8\left(1-\frac{5}{8}\right) = 3$$

d $r = \dfrac{3}{8}$

If $S_n = \dfrac{5\left(1-\left(\frac{3}{8}\right)^n\right)}{1-\frac{3}{8}} = 7.99$

$$\frac{5\left(1-\left(\frac{3}{8}\right)^n\right)}{\frac{5}{8}} = 7.99$$
$$1-0.375^n = 0.99875$$
$$0.375^n = 0.00125$$
$$n = \frac{\log 0.00125}{\log 0.375}$$
$$n = 6.815$$
$$n = 7$$

Challenge

a First series: $a + ar + ar^2 + ar^3 + ...$
Second series: $a^2 + a^2r^2 + a^2r^4 + ...$
Second series has first term a^2 and common ratio r^2 so is a geometric series.

b For the first series: $S_\infty = 7$

$$\frac{a}{1-r} = 7$$
$$a = 7(1-r)$$

For the second series: $S_\infty = 35$

$$\frac{a^2}{1-r^2} = 35$$
$$\frac{a^2}{(1-r)(1+r)} = 35$$
$$\frac{49(1-r)^2}{(1-r)(1+r)} = 35$$
$$49 - 49r = 35 + 35r$$

$14 = 84r$, so $r = \dfrac{1}{6}$

Sequences and series 3F

1 a i $\sum_{r=1}^{5}(3r+1)=4+7+10+13+16$

ii $S_5=50$

b i $\sum_{r=1}^{6}3r^2=3+12+27+48+75+108$

ii $S_6=273$

c i $\sum_{r=1}^{5}\sin(90r°)=1+0+(-1)+0+1$

ii $S_5=1$

d i $\sum_{r=5}^{8}2\left(-\frac{1}{3}\right)^r=-\frac{2}{243}+\frac{2}{729}-\frac{2}{2187}+\frac{2}{6561}$

ii $S_4=-\frac{40}{6561}$

2 a i $2+4+6+8=\sum_{r=1}^{4}2r$

ii $S_4=20$

b i $2+6+18+54+162=\sum_{r=1}^{5}\left(2\times 3^{r-1}\right)$

ii $S_5=242$

c i

$6+4.5+3+1.5+0-1.5=\sum_{r=1}^{6}\left(-\frac{3}{2}r+\frac{15}{2}\right)$

ii $S_6=13.5$

3 a i $7+13+19+\ldots+157=\sum_{r=1}^{n}(6r+1)$

$6n+1=157$
$n=26$

ii $\sum_{r=1}^{26}(6r+1)$

b i

$\frac{1}{3}+\frac{2}{15}+\frac{4}{75}+\ldots+\frac{64}{46\,875}=\sum_{r=1}^{n}\left(\frac{1}{3}\times\left(\frac{2}{5}\right)^{r-1}\right)$

$\frac{1}{3}\times\left(\frac{2}{5}\right)^{n-1}=\frac{64}{46\,875}$

$\left(\frac{2}{5}\right)^{n-1}=\frac{64}{15\,625}$

$n=\frac{\log(0.004096)}{\log(0.4)}+1\;=7$

ii $\sum_{r=1}^{7}\left(\frac{1}{3}\times\left(\frac{2}{5}\right)^{r-1}\right)$

c i $8-1-10-19-\ldots-127=\sum_{r=1}^{n}(17-9r)$

$17-9n=-127$
$n=16$

ii $\sum_{r=1}^{16}(17-9r)$

4 a $\sum_{r=1}^{20}(7-2r)=5+3+1+\ldots-33$

$a=5,\ l=-33,\ n=20$

$S_{20}=\frac{20}{2}(5-33)$

$=-280$

4 b $$\sum_{r=1}^{10} 3\times 4^r = 12+48+192+\ldots+3\,145\,728$$

$a=12$, $r=4$, $n=10$

$$S_{10}=\frac{12(4^{10}-1)}{4-1}$$

$= 4\,194\,300$

c $$\sum_{r=1}^{100}(2r-8) = -6-4-2+\ldots+192$$

$a=-6$, $l=192$, $n=100$

$$S_{100}=\frac{100}{2}(-6+192)$$

$=9300$

d $$\sum_{r=1}^{\infty}7\left(-\frac{1}{3}\right)^r = -\frac{7}{3}+\frac{7}{9}-\frac{7}{27}+\ldots$$

$a=-\frac{7}{3}$, $r=-\frac{1}{3}$

$$S_\infty=\frac{-\frac{7}{3}}{1+\frac{1}{3}}$$

$=-\frac{7}{4}$

5 a $$\sum_{r=9}^{30}\left(5r-\frac{1}{2}\right)=44\frac{1}{2}+49\frac{1}{2}+\ldots+149\frac{1}{2}$$

$a=44\frac{1}{2}$, $l=149\frac{1}{2}$, $n=22$

$$S_{22}=\frac{22}{2}\left(44\frac{1}{2}+149\frac{1}{2}\right)$$

$=2134$

b $$\sum_{r=100}^{200}(3r+4)=304+307+310+\ldots+604$$

$a=304$, $l=604$, $n=101$

$$S_{101}=\frac{101}{2}(304+604)$$

$=45\,854$

c

$$\sum_{r=5}^{100}3\times 0.5^r = 0.09375+0.046875+0.0234375+\ldots$$

$a=0.09375$, $r=0.5$, $n=96$

$$S_{96}=\frac{0.09375(1-0.5^{96})}{1-0.5}$$

$=0.1875$

d $$\sum_{i=5}^{100}1 = 1+1+1+\ldots+1$$

$a=1$, $l=1$, $n=96$

$$S_{96}=\frac{96}{2}(1+1)$$

$=96$

These are the answers to Q6 and Q7 in the 2020 update to the student book. The answers to the original questions are below.

6 $$\sum_{r=1}^{30}(r+2^r)=\sum_{r=1}^{30}r+\sum_{r=1}^{30}2^r$$

$$\sum_{r=1}^{30}r=\frac{30(30+1)}{2}=465$$

$$\sum_{r=1}^{30}2^r = 2+4+8+\ldots$$

$a=2$, $r=2$, $n=30$

$$S_n=\frac{a(r^n-1)}{r-1}$$

$$S_{30}=\frac{2(2^{30}-1)}{2-1}=2\,147\,483\,646$$

So $$\sum_{r=1}^{30}(r+2^r)=\sum_{r=1}^{30}r+\sum_{r=1}^{30}2^r$$

$=465+2\,147\,483\,646$

$=2\,147\,484\,111$

7 $$\sum_{r=1}^{12}(2r-5+3^r)=\sum_{r=1}^{12}(2r-5)+\sum_{r=1}^{12}3^r$$

$$\sum_{r=1}^{12}(2r-5)=(-3)+(-1)+1+\ldots+19$$

$\boldsymbol{a=-3, d=2, n=12}$

$$S_n=\frac{n}{2}(2a+(n-1)d)$$

$$S_{12}=\frac{12}{2}(2\times(-3)+(12-1)(2))=96$$

$$\sum_{r=1}^{12}3^r=3+9+27+\ldots$$

$\boldsymbol{a = 3,\ r = 3,\ n = 12}$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{12} = \frac{3(3^{12} - 1)}{3 - 1} = 797160$$

So

$$\sum_{r=1}^{12}(2r - 5 + 3^r) = \sum_{r=1}^{12}(2r - 5) + \sum_{r=1}^{12}3^r = 96 + 797160 = 797\,256$$

These are the answers to Q6 and Q7 in the original version of the student book.

6 $\displaystyle\sum_{r=1}^{n}2r = 2 + 4 + 6 + \ldots + 2n$

$a = 2,\ l = 2n$

$$S_n = \frac{n}{2}(2 + 2n)$$

$$= n + n^2$$

7 $\displaystyle\sum_{r=1}^{n}2r = n + n^2$

$$\sum_{r=1}^{n}(2r - 1) = 1 + 3 + 5 + \ldots + (2n - 1)$$

$a = 1,\ l = 2n - 1$

$$S_n = \frac{n}{2}(1 + 2n - 1)$$

$$= n^2$$

$$\sum_{r=1}^{n}2r - \sum_{r=1}^{n}(2r - 1) = n + n^2 - n^2 = n$$

8 a $\displaystyle\sum_{r=1}^{k}4(-2)^r = -8 + 16 + \ldots + 4(-2)^k$

$a = -8,\ r = -2$

$$S_k = \frac{-8(1 - (-2)^k)}{1 + 2}$$

$$= \frac{8}{3}\left((-2)^k - 1\right)$$

b $\displaystyle\sum_{r=1}^{k}(100 - 2r) = 98 + 96 + \ldots + (100 - 2k)$

$a = 98,\ l = 100 - 2k$

$$S_k = \frac{k}{2}(98 + 100 - 2k)$$

$$= 99k - k^2$$

8 c $\displaystyle\sum_{r=10}^{k}(7 - 2r) = -13 - 15 - \ldots + (7 - 2k)$

$a = -13,\ l = 7 - 2k$

$$S_{k-9} = \frac{k - 9}{2}(-13 + 7 - 2k)$$

$$= (k - 9)(-3 - k)$$

$$= 6k - k^2 + 27$$

9 $\displaystyle\sum_{r=10}^{\infty}200 \times \left(\frac{1}{4}\right)^r =$

$$\sum_{r=1}^{\infty}200 \times \left(\frac{1}{4}\right)^r - \sum_{r=1}^{r=9}200 \times \left(\frac{1}{4}\right)^r$$

$a = 50,\ r = \frac{1}{4}$

$$S_\infty - S_9 = \frac{50}{1 - \frac{1}{4}} - \frac{50\left(1 - \left(\frac{1}{4}\right)^9\right)}{1 - \frac{1}{4}}$$

$$= \frac{50 - 50\left(1 - \left(\frac{1}{4}\right)^9\right)}{\frac{3}{4}}$$

$$= \frac{200}{3}\left(\frac{1}{4}\right)^9$$

$$= \frac{25}{98\,304}$$

10 a $\displaystyle\sum_{r=1}^{k}(8 + 3r) = 11 + 14 + 17 + \ldots + (8 + 3k)$

$a = 11,\ l = 8 + 3k,\ n = k$

$$S_k = \frac{k}{2}(11 + 8 + 3k)$$

$$\frac{k}{2}(19 + 3k) = 377$$

$$19k + 3k^2 = 754$$

$$3k^2 + 19k - 754 = 0$$

$$(3k + 58)(k - 13) = 0$$

b As $k > 0,\ k = 13$

11 a $\sum_{r=1}^{k} 2\times 3^r = 59\ 046$

$a = 6,\ r = 3$

$$S_k = \frac{6(3^k - 1)}{3-1} = 59\ 046$$

$3(3^k - 1) = 59\ 046$

$3^k = 19\ 683$

$k \log 3 = \log 19\ 683$

$$k = \frac{\log 19\ 683}{\log 3}$$

b $k = 9$

$$\sum_{r=10}^{13} 2\times 3^r = \sum_{r=1}^{13} 2\times 3^r - \sum_{r=1}^{9} 2\times 3^r$$

$$= \frac{6(3^{13}-1)}{3-1} - 59\ 046$$

$= 4\ 782\ 966 - 59\ 046$

$= 4\ 723\ 920$

12 a $r = 3x$

As the series is convergent, $|3x| < 1$

$$|x| < \frac{1}{3}$$

b $S_\infty = \dfrac{1}{1-3x} = 2$

$1 = 2(1-3x)$

$6x = 1$

$$x = \frac{1}{6}$$

Challenge

$$\sum_{r=1}^{10}(a+(r-1)d) = a+(a+d)+\ldots+(a+9d)$$

$$= \frac{10}{2}(a+a+9d)$$

$= 5(2a+9d)$

$$\sum_{r=11}^{14}(a+(r-1)d) = (a+10d)+\ldots+(a+13d)$$

$$= \frac{4}{2}(a+10d+a+13d)$$

$= 2(2a+23d)$

As $\sum_{r=1}^{10}(a+(r-1)d) = \sum_{r=11}^{14}(a+(r-1)d)$

$5(2a+9d) = 2(2a+23d)$

$10a + 45d = 4a + 46d$

$d = 6a$

Sequences and series 3G

1 **a** $u_{n+1} = u_n + 3$, $u_1 = 1$

$$n=1 \Rightarrow u_2 = u_1 + 3 = 1 + 3 = 4$$
$$n=2 \Rightarrow u_3 = u_2 + 3 = 4 + 3 = 7$$
$$n=3 \Rightarrow u_4 = u_3 + 3 = 7 + 3 = 10$$

Terms are 1, 4, 7, 10, …

b $u_{n+1} = u_n - 5$, $u_1 = 9$

$$n=1 \Rightarrow u_2 = u_1 - 5 = 9 - 5 = 4$$
$$n=2 \Rightarrow u_3 = u_2 - 5 = 4 - 5 = -1$$
$$n=3 \Rightarrow u_4 = u_3 - 5 = -1 - 5 = -6$$

Terms are 9, 4, −1, −6, …

c $u_{n+1} = 2u_n$, $u_1 = 3$

$$n=1 \Rightarrow u_2 = 2u_1 = 2\times 3 = 6$$
$$n=2 \Rightarrow u_3 = 2u_2 = 2\times 6 = 12$$
$$n=3 \Rightarrow u_4 = 2u_3 = 2\times 12 = 24$$

Terms are 3, 6, 12, 24, …

d $u_{n+1} = 2u_n + 1$, $u_1 = 2$

$$n=1 \Rightarrow u_2 = 2u_1 + 1 = 2\times 2 + 1 = 5$$
$$n=2 \Rightarrow u_3 = 2u_2 + 1 = 2\times 5 + 1 = 11$$
$$n=3 \Rightarrow u_4 = 2u_3 + 1 = 2\times 11 + 1 = 23$$

Terms are 2, 5, 11, 23, …

e $u_{n+1} = \frac{u_n}{2}$, $u_1 = 10$

$$n=1 \Rightarrow u_2 = \frac{u_1}{2} = \frac{10}{2} = 5$$
$$n=2 \Rightarrow u_3 = \frac{u_2}{2} = \frac{5}{2} = 2.5$$
$$n=3 \Rightarrow u_4 = \frac{u_3}{2} = \frac{2.5}{2} = 1.25$$

Terms are 10, 5, 2.5, 1.25, …

f $u_{n+1} = (u_n)^2 - 1$, $u_1 = 2$

$$n=1 \Rightarrow u_2 = (u_1)^2 - 1 = 2^2 - 1 = 4 - 1 = 3$$
$$n=2 \Rightarrow u_3 = (u_2)^2 - 1 = 3^2 - 1 = 9 - 1 = 8$$
$$n=3 \Rightarrow u_4 = (u_3)^2 - 1 = 8^2 - 1 = 64 - 1 = 63$$

Terms are 2, 3, 8, 63, …

2 **a** $3 \underset{+2}{\rightarrow} 5 \underset{+2}{\rightarrow} 7 \underset{+2}{\rightarrow} 9 \ldots$

$u_{n+1} = u_n + 2$, $u_1 = 3$

b $20 \underset{-3}{\rightarrow} 17 \underset{-3}{\rightarrow} 14 \underset{-3}{\rightarrow} 11 \ldots$

$u_{n+1} = u_n - 3$, $u_1 = 20$

c $1 \underset{\times 2}{\rightarrow} 2 \underset{\times 2}{\rightarrow} 4 \underset{\times 2}{\rightarrow} 8 \ldots$

$u_{n+1} = 2 \times u_n$, $u_1 = 1$

d $100 \underset{\div 4}{\rightarrow} 25 \underset{\div 4}{\rightarrow} 6.25 \underset{\div 4}{\rightarrow} 1.5625 \ldots$

$$u_{n+1} = \frac{u_n}{4}, \quad u_1 = 100$$

2 **e** $1 \xrightarrow[\times(-1)]{} -1 \xrightarrow[\times(-1)]{} 1 \xrightarrow[\times(-1)]{} -1\ldots$

$u_{n+1} = (-1) \times u_n$, $u_1 = 1$

f $3 \xrightarrow[\times 2+1]{} 7 \xrightarrow[\times 2+1]{} 15 \xrightarrow[\times 2+1]{} 31\ldots$

$u_{n+1} = 2u_n + 1$, $u_1 = 3$

g $0 \xrightarrow[0^2+1]{} 1 \xrightarrow[1^2+1]{} 2 \xrightarrow[2^2+1]{} 5 \xrightarrow[5^2+1]{} 26\ldots$

$u_{n+1} = (u_n)^2 + 1$, $u_1 = 0$

h $26 \xrightarrow[+2\div 2]{} 14 \xrightarrow[+2\div 2]{} 8 \xrightarrow[+2\div 2]{} 5 \xrightarrow[+2\div 2]{} 3.5\ldots$

$$u_{n+1} = \frac{u_n + 2}{2}, \quad u_1 = 26$$

3 **a** $u_n = 2n - 1$.

Substituting $n = 1, 2, 3$ and 4 gives

$$u_1 = 1 \xrightarrow[+2]{} u_2 = 3 \xrightarrow[+2]{} u_3 = 5 \xrightarrow[+2]{} u_4 = 7$$

Recurrence formula is
$u_{n+1} = u_n + 2$, $u_1 = 1$.

b $u_n = 3n + 2$. Substituting $n = 1, 2, 3$ and 4 gives

$$u_1 = 5 \xrightarrow[+3]{} u_2 = 8 \xrightarrow[+3]{} u_3 = 11 \xrightarrow[+3]{} u_4 = 14$$

Recurrence formula is
$u_{n+1} = u_n + 3$, $u_1 = 5$.

c $u_n = n + 2$. Substituting $n = 1, 2, 3$ and 4 gives

$$u_1 = 3 \xrightarrow[+1]{} u_2 = 4 \xrightarrow[+1]{} u_3 = 5 \xrightarrow[+1]{} u_4 = 6$$

Recurrence formula is
$u_{n+1} = u_n + 1$, $u_1 = 3$.

d $u_n = \frac{n+1}{2}$. Substituting $n = 1, 2, 3$ and 4 gives

$$u_1 = 1 \xrightarrow[+\frac{1}{2}]{} u_2 = \frac{3}{2} \xrightarrow[+\frac{1}{2}]{} u_3 = 2 \xrightarrow[+\frac{1}{2}]{} u_4 = \frac{5}{2}$$

Recurrence formula is
$u_{n+1} = u_n + \frac{1}{2}$, $u_1 = 1$.

e $u_n = n^2$. Substituting $n = 1, 2, 3$ and 4:

$$u_1 = 1 \xrightarrow[+3]{} u_2 = 4 \xrightarrow[+5]{} u_3 = 9 \xrightarrow[+7]{} u_4 = 16$$

Differences are
$2 \times 1 + 1$, $2 \times 2 + 1$, $2 \times 3 + 1$

$u_{n+1} = u_n + 2n + 1$, $u_1 = 1$.

f $u_n = 3^n - 1$
$u_1 = 3^1 - 1 = 2$
$u_2 = 3^2 - 1 = 8$
$u_3 = 3^3 - 1 = 26$
$u_4 = 3^4 - 1 = 80$
$u_{n+1} = 3u_n + 2$, $u_1 = 2$

4 **a** $u_{n+1} = ku_n + 2$,
$u_1 = 3$
$u_2 = ku_1 + 2$
$= 3k + 2$

b $u_3 = ku_2 + 2$
$= k(3k + 2) + 2$
$= 3k^2 + 2k + 2$

4 c $u_3 = 42$, so $3k^2 + 2k + 2 = 42$
$3k^2 + 2k - 40 = 0$
$(k + 4)(3k - 10) = 0$
So $k = -4$ or $k = \frac{10}{3}$

5 $u_{n+1} = pu_n + q$
$u_1 = 2$
$u_2 = 2p + q = -1$, so $q = -2p - 1$
$u_3 = p(2p + q) + q = 2p^2 + pq + q = 11$
$2p^2 + p(-2p - 1) - 2p - 1 = 11$
$2p^2 - 2p^2 - p - 2p - 1 = 11$
$-3p = 12$
$p = -4$
$q = -2(-4) - 1 = 7$
$p = -4$ and $q = 7$

6 a $x_{n+1} = x_n(p - 3x_n)$
$x_1 = 2$
$x_2 = 2(p - 3 \times 2) = 2p - 12$
$x_3 = (2p - 12)(p - 3(2p - 12))$
$= (2p - 12)(-5p + 36)$
$= -10p^2 + 132p - 432$

b $-10p^2 + 132p - 432 = -288$
$-10p^2 + 132p - 144 = 0$
$5p^2 - 66p + 72 = 0$
$(5p - 6)(p - 12) = 0$
$p = \frac{6}{5}$ or $p = 12$
As p is an integer, $p = 12$

c $x_4 = -288(12 - 3(-288)) = -252\,288$

7 a $a_1 = k$
$a_2 = 4k + 5$
$a_3 = 4(4k + 5) + 5 = 16k + 25$

b $a_4 = 4(16k + 25) + 5 = 64k + 105$
$$\sum_{r=1}^{4} a_r = k + 4k + 5 + 16k + 25 + 64k + 105$$
$= 85k + 135$
$= 5(17k + 27)$
Therefore, $\sum_{r=1}^{4} a_r$ is a multiple of 5.

Sequences and series 3H

1 a i The sequence is increasing.

b i The sequence is decreasing.

c i The sequence is increasing.

d i The sequence is periodic.

ii Order 2

2 a i $u_n = 20 - 3n$
$u_1 = 20 - 3(1) = 17$
$u_2 = 20 - 3(2) = 14$
$u_3 = 20 - 3(3) = 11$
$u_4 = 20 - 3(4) = 8$
$u_5 = 20 - 3(5) = 5$

ii The sequence is decreasing.

b i $u_n = 2^{n-1}$
$u_1 = 2^{1-1} = 1$
$u_2 = 2^{2-1} = 2$
$u_3 = 2^{3-1} = 4$
$u_4 = 2^{4-1} = 8$
$u_5 = 2^{5-1} = 16$

ii The sequence is increasing.

c i $u_n = \cos(180n°)$
$u_1 = \cos(180(1)°) = -1$
$u_2 = \cos(180(2)°) = 1$
$u_3 = \cos(180(3)°) = -1$
$u_4 = \cos(180(4)°) = 1$
$u_5 = \cos(180(5)°) = -1$

ii The sequence is periodic.

iii Order 2

d i $u_n = (-1)^n$
$u_1 = (-1)^1 = -1$
$u_2 = (-1)^2 = 1$
$u_3 = (-1)^3 = -1$
$u_4 = (-1)^4 = 1$
$u_5 = (-1)^5 = -1$

ii The sequence is periodic.

iii Order 2

e i $u_{n+1} = u_n - 5$
$u_1 = 20$
$u_2 = 20 - 5 = 15$
$u_3 = 15 - 5 = 10$
$u_4 = 10 - 5 = 5$
$u_5 = 5 - 5 = 0$

ii The sequence is decreasing.

f i $u_{n+1} = 5 - u_n$
$u_1 = 20$
$u_2 = 5 - 20 = -15$
$u_3 = 5 + 15 = 20$
$u_4 = 5 - 20 = -15$
$u_5 = 5 - 5 = 20$

ii The sequence is periodic.

iii Order 2

g i $u_{n+1} = \frac{2}{3}u_n$

$u_1 = k$

$u_2 = \frac{2k}{3}$

$u_3 = \frac{2}{3}\left(\frac{2k}{3}\right) = \frac{4k}{9}$

$u_4 = \frac{2}{3}\left(\frac{4k}{9}\right) = \frac{8k}{27}$

$u_5 = \frac{2}{3}\left(\frac{8k}{27}\right) = \frac{16k}{81}$

2 g ii The sequence is dependent on the value of k.

3 $u_{n+1} = ku_n$

$u_1 = 5$

$u_2 = 5k$

$u_3 = 5k^2$

If $k \geqslant 1$ the sequence is increasing.

If $k \leqslant 0$ the sequence is periodic.

If $0 < k < 1$ the sequence is decreasing.

4 $u_{n+1} = pu_n + 10$

$u_1 = 5$

$u_2 = 5p + 10$

$u_3 = p(5p + 10) + 10$

As the sequence is periodic with order 2,

$p(5p + 10) + 10 = 5$

$5p^2 + 10p + 5 = 0$

$p^2 + 2p + 1 = 0$

$(p + 1)^2 = 0$

$p = -1$

5 a $a_n = \cos(90n°)$

$a_1 = \cos(90(1)°) = 0$

$a_2 = \cos(90(2)°) = -1$

$a_3 = \cos(90(3)°) = 0$

$a_4 = \cos(90(4)°) = 1$

$a_5 = \cos(90(5)°) = 0$

$a_6 = \cos(90(6)°) = -1$

Order 4

b $\sum_{r=1}^{444} a_r = 111(0 - 1 + 0 + 1) = 0$

Challenge

a $u_{n+2} = \dfrac{1 + u_{n+1}}{u_n}$

$u_1 = a$

$u_2 = b$

$$u_3 = \frac{1+b}{a}$$

$$u_4 = \frac{1 + \frac{1+b}{a}}{b} = \frac{a+b+1}{ab}$$

$$u_5 = \frac{1 + \frac{a+b+1}{ab}}{\frac{1+b}{a}} = \frac{ab+a+b+1}{b(1+b)}$$

$$= \frac{a(b+1)+b+1}{b(1+b)} = \frac{a+1}{b}$$

$$u_6 = \frac{1 + \frac{a+1}{b}}{\frac{a+b+1}{ab}} = \frac{a+b+1}{b} \times \frac{ab}{a+b+1} = a$$

$$u_7 = \frac{1+a}{\frac{a+1}{b}} = (1+a) \times \frac{b}{a+1} = b$$

Therefore, the sequence is periodic and order 5

b When $a = 2$ and $b = 9$

$u_1 = 2$

$u_2 = 9$

$$u_3 = \frac{1+9}{2} = 5$$

$$u_4 = \frac{2+9+1}{2 \times 9} = \frac{2}{3}$$

$$u_5 = \frac{2+1}{9} = \frac{1}{3}$$

$$\sum_{r=1}^{5} u_r = 2 + 9 + 5 + \frac{2}{3} + \frac{1}{3} = 17$$

Series is periodic so $\sum_{r=1}^{5} u_r = \sum_{r=6}^{10} u_r = \sum_{r=11}^{15} u_r$ and so on.

c So $\sum_{r=1}^{100} u_r = 20 \times \sum_{r=1}^{5} u_r = 20 \times 17 = 340$

Sequences and series 3I

1 **a** Initial amount = £4000
(start of month 1)

Start of month 2 = £(4000 + 200)

Start of month 3 = £(4000 + 200 + 200)
= £(4000 + 2 × 200)

Start of month 10 = £(4000 + 9 ×200)
= £(4000 + 1800)
= £5800

b Start of *m*th month
= £(4000 + (*m* − 1) × 200)
= £(4000 + 200*m* − 200)
= £(3800 + 200*m*)

2

$$\begin{array}{ccccccc} 20\,000 & + & 20\,500 & + & 21\,000 & + & 21\,500+\ldots \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{Year 1} & \underset{\text{1st increment}}{\rightarrow} & \text{Year 2} & \underset{\text{2nd increment}}{\rightarrow} & \text{Year 3} & \underset{\text{3rd increment}}{\rightarrow} & \text{Year 4} \end{array}$$

Carol will reach her maximum salary after

$$\frac{25\,000-20\,000}{500}=10 \text{ increments}$$

This will be after 11 years.

a Total amount after 10 years

$$=\underbrace{20\,000\ +\ 20\,500\ +\ 21\,000\ +\ \ldots}_{}$$

This is an arithmetic series with
a = 20 000, *d* = 500 and *n* = 10. Use
$S_n=\frac{n}{2}\left(2a+(n-1)d\right)$.

$$=\frac{10}{2}\left(40\,000+9\times500\right)$$

$$=5\times44\,500$$

$$=£222\,500$$

b From year 11 to year 15 she will continue to earn £25 000.

Total in this time = 5 × 25 000
= £125 000.

Total amount in the first 15 years is

£222 500 + £125 000 = £347 500

c It is unlikely her salary will rise by the same amount each year.

3 Amount saved by James

$$=\underbrace{1\ +\ 2\ +\ 3\ +\ \ldots\ +\ 42}_{}$$

This is an arithmetic series with *a* = 1, *d* = 1, *n* = 42 and *L* = 42.

a Use $S_n=\frac{n}{2}\left(a+L\right)$

$$=\frac{42}{2}\left(1+42\right)$$

$$=21\times43$$

$$=903\text{p}$$

$$=£9.03$$

b To save £100 we need

$$\underbrace{1+2+3+\ldots}_{\text{Sum to } n \text{ terms}}=10\,000$$

$$\frac{n}{2}\left(2\times1+\left(n-1\right)\times1\right)=10\,000$$

$$\frac{n}{2}\left(n+1\right)=10\,000$$

$$n\left(n+1\right)=20\,000$$

$$n^2+n-20\,000=0$$

$$n=\frac{-1\pm\sqrt{\left(1\right)^2-4\times1\times\left(-20\,000\right)}}{2}$$

$$n=140.9 \text{ or } -141.9$$

It takes James 141 days to save £100.

4 A growth of 10% a year gives a multiplication factor of 1.1.

a After 1 year number is $200 \times 1.1 = 220$

b After 2 years number is $200 \times 1.1^2 = 242$

c After 3 years number is

$200 \times 1.1^3 = 266.2 = 266$
(to nearest whole number)

d After 10 years number is

$200 \times 1.1^{10} = 518.748\ldots = 519$
(to nearest whole number)

5 Let maximum speed in bottom gear be a km h^{-1}

This gives maximum speeds in each successive gear **of** ar, ar^2, ar^3, where r is the common ratio.

We are given

$a = 40$ **(1)**

$ar^3 = 120$ **(2)**

Substitute (**1**) into (**2**):

$$40r^3 = 120 \quad (\div 40)$$
$$r^3 = 3$$
$$r = \sqrt[3]{3}$$
$$r = 1.442\ldots \ (3\text{ d.p.})$$

Maximum speed in 2nd gear is
$ar = 40 \times 1.442\ldots = 57.7$ km h^{-1}

Maximum speed in 3rd gear is
$ar^2 = 40 \times (1.442\ldots)^2 = 83.2$ km h^{-1}

6 a $r = 0.85$
$a \times 0.85^3 = 11\,054.25$
$a = £18\,000$

b $18\,000 \times 0.85^n > 5000$

$$0.85^n > \frac{5}{18}$$
$$n > \frac{\log\left(\frac{5}{18}\right)}{\log(0.85)}$$
$$n > 7.88$$

7 a Total commission

$$= \underbrace{10 + 20 + 30 + \ldots + 520}$$

Arithmetic series with $a = 10$, $d = 10$, $n = 52$.

$$= \frac{52}{2}\left(2\times 10 + (52-1)\times 10\right) \text{ using}$$
$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$
$$= 26(20 + 51\times 10)$$
$$= 26(20 + 510)$$
$$= 26 \times 530$$
$$= £13\,780$$

b Commission = policies for year 1 + policies for 2nd week of year 2
= 520 + 22 = £542

c Total commission for year 2

= Commission for year 1 policies + Commission for year 2 policies

$= 520 \times 52 + (11 + 22 + 33 + \ldots 52 \times 11)$

Use $S_n = \frac{n}{2} = \left(2a + (n-1)d\right)$
with $n = 52,\ a = 11,\ d = 11$

$$= 27\,040 + \frac{52}{2}\left(2\times 11 + (52-1)\times 11\right)$$
$$= 27\,040 + 26\times(22 + 51\times 11)$$
$$= 27\,040 + £\,15158$$
$$= £\,42198$$

8 **a** Cost of drilling to 500 m

$$= \underset{\substack{\text{1st}\\ 50\,\text{m}}}{\underset{\uparrow}{500}} + \underset{\substack{\text{2nd}\\ 50\,\text{m}}}{\underset{\uparrow}{640}} + \underset{\substack{\text{3rd}\\ 50\,\text{m}}}{\underset{\uparrow}{780}} + \ldots$$

There would be 10 terms because there are 10 lots of 50 m in 500 m.

Arithmetic series with $a = 500$, $d = 140$ and $n = 10$.

$$\text{Using } S_n = \frac{n}{2}\left(2a+(n-1)d\right)$$
$$= \frac{10}{2}\left(2\times500+(10-1)\times140\right)$$
$$= 5\left(1000+9\times140\right)$$
$$= 5\times2260$$
$$= £11300$$

b This time we are given $S = 76\,000$. The first term will still be 500 and d remains 140.

Use $S = \frac{n}{2}\left(2a+(n-1)d\right)$ with $S = 76\,000$, $a = 500$, $d = 140$, and solve for n.

$$76000 = \frac{n}{2}\left(2\times500+(n-1)\times140\right)$$
$$76000 = \frac{n}{2}\left(1000+140(n-1)\right)$$
$$76000 = n\left(500+70(n-1)\right)$$
$$76000 = n\left(500+70n-70\right)$$
$$76000 = n\left(70n+430n\right)\ \left(\text{multiply out}\right)$$
$$76000 = 70n^2+430n\ \left(\div10\right)$$
$$7600 = 7n^2+43n$$

$$0 = 7n^2+43n-7600$$
$$n = \frac{-43\pm\sqrt{(43)^2-4\times7\times(-7600)}}{2\times7}$$
$$n = 30.02,\ \left(-36.16\right)$$

Only accept the positive answer, so there are 30 terms (to the nearest term).

So the greatest depth that can be drilled is 30 × 50 = 1500 m (to the nearest 50 m).

9 **a** 1st year = 500

2nd year = 550 = 500 + 1 × 50

3rd year = 600 = 500 + 2 × 50

$$\vdots$$

40th year = 500 + 39 × 50 = £2450

b Total amount paid in

$$= \underbrace{£500 + £550 + £600 + \ldots + £2450}$$

This is an arithmetic series with $a = 500$, $d = 50$, $L = 2450$ and $n = 40$.

$$S_n = \frac{n}{2}\left(a+L\right)$$
$$S_{40} = \frac{40}{2}\left(500+2450\right)$$
$$= 20\times2950$$
$$= £59000$$

c Brian's amount

$$= \underbrace{890 + \left(890+d\right) + \left(890+2d\right) + \ldots}_{40\,\text{years}}$$

Use $S_n = \frac{n}{2}\left(2a+(n-1)d\right)$ with $n = 40$, $a = 890$ and d.

$$S_{40} = \frac{40}{2}\left(2\times890+(40-1)d\right)$$
$$= 20\left(1780+39d\right)$$

Use the fact that

Brian's saving = Anne's savings

$$20\left(1780+39d\right) = 59000 \quad \left(\div20\right)$$
$$1780+39d = 2950 \quad \left(-1780\right)$$
$$39d = 1170 \quad \left(\div39\right)$$
$$d = 30$$

10 If the number of people infected increases by 4% the multiplication factor is 1.04.

After n days $100 \times (1.04)^n$ people will be infected.

If 1000 people are infected

$$\begin{aligned} 100\times(1.04)^n &= 1000 \\ (1.04)^n &= 10 \\ \log(1.04)^n &= \log 10 \\ n\log(1.04) &= 1 \\ n &= \frac{1}{\log(1.04)} \\ n &= 58.708\ldots \end{aligned}$$

It would take 59 days.

11 If the increase is 3.5% per annum the multiplication factor is 1.035.

Therefore after n years I will have £$A \times (1.035)^n$.

If the money is doubled it will equal $2A$, therefore

$$\begin{aligned} A\times(1.035)^n &= 2A \\ (1.035)^n &= 2 \\ \log(1.035)^n &= \log 2 \\ n\log(1.035) &= \log 2 \\ n &= \frac{\log 2}{\log(1.035)} = 20.14879\ldots \end{aligned}$$

My money will double after 20.15 years.

12 The reduction is 6% which gives a multiplication factor of 0.94.

Let the number of fish now be F.

After n years there will be $F \times (0.94)^n$.

When their number is halved the number will be $\frac{1}{2}F$.

Set these equal to each other:

$$\begin{aligned} F\times(0.94)^n &= \frac{1}{2}F \\ (0.94)^n &= \frac{1}{2} \\ \log(0.94)^n &= \log\left(\frac{1}{2}\right) \\ n\log(0.94) &= \log\left(\frac{1}{2}\right) \\ n &= \frac{\log\left(\frac{1}{2}\right)}{\log(0.94)} \\ n &= 11.2 \end{aligned}$$

The fish stocks will halve in 11.2 years.

13 No. grains $= \underbrace{1 + 2 + 4 + 8 + \ldots}_{\text{64 terms}}$

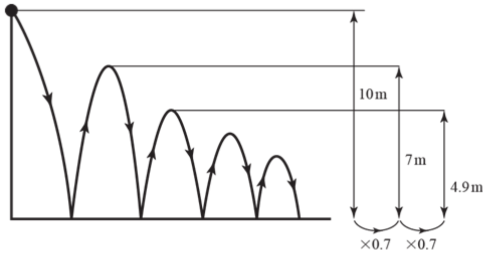
This is a geometric series with $a = 1,\ r = 2$ and $n = 64$.

As $|r| > 1$ use $S_n = \frac{a(r^n - 1)}{r - 1}$.

$$\text{Number of grains} = \frac{1(2^{64}-1)}{2-1} = 2^{64} - 1$$

$$= 1.84\times10^{19}$$

14 a



After the 1st bounce it bounces to 7 cm

After the 2nd bounce it bounces to 4.9 cm (× 0.7)

After the 3rd bounce it bounces to 3.43 cm (× 0.7)

After the 4th bounce it bounces to 2.401 cm (× 0.7)

14 b Total distance travelled

$$= \underbrace{10}_{\substack{\text{1st}\\\text{bounce}}} + 7 + \underbrace{7}_{\substack{\text{2nd}\\\text{bounce}}} + 4.9 + \underbrace{4.9}_{\substack{\text{3rd}\\\text{bounce}}} + \ldots$$

$$= 2\times\underbrace{(10 + 7 + 4.9 + \ldots)}_{\substack{\text{6 terms}\\ a=10,\ r=0.7,\ n=6}} - 10$$

$$= 2\times\frac{10\left(1-0.7^6\right)}{1-0.7} - 10$$

$$= 48.8234\,\text{m}$$

15 a $a = 10$, $r = 1.1$

$$S_n = \frac{10(1.1^n - 1)}{1.1 - 1} = 1000$$

$$1.1^n - 1 = 10$$

$$1.1^n = 11$$

$$n = \frac{\log 11}{\log 1.1}$$

$= 25.16$

So 26 days

b On the 25th day:

$ar^{24} = 10 \times 1.1^{24} = 98.5$ miles

16 Jan. 1st, year 1 = £500

Dec. 31st, year 1 = 500×1.035

Jan. 1st, year 2 = $500 \times 1.035 + 500$

Dec. 31st, year 2

$= (500 \times 1.035 + 500) \times 1.035$

$= 500 \times 1.035^2 + 500 \times 1.035$

$\vdots$

Dec. 31st, year n

$= 500 \times 1.035^n + \ldots + 500 \times 1.035^2$
$+ 500 \times 1.035$

$$= 500\times\underbrace{\left(1.035^n + \ldots + 1.035^2 + 1.035\right)}_{}$$

A geometric series with $a = 1.035$, $r = 1.035$ and n.

Use $S_n = \dfrac{a\left(r^n - 1\right)}{r-1}$.

Dec. 31st year $n = 500\times\dfrac{1.035\left(1.035^n - 1\right)}{1.035 - 1}$

Set this equal to £20 000.

$$20\,000 = 500\times\frac{1.035\left(1.035^n - 1\right)}{1.035 - 1}$$

$$\left(1.035^n - 1\right) = \frac{20\,000\times(1.035 - 1)}{500\times 1.035}$$

$$1.035^n - 1 = 1.3526570\ldots$$

$$1.035^n = 2.3526570\ldots$$

$$\log(1.035^n) = \log 2.3526570\ldots$$

$$n\log(1.035) = \log 2.3526570\ldots$$

$$n = \frac{\log 2.3526570\ldots}{\log 1.035}$$

$$n = 24.9 \text{ years (3 s.f.)}$$

It takes Alan 25 years to save £20 000.

Sequences and series Mixed exercise 3

1 **a** Let a = first term and r = common ratio.

3rd term = 27 ⇒ $ar^2 = 27$ (**1**)

6th term = 8 ⇒ $ar^5 = 8$ (**2**)

Equation (**2**) ÷ Equation (**1**):

$$\frac{\cancel{a}r^5}{\cancel{a}r^2} = \frac{8}{27} \quad \left(\frac{r^5}{r^2} = r^{5-2}\right)$$

$$r^3 = \frac{8}{27}$$

$$r = \sqrt[3]{\frac{8}{27}} = \frac{2}{3}$$

The common ratio is $\frac{2}{3}$.

b Substitute $r = \frac{2}{3}$ back into Equation (**1**):

$$a \times \left(\frac{2}{3}\right)^2 = 27$$

$$a \times \frac{4}{9} = 27$$

$$a = \frac{27 \times 9}{4}$$

$$a = 60.75$$

The first term is 60.75.

c Sum to infinity $= \frac{a}{1-r}$

$$\Rightarrow S_\infty = \frac{60.75}{1-\frac{2}{3}} = \frac{60.75}{\frac{1}{3}} = 182.25$$

Sum to infinity is 182.25.

d Sum to ten terms $\frac{a\left(1-r^{10}\right)}{1-r}$

So

$$S_{10} = \frac{60.75\left(1-\left(\frac{2}{3}\right)^{10}\right)}{\left(1-\frac{2}{3}\right)} = \frac{60.75\left(1-\left(\frac{2}{3}\right)^{10}\right)}{\frac{1}{3}}$$

$$= 179.0895\ldots$$

Difference between S_{10} and S_∞ = 182.25 − 179.0895 = 3.16 (3 s.f.)

2 **a** 2nd term is 80 ⇒ $ar^{2-1} = 80$

$ar = 80$ (**1**)

5th term is 5.12 ⇒ $ar^{5-1} = 5.12$

$ar^4 = 5.12$ (**2**)

Equation (**2**) ÷ Equation (**1**):

$$\frac{\cancel{a}r^4}{\cancel{a}r} = \frac{5.12}{80}$$

$$r^3 = 0.064\left(\sqrt[3]{\ }\right)$$

$$r = 0.4$$

Hence common ratio = 0.4.

b Substitute $r = 0.4$ into Equation (**1**):

$$a \times 0.4 = 80\ (\div 0.4)$$

$$a = 200$$

The first term in the series is 200.

c $S_\infty = \frac{a}{1-r} = \frac{200}{1-0.4} = \frac{200}{0.6} = 333\frac{1}{3}$

2 d Sum to n terms $=\dfrac{a(1-r^n)}{1-r}$

So $S_{14}=\dfrac{200(1-0.4^{14})}{(1-0.4)}=333.3324385$

Required difference

$$S_{14}-S_\infty=333.3324385-333\frac{1}{3}$$
$$=0.0008947=8.95\times10^{-4}\,(3\text{ s.f.})$$

3 a $u_n=95\left(\dfrac{4}{5}\right)^n$

Replace n with 1 ⇒ $u_1=95\left(\dfrac{4}{5}\right)^1=76$

Replace n with 2 ⇒ $u_2=95\left(\dfrac{4}{5}\right)^2=60.8$

b Replace n with 21 ⇒

$u_{21}=95\left(\dfrac{4}{5}\right)^{21}=0.876\ (3\text{ s.f.})$

c $$\sum_{n=1}^{15}u_n=\underbrace{76+60.8+\ldots+95\left(\frac{4}{5}\right)^{15}}_{15\text{ terms}}$$

A geometric series with a = 76, $r=\dfrac{4}{5}$.

Use $S_n=\dfrac{a(1-r^n)}{1-r}$

$$\sum_{n=1}^{15}u_n=\frac{76\left(1-\left(\frac{4}{5}\right)^{15}\right)}{1-\frac{4}{5}}=\frac{76\left(1-\left(\frac{4}{5}\right)^{15}\right)}{\frac{1}{5}}$$

$\left(\div\dfrac{1}{5}\text{ is equivalent to}\times5\right)$

$$\sum_{n=1}^{15}u_n=76\times5\times\left(1-\left(\frac{4}{5}\right)^{15}\right)$$
$$=366.63=367\ (\text{to }3\text{ s.f.})$$

d $$S_\infty=\frac{a}{1-r}=\frac{76}{1-\frac{4}{5}}=\frac{76}{\frac{1}{5}}=76\times5=380$$

Sum to infinity is 380.

4 a $u_n=3\left(\dfrac{2}{3}\right)^n-1$

Replace n with 1 ⇒

$u_1=3\times\left(\dfrac{2}{3}\right)^1-1=2-1=1$

Replace n with 2 ⇒

$u_2=3\times\left(\dfrac{2}{3}\right)^2-1=3\times\dfrac{4}{9}-1=\dfrac{1}{3}$

Replace n with 3 ⇒

$u_3=3\times\left(\dfrac{2}{3}\right)^3-1=3\times\dfrac{8}{27}-1=-\dfrac{1}{9}$

4 b

$$\sum_{n=1}^{15} u_n = \left(3\times\left(\frac{2}{3}\right)-1\right)+\left(3\times\left(\frac{2}{3}\right)^2-1\right)$$

$$+\left(3\times\left(\frac{2}{3}\right)^3-1\right)+\ldots+\left(3\times\left(\frac{2}{3}\right)^{15}-1\right)$$

$$=\underbrace{3\times\left(\frac{2}{3}\right)+3\times\left(\frac{2}{3}\right)^2+3\times\left(\frac{2}{3}\right)^3+\ldots+3\times\left(\frac{2}{3}\right)^{15}}_{\text{a geometric series with 15 terms}}$$

$$\underbrace{-1-1-1-\ldots-1}_{\text{15 times}}$$

where $a=3\times\frac{2}{3}=2$ and $r=\frac{2}{3}$

Use $S_n=\frac{a(1-r^n)}{1-r}$

$$\sum_{n=1}^{15} u_n = \frac{2\left(1-\left(\frac{2}{3}\right)^{15}\right)}{1-\frac{2}{3}}-15=5.986\ldots-15$$

$$=-9.0137\ldots=-9.014\,(4\text{ s.f.})$$

c

$$u_{n+1}=3\times\left(\frac{2}{3}\right)^{n+1}-1$$

$$=3\times\frac{2}{3}\times\left(\frac{2}{3}\right)^{n}-1$$

$$=2\left(\frac{2}{3}\right)^{n}-1=\frac{2u_n-1}{2}$$

5 a Let a = first term and r = the common ratio of the series.

We are given

3rd term = 6.4 ⇒ $ar^2 = 6.4$ **(1)**

4th term = 5.12 ⇒ $ar^3 = 5.12$ **(2)**

Equation **(2)** ÷ Equation **(1)**:

$$\frac{\cancel{a}r^3}{\cancel{a}r^2}=\frac{5.12}{6.4}$$

$$r=0.8$$

The common ratio is 0.8.

b Substitute r = 0.8 into Equation (**1**):

$$a\times 0.8^2 = 6.4$$

$$a=\frac{6.4}{0.8^2}$$

$$a = 10$$

The first term is 10.

c Use $S_\infty=\frac{a}{1-r}$ with a = 10 and r = 0.8.

$$S_\infty=\frac{10}{1-0.8}=\frac{10}{0.2}=50$$

Sum to infinity is 50.

d $$S_{25}=\frac{a(1-r^{25})}{1-r}=\frac{10(1-0.8^{25})}{1-0.8}$$

$$=49.8111\ldots$$

$$S_\infty - S_{25} = 50 - 49.8111\ldots$$

$$= 0.189 \text{ (3 s.f.)}$$

6 a $u_5=20\,000\times0.85^5=£8874.11$

b $20\,000\times0.85^n<4000$

$0.85^n<0.2$

$n>\frac{\log 0.2}{\log 0.85}$

$n>9.9$

So the value will be less than £4000 after 9.9 years.

7 **a** $$\frac{p(2q+2)}{p(3q+1)}=\frac{p(2q-1)}{p(2q+2)}$$

$$(2q+2)^2=(2q-1)(3q+1)$$

$$4q^2+8q+4=6q^2-q-1$$

$$2q^2-9q-5=0$$

$$(q-5)(2q+1)=0$$

$$q=5 \text{ or } q=-\frac{1}{2}$$

b $q=5$, $S_\infty=896$, a = 16p, r = 0.75

$$\frac{16p}{1-0.75}=896$$

$$p=14$$

a = 224

$$S_{12}=\frac{224(1-0.75^{12})}{1-0.75}$$

= 867.62

8 **a** $S = a + (a + d) + (a + 2d) + \ldots + (a + (n - 2)d) + (a+(n - 1)d)$

Turning series around:

$S = (a + (n - 1)d) + (a + (n - 2)d) + \ldots (a + d) + a$

Adding the two sums:

$2S = (2a + (n - 1)d) + (2a + (n - 1)d) + \ldots (2a + (n - 1)d) + (2a + (n - 1)d)$

There are n lots of $(2a + (n - 1)d)$:

$2S = n \times (2a + (n - 1)d)$

$$(\div 2):\quad S=\frac{n}{2}\left(2a+(n-1)d\right)$$

b The first 100 natural numbers are 1, 2, 3, . . . 100.

We need to find
$S = 1 + 2 + 3 + \ldots 99 + 100$.

This series is arithmetic with a = 1, d = 1, n = 100.

Using $S=\frac{n}{2}\left(2a+(n-1)d\right)$ with

a =1, d = 1 and n = 100 gives

$$S=\frac{100}{2}\left(2\times1+(100-1)\times1\right)$$

$$=\frac{100}{2}\left(2+99\times1\right)$$

$$=50\times101=5050$$

9 $$\sum_{r=1}^{n}(4r-3)=(4\times1-3)+(4\times2-3)+(4\times3-3)+\ldots+(4\times n-3)$$

$$=\underbrace{1+5+9+\ldots+(4n-3)}$$

Arithmetic series with a = 1, d = 4.

Using $S_n=\frac{n}{2}\left(2a+(n-1)d\right)$ with a = 1, d = 4 gives

$$S_n=\frac{n}{2}\left(2\times1+(n-1)\times4\right)=\frac{n}{2}\left(2+4n-4\right)$$

$$=\frac{n}{2}\left(4n-2\right)$$

$$=n(2n-1)$$

Solve S_n = 2000:

$$n(2n-1)=2000$$

$$2n^2-n=2000$$

$$2n^2-n-2000=0$$

$$n=\frac{1\pm\sqrt{1-4\times2\times-2000}}{2\times2}=31.87\text{ or }-31.37$$

n must be positive, so n = 31.87.

If the sum has to be greater than 2000 then n = 32.

10 a Let a = first term and d = common difference.

Sum of the first two terms = 47

$\Rightarrow a + a + d = 47$

$\Rightarrow 2a + d = 47$

30th term = −62

Using nth term = $a + (n - 1)d$

$\Rightarrow a + 29d = -62$

(Note: $a + 12d$ is a common error here)

Our two simultaneous equations are

$$\begin{aligned} 2a + d &= 47 && (\mathbf{1}) \\ a + 29d &= -62 && (\mathbf{2}) \\ 2a + 58d &= -124 && (\mathbf{3}) \quad ((\mathbf{2}) \times 2) \\ 57d &= -171 && ((\mathbf{3}) - (\mathbf{1})) \\ d &= -3 && (\div 57) \end{aligned}$$

Substitute $d = -3$ into (**1**):

$2a - 3 = 47 \Rightarrow 2a = 50 \Rightarrow a = 25$

Therefore, first term = 25 and common difference = −3.

b using $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$

$$S_{60} = \frac{60}{2}\left(2a + (60-1)d\right) = 30(2a + 59d)$$

Substituting $a = 25$, $d = -3$ gives

$S_{60} = 30(2 \times 25 + 59 \times (-3))$

$= 30(50 - 177) = 30 \times (-127)$

$= -3810$

11 a Sum of integers divisible by 3 which lie between 1 and 400

$= 3 + 6 + 9 + 12 + \ldots + 399$

This is an arithmetic series with a = 3, d = 3 and L = 399.

Using $L = a + (n-1)d$

$$\begin{aligned} 399 &= 3 + (n-1) \times 3 \\ 399 &= 3 + 3n - 3 \\ 399 &= 3n \\ n &= 133 \end{aligned}$$

Therefore, there are 133 of these integers up to 400.

$$S_n = \frac{n}{2}(a + L) = \frac{133}{2}(3 + 399)$$

$$= \frac{133}{2} \times 402 = 26\,733$$

11 b Sum of integers not divisible by 3

$= 1 + 2 + 4 + 5 + 7 + 8 + 10 + \ldots + 400$

$$= \underbrace{(1 + 2 + 3 + 4 + \ldots + 399 + 400)}_{\text{Arithmetic series with } a=1, L=400, n=400}$$

$$- \underbrace{(3 + 6 + 9 + \ldots + 399)}_{\text{From part } \mathbf{a}\text{, this equals } 26\,733}$$

$$\begin{aligned} S_n &= \frac{400}{2}(1 + 400) \\ &= 200 \times 401 \\ &= 80\,200 \end{aligned}$$

So sum of integers not divisible by 3

= 80 200 – 26 733

= 53 467

12 Let the shortest side be x.

$$S_{10} = \frac{10}{2}(x+2x) = 675$$

$$5(3x) = 675$$

$$15x = 675$$

$$x = 45$$

Length of shortest side is 45 cm.

13

$$\text{Sum} = \underset{\substack{\uparrow\\ \text{1st}}}{4} + \underset{\substack{\uparrow\\ \text{2nd}}}{8} + \underset{\substack{\uparrow\\ \text{3rd}}}{12} + \ldots + \underset{\substack{\uparrow\\ 2n\text{th}}}{8n}$$

This is an arithmetic series with $a = 4$, $d = 4$ and $n = 2n$.

Using $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$:

$$S_{2n} = \frac{\cancel{2}n}{\cancel{2}}\left(2\times 4 + (2n-1)\times 4\right)$$

$$= n(8 + 8n - 4)$$

$$= n(8n + 4)$$

$$= n \times 4(2n+1)$$

$$= 4n(2n+1)$$

14 a Replacing n with 1 ⇒ $U_2 = ku_1 - 4$

$u_1 = 2 \Rightarrow u_2 = 2k - 4$

Replacing n with 2 ⇒ $u_3 = ku_2 - 4$

$u_2 = 2k - 4 \Rightarrow u_3 = k(2k - 4) - 4$

$\Rightarrow u_3 = 2k^2 - 4k - 4$

b Substitute $u_3 = 26$

$\Rightarrow 2k^2 - 4k - 4 = 26$

$\Rightarrow 2k^2 - 4k - 30 = 0$ (÷ 2)

$\Rightarrow k^2 - 2k - 15 = 0$ (factorise)

$\Rightarrow (k - 5)(k + 3) = 0$

$\Rightarrow k = 5, -3$

15 a Use nth term = $a + (n - 1)d$:

5th term is 14 ⇒ $a + 4d = 14$

Use 1st term = a, 2nd term = $a + d$, 3rd term = $a + 2d$:

sum of first three terms = −3

$\Rightarrow a + a + d + a + 2d = -3$

$\Rightarrow 3a + 3d = -3$ (÷3)

$\Rightarrow a + d = -1$

Our simultaneous equations are

$a + 4d = 14$ **(1)**

$a + d = -1$ **(2)**

(1) − **(2)**: $3d = 15$ (÷3)

$d = 5$

Common difference = 5

Substitute $d = 5$ back into **(2)**:

$a + 5 = -1$

$a = -6$

First term = −6

b nth term must be greater than 282

$\Rightarrow a + (n - 1)d > 282$

$\Rightarrow -6 + 5(n - 1) > 282$ (+6)

$\Rightarrow 5(n - 1) > 288$ (÷5)

$\Rightarrow (n - 1) > 57.6$ (+1)

$\Rightarrow n > 58.6$

∴ least value of $n = 59$

16 a We know nth term = $a + (n - 1)d$

4th term is $3k$

$\Rightarrow \quad a + (4 - 1)d = 3k$

$\Rightarrow \quad a + 3d = 3k$

We know $S_n = \frac{n}{2}\left(2a+\left(n-1\right)d\right)$

Sum to 6 terms is $7k + 9$, therefore

$$\frac{6}{2}\left(2a+\left(6-1\right)d\right)=7k+9$$

$3(2a + 5d) = 7k + 9$

$6a + 15d = 7k + 9$

The simultaneous equations are

$a + 3d = 3k$ **(1)**

$6a + 15d = 7k + 9$ **(2)**

(1) × 5: $5a + 15d = 15k$ **(3)**

(2) − **(3)** : $1a = -8k + 9$

$\Rightarrow \quad a = 9 - 8k$

First term is $9 - 8k$.

b Substituting this in (**1**) gives

$$\begin{aligned}9-8k+3d&=3k\\3d&=11k-9\\d&=\frac{11k-9}{3}\end{aligned}$$

Common difference is $\frac{11k-9}{3}$.

c If the 7th term is 12, then

$a + 6d = 12$

Substitute values of a and d:

$$\begin{aligned}-8k+9+6\times\left(\frac{11k-9}{3}\right)&=12\\-8k+9+2\left(11k-9\right)&=12\\-8k+9+22k-18&=12\\14k-9&=12\\14k&=21\\k&=\frac{21}{14}\\&=1.5\end{aligned}$$

d Calculate values of a and d first:

$a = 9 - 8k = 9 - 8 \times 1.5 = 9 - 12 = -3$

$$d=\frac{11k-9}{3}=\frac{11\times1.5-9}{3}=\frac{16.5-9}{3}=\frac{7.5}{3}$$
$$=2.5$$

$$\begin{aligned}S_{20}&=\frac{20}{2}\left(2a+\left(20-1\right)d\right)\\&=10\left(2a+19d\right)\\&=10\left(2\times\left(-3\right)+19\times2.5\right)\\&=10\left(-6+47.5\right)\\&=10\times41.5\\&=415\end{aligned}$$

Sum to 20 terms is 415.

17 a $a_1 = p$

$$a_2=\frac{1}{p}$$

$$a_3=\frac{1}{\frac{1}{p}}=1\times\frac{p}{1}=p$$

$$a_4=\frac{1}{p}$$

So the sequence is periodic with order 2.

17 b $\sum_{r=1}^{1000} a_n = \frac{1000}{2}\left(p+\frac{1}{p}\right)$

$= 500\left(p+\frac{1}{p}\right)$

18 a $a_1 = k$

$a_2 = 2k+6$

$a_3 = 2(2k+6)+6 = 4k+18$

As the sequence is increasing:

$a_1 < a_2 < a_3$

$k < 2k+6 < 4k+18$

$k > -6$

b $a_4 = 2(4k+18)+6 = 8k+42$

c $\sum_{r=1}^{4} a_r = k+2k+6+4k+18+8k+42$

$= 15k+66$

$= 3(5k+22)$

Therefore, $\sum_{r=1}^{4} a_r$ is divisible by 3.

19 a $a = 130$, $S_\infty = 650$

$\frac{130}{1-r} = 650$

$130 = 650 - 650r$

$-520 = -650r$

$r = \frac{4}{5}$

b $u_7 - u_8 = ar^6 - ar^7$

$= 130\left(\frac{4}{5}\right)^6 - 130\left(\frac{4}{5}\right)^7$

$= 6.82$

c $S_7 = \frac{130(1-0.8^7)}{1-0.8}$

= 513.69 (2 d.p.)

d $\frac{130(1-0.8^n)}{1-0.8} > 600$

$\frac{130(1-0.8^n)}{0.2} > 600$

$1-0.8^n > \frac{12}{13}$

$0.8^n < \frac{1}{13}$

$n\log 0.8 < -\log 13$

$n > \frac{-\log 13}{\log 0.8}$

20 a $a = 25\,000$, $r = 1.02$

$ar^2 = 25\,000 \times 1.02^2$

$= 26\,010$

b $25\,000 \times 1.02^n > 50\,000$

$1.02^n > 2$

$n \log 1.02 > \log 2$

$n > \frac{\log 2}{\log 1.02}$

c $n > 35.003$

Initial year was 2012, and n is an integer, so 2048.

d $S_8 = \frac{25\,000(1.02^8 - 1)}{1.02-1} = 214\,574.22$

$= 214\,574$

e People may visit the doctor more frequently than once a year, some may not visit at all. It depends on their state of health.

21 a 3, 5, 7, …

nth term = $(3 + (n - 1)2) = 2n + 1$

b $2k + 1 = 301$

$k = 150$

c i $S_q = \frac{q}{2}(2\times 3+(q-1)2) = p$

$q(q+2) = p$

$q^2 + 2q = p$

$q^2 + 2q - p = 0$

21 c ii $p > 1520$

$$q^2 + 2q = p$$

$$q^2 + 2q > 1520$$

$$q^2 + 2q - 1520 > 0$$

$$q^2 + 2q - 1520 = 0$$

$$(q-38)(q+40) = 0$$

$q = 38$ or -40

As $q^2 + 2q - 1520 > 0$, $q > 38$

minimum numbers of rows is 39.

22 a $ar = -3,\ S_\infty = 6.75$

$$a = -\frac{3}{r}$$

$$\frac{a}{1-r} = 6.75$$

$$-\frac{3}{r} \times \frac{1}{1-r} = 6.75$$

$$\frac{-3}{r-r^2} = 6.75$$

$$6.75r - 6.75r^2 + 3 = 0$$

$$27r^2 - 27r - 12 = 0$$

b $9r^2 - 9r - 4 = 0$

$(3r - 4)(3r + 1) = 0$

$r = \frac{4}{3}$ or $r = -\frac{1}{3}$

As the series is convergent, $|r| < 1$ so

$$r = -\frac{1}{3}$$

22 c $ar = -3$ so $a = 9$

$$S_5 = \frac{9\left(1 - \left(-\frac{1}{3}\right)^5\right)}{1 + \frac{1}{3}}$$

$$= \frac{27}{4}\left(1 - \left(-\frac{1}{3}\right)^5\right)$$

$$= 6.78$$

Challenge

a $u_{n+2} = 5u_{n+1} - 6u_n$

$$= 5[p(3^{n+2}) + q(2^{n+2})] - 6[p(3^n) + q(2^n)]$$

$$= 5\left(p\left(\frac{1}{3}\right)(3^{n+2}) + q\left(\frac{1}{2}\right)(2^{n+2})\right)$$

$$-6\left(p\left(\frac{1}{3}\right)^2(3^{n+2}) + q\left(\frac{1}{2}\right)^2(2^{n+2})\right)$$

$$= \left(\frac{5}{3}p - \frac{6}{9}p\right)(3^{n+2}) + \left(\frac{5}{2}q - \frac{6}{4}q\right)(2^{n+2})$$

$$= p(3^{n+2}) + q(2^{n+2})$$

b $u_1 = 5 = p(3^1) + q(2^1)$

$u_2 = 12 = p(3^2) + q(2^2)$

$5 = 3p + 2q$

$12 = 9p + 4q$

Solving simultaneously:

$10 = 6p + 4q$ **(1)**

$12 = 9p + 4q$ **(2)**

(2) − **(1)**:

$2 = 3p$

$$p = \frac{2}{3}$$

$2q = 5 - 2 = 3$

$$q = \frac{3}{2}$$

Therefore, $u_n = \left(\frac{2}{3}\right)3^n + \left(\frac{3}{2}\right)2^n$

c $u_{100} = \left(\frac{2}{3}\right)3^{100} + \left(\frac{3}{2}\right)2^{100}$

$= 3.436 \times 10^{47}$

So it contains 48 digits.

Binomial expansion 4A

1 a i $(1+x)^{-4}=1+(-4)x+\frac{(-4)(-5)}{2!}x^2+\frac{(-4)(-5)(-6)}{3!}x^3+\ldots$

$=1-4x+10x^2-20x^3+\ldots$

ii $|x|<1$

b i $(1+x)^{-6}=1+(-6)x+\frac{(-6)(-7)}{2!}x^2+\frac{(-6)(-7)(-8)}{3!}x^3+\ldots$

$=1-6x+21x^2-56x^3+\ldots$

ii $|x|<1$

c i $(1+x)^{\frac{1}{2}}=1+\left(\frac{1}{2}\right)x+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}x^2+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3+\ldots$

$=1+\left(\frac{1}{2}\right)x+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}x^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}x^3+\ldots$

$=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\ldots$

ii $|x|<1$

d i $(1+x)^{\frac{5}{3}}=1+\left(\frac{5}{3}\right)x+\frac{\left(\frac{5}{3}\right)\left(\frac{5}{3}-1\right)}{2!}x^2+\frac{\left(\frac{5}{3}\right)\left(\frac{5}{3}-1\right)\left(\frac{5}{3}-2\right)}{3!}x^3+\ldots$

$(1+x)^{\frac{5}{3}}=1+\left(\frac{5}{3}\right)x+\frac{\left(\frac{5}{3}\right)\left(\frac{2}{3}\right)}{2}x^2+\frac{\left(\frac{5}{3}\right)\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)}{6}x^3+\ldots$

$=1+\frac{5}{3}x+\frac{5}{9}x^2-\frac{5}{81}x^3+\ldots$

ii $|x|<1$

1 e i $$(1+x)^{-\frac{1}{4}}=1+\left(-\frac{1}{4}\right)x+\frac{\left(-\frac{1}{4}\right)\left(-\frac{1}{4}-1\right)}{2!}x^2+\frac{\left(-\frac{1}{4}\right)\left(-\frac{1}{4}-1\right)\left(-\frac{1}{4}-2\right)}{3!}x^3+\ldots$$

$$=1+\left(-\frac{1}{4}\right)x+\frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{2}x^2+\frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{9}{4}\right)}{6}x^3+\ldots$$

$$=1-\frac{1}{4}x+\frac{5}{32}x^2-\frac{15}{128}x^3+\ldots$$

ii $|x|<1$

f i $$(1+x)^{-\frac{3}{2}}=1+\left(-\frac{3}{2}\right)x+\frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)}{2!}x^2+\frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)\left(-\frac{3}{2}-2\right)}{3!}x^3+\ldots$$

$$=1+\left(-\frac{3}{2}\right)x+\frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{2}x^2+\frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{6}x^3+\ldots$$

$$=1-\frac{3}{2}x+\frac{15}{8}x^2-\frac{35}{16}x^3+\ldots$$

ii $|x|<1$

2 a i $$(1+3x)^{-3}=1+(-3)(3x)+\frac{(-3)(-4)}{2!}(3x)^2+\frac{(-3)(-4)(-5)}{3!}(3x)^3+\ldots$$

$$=1+(-3)(3x)+\frac{(-3)(-4)}{2}9x^2+\frac{(-3)(-4)(-5)}{6}27x^3+\ldots$$

$$=1-9x+54x^2-270x^3+\ldots$$

ii $|3x|<1$

$|x|<\frac{1}{3}$

b i $$\left(1+\frac{1}{2}x\right)^{-5}=1+(-5)\left(\frac{1}{2}x\right)+\frac{(-5)(-6)}{2!}\left(\frac{1}{2}x\right)^2+\frac{(-5)(-6)(-7)}{3!}\left(\frac{1}{2}x\right)^3+\ldots$$

$$=1+(-5)\left(\frac{1}{2}x\right)+\frac{(-5)(-6)}{2}\frac{1}{4}x^2+\frac{(-5)(-6)(-7)}{6}\frac{1}{8}x^3+\ldots$$

$$=1-\frac{5}{2}x+\frac{15}{4}x^2-\frac{35}{8}x^3+\ldots$$

ii $\left|\frac{1}{2}x\right|<1$

$|x|<2$

2 c i $$(1+2x)^{\frac{3}{4}}=1+\left(\frac{3}{4}\right)(2x)+\frac{\left(\frac{3}{4}\right)\left(\frac{3}{4}-1\right)}{2!}(2x)^2+\frac{\left(\frac{3}{4}\right)\left(\frac{3}{4}-1\right)\left(\frac{3}{4}-2\right)}{3!}(2x)^3+\ldots$$

$$=1+\left(\frac{3}{4}\right)(2x)+\frac{\left(\frac{3}{4}\right)\left(-\frac{1}{4}\right)}{2}4x^2+\frac{\left(\frac{3}{4}\right)\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{6}8x^3+\ldots$$

$$=1+\frac{3}{2}x-\frac{3}{8}x^2+\frac{5}{16}x^3+\ldots$$

ii $|2x|<1$

$|x|<\frac{1}{2}$

d i $$(1-5x)^{\frac{7}{3}}=1+\left(\frac{7}{3}\right)(-5x)+\frac{\left(\frac{7}{3}\right)\left(\frac{7}{3}-1\right)}{2!}(-5x)^2+\frac{\left(\frac{7}{3}\right)\left(\frac{7}{3}-1\right)\left(\frac{7}{3}-2\right)}{3!}(-5x)^3+\ldots$$

$$=1-\left(\frac{7}{3}\right)5x+\frac{\left(\frac{7}{3}\right)\left(\frac{4}{3}\right)}{2}25x^2-\frac{\left(\frac{7}{3}\right)\left(\frac{4}{3}\right)\left(\frac{1}{3}\right)}{6}125x^3+\ldots$$

$$=1-\frac{35}{3}x+\frac{350}{9}x^2-\frac{1750}{81}x^3+\ldots$$

ii $|-5x|<1$

$|x|<\frac{1}{5}$

e i $$(1+6x)^{-\frac{2}{3}}=1+\left(-\frac{2}{3}\right)(6x)+\frac{\left(-\frac{2}{3}\right)\left(-\frac{2}{3}-1\right)}{2!}(6x)^2+\frac{\left(-\frac{2}{3}\right)\left(-\frac{2}{3}-1\right)\left(-\frac{2}{3}-2\right)}{3!}(6x)^3+\ldots$$

$$=1+\left(-\frac{2}{3}\right)(6x)+\frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{2}36x^2+\frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{6}216x^3+\ldots$$

$$=1-4x+20x^2-\frac{320}{3}x^3+\ldots$$

ii $|6x|<1$

$|x|<\frac{1}{6}$

2 f i

$$\left(1-\frac{3}{4}x\right)^{-\frac{5}{3}}=1+\left(-\frac{5}{3}\right)\left(-\frac{3}{4}x\right)+\frac{\left(-\frac{5}{3}\right)\left(-\frac{5}{3}-1\right)}{2!}\left(-\frac{3}{4}x\right)^{2}+\frac{\left(-\frac{5}{3}\right)\left(-\frac{5}{3}-1\right)\left(-\frac{5}{3}-2\right)}{3!}\left(-\frac{3}{4}x\right)^{3}+\ldots$$

$$=1+\left(-\frac{5}{3}\right)\left(-\frac{3}{4}x\right)+\frac{\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{2}\frac{9}{16}x^{2}-\frac{\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)\left(-\frac{11}{3}\right)}{6}\frac{27}{64}x^{3}+\ldots$$

$$=1+\frac{5}{4}x+\frac{5}{4}x^{2}+\frac{55}{48}x^{3}+\ldots$$

ii $\left|-\frac{3}{4}x\right|<1$

$|x|<\frac{4}{3}$

3 a i $\frac{1}{(1+x)^{2}}=(1+x)^{-2}=1+(-2)x+\frac{(-2)(-3)}{2!}x^{2}+\frac{(-2)(-3)(-4)}{3!}x^{3}+\ldots$

$$=1-2x+3x^{2}-4x^{3}+\ldots$$

ii $|x|<1$

b i $\frac{1}{(1+3x)^{4}}=(1+3x)^{-4}=1+(-4)(3x)+\frac{(-4)(-5)}{2!}(3x)^{2}+\frac{(-4)(-5)(-6)}{3!}(3x)^{3}+\ldots$

$$=1+(-4)(3x)+\frac{(-4)(-5)}{2}9x^{2}+\frac{(-4)(-5)(-6)}{6}27x^{3}+\ldots$$

$$=1-12x+90x^{2}-540x^{3}+\ldots$$

ii $|3x|<1$

$|x|<\frac{1}{3}$

3 c i $\sqrt{1-x}=(1-x)^{\frac{1}{2}}=1+\left(\frac{1}{2}\right)(-x)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(-x)^{2}+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(-x)^{3}+\ldots$

$$=1-\left(\frac{1}{2}\right)x+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}x^{2}-\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}x^{3}+\ldots$$

$$=1-\frac{1}{2}x-\frac{1}{8}x^{2}-\frac{1}{16}x^{3}+\ldots$$

ii $|-x|<1$

$|x|<1$

3 d i $\sqrt[3]{1-3x}=(1-3x)^{\frac{1}{3}}=1+\left(\frac{1}{3}\right)(-3x)+\frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)}{2!}(-3x)^2+\frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}(-3x)^3+\ldots$

$=1-\left(\frac{1}{3}\right)3x+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2}9x^2-\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{6}27x^3+\ldots$

$=1-x-x^2-\frac{5}{3}x^3+\ldots$

ii $|-3x|<1$

$|x|<\frac{1}{3}$

e i

$$\frac{1}{\sqrt{1+\frac{1}{2}x}}=\left(1+\frac{1}{2}x\right)^{-\frac{1}{2}}=1+\left(-\frac{1}{2}\right)\left(\frac{1}{2}x\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(\frac{1}{2}x\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}\left(\frac{1}{2}x\right)^3+\ldots$$

$=1+\left(-\frac{1}{2}\right)\left(\frac{1}{2}x\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\frac{1}{4}x^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{6}\frac{1}{8}x^3+\ldots$

$=1-\frac{1}{4}x+\frac{3}{32}x^2-\frac{5}{128}x^3+\ldots$

ii $\left|\frac{1}{2}x\right|<1$

$|x|<2$

3 f i

$$\frac{\sqrt[3]{1-2x}}{1-2x}=(1-2x)^{-\frac{2}{3}}=1+\left(-\frac{2}{3}\right)(-2x)+\frac{\left(-\frac{2}{3}\right)\left(-\frac{2}{3}-1\right)}{2!}(-2x)^2+\frac{\left(-\frac{2}{3}\right)\left(-\frac{2}{3}-1\right)\left(-\frac{2}{3}-2\right)}{3!}(-2x)^3+\ldots$$

$=1-\left(-\frac{2}{3}\right)2x+\frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{2}4x^2-\frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{6}8x^3+\ldots$

$=1+\frac{4}{3}x+\frac{20}{9}x^2+\frac{320}{81}x^3+\ldots$

ii $|-2x|<1$

$|x|<\frac{1}{2}$

4 a $\frac{1+x}{1-2x}=(1+x)(1-2x)^{-1}$ Expand $(1-2x)^{-1}$ using binomial expansion

$$=(1+x)\left(1+(-1)(-2x)+\frac{(-1)(-2)(-2x)^2}{2!}+\frac{(-1)(-2)(-3)(-2x)^3}{3!}+\ldots\right)$$

$=(1+x)(1+2x+4x^2+8x^3+\ldots)$ Multiply out

$=1+2x+4x^2+8x^3+\ldots+x+2x^2+4x^3+8x^4+\ldots$ Add like terms

$=1+3x+6x^2+12x^3+\ldots$

b $(1-2x)^{-1}$ is only valid when $|-2x|<1\Rightarrow|x|<\frac{1}{2}$

So expansion of $\frac{1+x}{1-2x}$ is only valid when $|x|<\frac{1}{2}$

5 a $\mathrm{f}(x)=(1+3x)^{\frac{1}{2}}=1+\left(\frac{1}{2}\right)(3x)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(3x)^2+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(3x)^3+\ldots$

$$=1+\left(\frac{1}{2}\right)(3x)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}9x^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}27x^3+\ldots$$

$$=1+\frac{3x}{2}-\frac{9x^2}{8}+\frac{27x^3}{16}+\ldots$$

b When $x=\frac{1}{100}$, $\mathrm{f}(x)=\sqrt{1+3\left(\frac{1}{100}\right)}=\sqrt{\frac{103}{100}}=\frac{\sqrt{103}}{10}$

c Using the expansion:

$$\mathrm{f}(0.01)\approx1+\frac{3(0.01)}{2}-\frac{9(0.01)^2}{8}+\frac{27(0.01)^3}{16}$$

$$=1.014889188...$$

Percentage error $=\frac{1.014889188-\frac{\sqrt{103}}{10}}{\frac{\sqrt{103}}{10}}\times100=0.0000031\%$

6 a $(1+ax)^{-\frac{1}{2}}=1+\left(-\frac{1}{2}\right)(ax)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(ax)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}(ax)^3+\ldots$

$$=1+\left(-\frac{1}{2}\right)(ax)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}a^2x^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{6}a^3x^3+\ldots$$

$$=1-\frac{ax}{2}+\frac{3a^2x^2}{8}-\frac{5a^3x^3}{16}+\ldots$$

$\frac{3a^2}{8}=24$

$a^2=64$

$a=\pm8$

b When $a = 8,\ -\frac{5(8)^3}{16} = -160$

When $a = -8,\ -\frac{5(-8)^3}{16} = 160$

7 For small values of x, ignore terms in x^3 and higher.

$$\sqrt{\frac{1+x}{1-x}} = (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$$

$$(1+x)^{\frac{1}{2}} = 1+\left(\frac{1}{2}\right)x+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}x^2$$

$$=1+\left(\frac{1}{2}\right)x+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}x^2$$

$$=1+\frac{1}{2}x-\frac{1}{8}x^2$$

$$(1-x)^{-\frac{1}{2}} = 1+\left(-\frac{1}{2}\right)(-x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(-x)^2$$

$$=1+\frac{1}{2}x+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}x^2$$

$$=1+\frac{1}{2}x+\frac{3}{8}x^2$$

$$\sqrt{\frac{1+x}{1-x}} = (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}} = \left(1+\frac{1}{2}x-\frac{1}{8}x^2\right)\left(1+\frac{1}{2}x+\frac{3}{8}x^2\right)$$

$$=1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{1}{2}x+\frac{1}{4}x^2-\frac{1}{8}x^2$$

$$=1+x+\frac{1}{2}x^2$$

8 a $\mathrm{h}(x)=\frac{6}{1+5x}-\frac{4}{1-3x}=6(1+5x)^{-1}-4(1-3x)^{-1}$

$$(1+5x)^{-1}=1+(-1)(5x)+\frac{(-1)(-2)}{2!}(5x)^2+\ldots$$

$$=1+(-1)(5x)+\frac{(-1)(-2)}{2}25x^2+\ldots$$

$$=1-5x+25x^2+\ldots$$

$$(1-3x)^{-1}=1+(-1)(-3x)+\frac{(-1)(-2)}{2!}(-3x)^2+\ldots$$

$$=1+(-1)(-3x)+\frac{(-1)(-2)}{2}9x^2+\ldots$$

$$=1+3x+9x^2+\ldots$$

$$\mathrm{h}(x)=6(1+5x)^{-1}-4(1-3x)^{-1}$$

$$=6(1-5x+25x^2+\ldots)-4(1+3x+9x^2+\ldots)$$

$$=6-30x+150x^2-4-12x-36x^2+\ldots$$

$$=2-42x+114x^2+\ldots$$

b $\mathrm{h}(0.01)=\frac{6}{1+5(0.01)}-\frac{4}{1-3(0.01)}=1.590574374$

$\mathrm{h}(0.01)\ =2-42(0.01)+114(0.01)^2=1.5914$

$$\frac{1.5914-1.590574374}{1.590574374}\times100=0.052\%$$

c The expansion is only valid for $|x|<\frac{1}{5}$. $|0.5|$ is not less than $\frac{1}{5}$

9 a $(1-3x)^{\frac{3}{2}}=1+\left(\frac{3}{2}\right)(-3x)+\frac{\left(\frac{3}{2}\right)\left(\frac{3}{2}-1\right)}{2!}(-3x)^2+\frac{\left(\frac{3}{2}\right)\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{3!}(-3x)^3+\ldots$

$$=1+\left(\frac{3}{2}\right)(-3x)+\frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{2}9x^2-\frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{6}27x^3+\ldots$$

$$=1-\frac{9x}{2}+\frac{27x^2}{8}+\frac{27x^3}{16}+\ldots$$

b When $x=\frac{1}{100}$, $\left(1-3\left(\frac{1}{100}\right)\right)^{\frac{3}{2}}=\left(\frac{97}{100}\right)^{\frac{3}{2}}=\left(\frac{\sqrt{97}}{10}\right)^3=\frac{97\sqrt{97}}{1000}$

c $\left(\sqrt{0.97}\right)^3=1-\frac{9(0.01)}{2}+\frac{27(0.01)^2}{8}+\frac{27(0.01)^3}{16}=0.955339...$

$$\sqrt{0.97}=\sqrt[3]{0.955339...}=0.984886$$

$$\sqrt{97}=\sqrt{0.97\times100}=0.984886\times10=9.84886$$

d To improve the accuracy of this approximation, use more terms from the binomial expansion of $(1-3x)^{\frac{3}{2}}$

Challenge

a $\left(1+\frac{1}{x}\right)^{-\frac{1}{2}}=1+\left(-\frac{1}{2}\right)\left(\frac{1}{x}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(\frac{1}{x}\right)^2+\ldots$

$$=1+\left(-\frac{1}{2}\right)\left(\frac{1}{x}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\frac{1}{x^2}+\ldots$$

$$=1-\frac{1}{2x}+\frac{3}{8x^2}+\ldots$$

b $\mathrm{h}(9)=\left(1+\frac{1}{9}\right)^{-\frac{1}{2}}=\left(\frac{10}{9}\right)^{-\frac{1}{2}}=\left(\frac{9}{10}\right)^{\frac{1}{2}}=\frac{3}{\sqrt{10}}=\frac{3\sqrt{10}}{10}$

c $\mathrm{h}(9)=\frac{3\sqrt{10}}{10}$

$$\frac{10}{3}\mathrm{h}(9)=\sqrt{10}$$

So $\sqrt{10}=\frac{10}{3}\left(1-\frac{1}{2(9)}+\frac{3}{8(9)^2}\right)$

$$=\frac{10}{3}\left(1-\frac{1}{18}+\frac{3}{648}\right)$$

$$=3\frac{52}{324}=3.16$$

Binomial expansion 4B

1 a i $\sqrt{(4+2x)}$ Write in index form.

$=(4+2x)^{\frac{1}{2}}$ Take out a factor of 4

$=\left(4\left(1+\frac{2x}{4}\right)\right)^{\frac{1}{2}}$ Remember to put the 4 to the power $\frac{1}{2}$

$=4^{\frac{1}{2}}\left(1+\frac{x}{2}\right)^{\frac{1}{2}}$ $4^{\frac{1}{2}}=2$

$=2\left(1+\frac{x}{2}\right)^{\frac{1}{2}}$ Use the expansion with $n=\frac{1}{2}$ and $x=\frac{x}{2}$

$$=2\left(1+\left(\frac{1}{2}\right)\left(\frac{x}{2}\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{x}{2}\right)^2}{2!}+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x}{2}\right)^3}{3!}+\ldots\right)$$

$=2\left(1+\frac{x}{4}-\frac{x^2}{32}+\frac{x^3}{128}+\ldots\right)$ Multiply by the 2

$=2+\frac{x}{2}-\frac{x^2}{16}+\frac{x^3}{64}+\ldots$

ii Valid if $\left|\frac{x}{2}\right|<1\Rightarrow|x|<2$

b i $\frac{1}{2+x}$ Write in index form

$=(2+x)^{-1}$ Take out a factor of 2

$=\left(2\left(1+\frac{x}{2}\right)\right)^{-1}$ Remember to put 2 to the power -1

$=2^{-1}\left(1+\frac{x}{2}\right)^{-1}$, $2^{-1}=\frac{1}{2}$. Use the binomial expansion with $n=-1$ and $x=\frac{x}{2}$

$$=\frac{1}{2}\left(1+(-1)\left(\frac{x}{2}\right)+\frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3+\ldots\right)$$

$=\frac{1}{2}\left(1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\ldots\right)$ Multiply by the $\frac{1}{2}$

$=\frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}+\ldots$

ii Valid if $\left|\frac{x}{2}\right|<1\Rightarrow|x|<2$

1 c i $\frac{1}{(4-x)^2}$ Write in index form

$=(4-x)^{-2}$ Take 4 out as a factor

$=\left(4\left(1-\frac{x}{4}\right)\right)^{-2}$

$=4^{-2}\left(1-\frac{x}{4}\right)^{-2}$, $4^{-2}=\frac{1}{16}$. Use the binomial expansion with $n=-2$ and $x=-\frac{x}{4}$

$=\frac{1}{16}\left(1+(-2)\left(-\frac{x}{4}\right)+\frac{(-2)(-3)}{2!}\left(-\frac{x}{4}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(-\frac{x}{4}\right)^3+\ldots\right)$

$=\frac{1}{16}\left(1+\frac{x}{2}+\frac{3x^2}{16}+\frac{x^3}{16}+\ldots\right)$ Multiply by $\frac{1}{16}$

$=\frac{1}{16}+\frac{x}{32}+\frac{3x^2}{256}+\frac{x^3}{256}+\ldots$

ii Valid if $\left|\frac{x}{4}\right|<1\Rightarrow|x|<4$

d i $\sqrt{9+x}$ Write in index form

$=(9+x)^{\frac{1}{2}}$ Take 9 out as a factor

$=\left(9\left(1+\frac{x}{9}\right)\right)^{\frac{1}{2}}$

$=9^{\frac{1}{2}}\left(1+\frac{x}{9}\right)^{\frac{1}{2}}$, $9^{\frac{1}{2}}=3$. Use binomial expansion with $n=\frac{1}{2}$ and $x=\frac{x}{9}$

$=3\left(1+\left(\frac{1}{2}\right)\left(\frac{x}{9}\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{x}{9}\right)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(\frac{x}{9}\right)^3+\ldots\right)$

$=3\left(1+\frac{x}{18}-\frac{x^2}{648}+\frac{x^3}{11\,664}+\ldots\right)$ Multiply by 3

$=3+\frac{x}{6}-\frac{x^2}{216}+\frac{x^3}{3888}+\ldots$

ii Valid for $\left|\frac{x}{9}\right|<1\Rightarrow|x|<9$

1 e i $\frac{1}{\sqrt{2+x}}$ Write in index form

$$=(2+x)^{-\frac{1}{2}}$$ Take out a factor of 2

$$=\left(2\left(1+\frac{x}{2}\right)\right)^{-\frac{1}{2}}$$

$$=2^{-\frac{1}{2}}\left(1+\frac{x}{2}\right)^{-\frac{1}{2}},\quad 2^{-\frac{1}{2}}=\frac{1}{2^{\frac{1}{2}}}=\frac{1}{\sqrt{2}}.$$ Use binomial expansion with $n=-\frac{1}{2}$ and $x=\frac{x}{2}$

$$=\frac{1}{\sqrt{2}}\left(1+\left(-\frac{1}{2}\right)\left(\frac{x}{2}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x}{2}\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{x}{2}\right)^3+\ldots\right)$$

$$=\frac{1}{\sqrt{2}}\left(1-\frac{x}{4}+\frac{3x^2}{32}-\frac{5x^3}{128}+\ldots\right)$$ Multiply by $\frac{1}{\sqrt{2}}$

$$=\frac{1}{\sqrt{2}}-\frac{x}{4\sqrt{2}}+\frac{3x^2}{32\sqrt{2}}-\frac{5x^3}{128\sqrt{2}}+\ldots$$ Rationalise surds

$$=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}x}{8}+\frac{3\sqrt{2}x^2}{64}-\frac{5\sqrt{2}x^3}{256}+\ldots$$

ii Valid if $\left|\frac{x}{2}\right|<1\Rightarrow|x|<2$

f i $\frac{5}{3+2x}$ Write in index form

$$=5(3+2x)^{-1}$$ Take out a factor of 3

$$=5\left(3\left(1+\frac{2x}{3}\right)\right)^{-1}$$

$$=5\times3^{-1}\left(1+\frac{2x}{3}\right)^{-1},\quad 3^{-1}=\frac{1}{3}.$$ Use binomial expansion with $n=-1$ and $x=\frac{2x}{3}$

$$=\frac{5}{3}\left(1+(-1)\left(\frac{2x}{3}\right)+\frac{(-1)(-2)}{2!}\left(\frac{2x}{3}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{2x}{3}\right)^3+\ldots\right)$$

$$=\frac{5}{3}\left(1-\frac{2x}{3}+\frac{4x^2}{9}-\frac{8x^3}{27}+\ldots\right)$$ Multiply by $\frac{5}{3}$

$$=\frac{5}{3}-\frac{10x}{9}+\frac{20x^2}{27}-\frac{40x^3}{81}+\ldots$$

ii Valid if $\left|\frac{2x}{3}\right|<1\Rightarrow|x|<\frac{3}{2}$

1 g i

$\frac{1+x}{2+x}=1-\frac{1}{2+x}$ Write $\frac{1}{2+x}$ in index form

$=1-(2+x)^{-1}$ Take out a factor of 2

$=1-\left(2\left(1+\frac{x}{2}\right)\right)^{-1}$

$=1-\left(2^{-1}\left(1+\frac{x}{2}\right)^{-1}\right)$ Expand $\left(1+\frac{x}{2}\right)^{-1}$ using the binomial expansion

$=1-\left(\frac{1}{2}\left(1+(-1)\left(\frac{x}{2}\right)+\frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3+...\right)\right)$

$=1-\left(\frac{1}{2}\left(1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\ldots\right)\right)$ Multiply $\left(1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\ldots\right)$ by $\frac{1}{2}$

$=1-\left(\frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}+\ldots\right)$

$=\frac{1}{2}+\frac{1}{4}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\ldots$

ii Valid for $\left|\frac{x}{2}\right|<1 \Rightarrow |x|<2$

1 h i $\sqrt{\dfrac{2+x}{1-x}}$

$= (2+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$ Put both in index form

$= 2^{\frac{1}{2}}\left(1+\dfrac{x}{2}\right)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$ Expand both using the binomial expansion

$$= \sqrt{2}\left(1+\left(\frac{1}{2}\right)\left(\frac{x}{2}\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{x}{2}\right)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(\frac{x}{2}\right)^3+\ldots\right)$$

$$\times\left(1+\left(-\frac{1}{2}\right)(-x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-x)^2}{2!}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-x)^3}{3!}+\ldots\right)$$

$= \sqrt{2}\left(1+\dfrac{1}{4}x-\dfrac{1}{32}x^2+\dfrac{1}{128}x^3+\ldots\right)\left(1+\dfrac{1}{2}x+\dfrac{3}{8}x^2+\dfrac{5}{16}x^3+\ldots\right)$ Multiply out

$$= \sqrt{2}\left(\begin{array}{l}1\left(1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{15}{16}x^3+\ldots\right)+\frac{1}{4}x\left(1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\ldots\right)\\ -\frac{1}{32}x^2\left(1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\ldots\right)+\frac{1}{128}x^3\left(1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\ldots\right)+\ldots\end{array}\right)$$

$$= \sqrt{2}\left(\begin{array}{l}1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\frac{1}{4}x+\frac{1}{8}x^2\\ +\frac{3}{32}x^3-\frac{1}{32}x^2-\frac{1}{64}x^3+\frac{1}{128}x^3+\ldots\end{array}\right)$$

Collect like terms

$= \sqrt{2}\left(1+\dfrac{3}{4}x+\dfrac{15}{32}x^2+\dfrac{51}{128}x^3+\ldots\right)$ Multiply by $\sqrt{2}$

$$= \sqrt{2}+\frac{3\sqrt{2}}{4}x+\frac{15\sqrt{2}}{32}x^2+\frac{51\sqrt{2}}{128}x^3+\ldots$$

ii Valid if $\left|\dfrac{x}{2}\right|<1$ and $|-x|<1 \Rightarrow |x|<1$ for both to be valid.

2 $$(5+4x)^{-2} = \left(5\left(1+\frac{4}{5}x\right)\right)^{-2} = 5^{-2}\left(1+\frac{4}{5}x\right)^{-2} = \frac{1}{25}\left(1+\frac{4}{5}x\right)^{-2}$$

$$= \frac{1}{25}\left(1+(-2)\left(\frac{4}{5}x\right)+\frac{(-2)(-3)}{2!}\left(\frac{4}{5}x\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{4}{5}x\right)^3+\ldots\right)$$

$$= \frac{1}{25}\left(1+(-2)\left(\frac{4}{5}x\right)+\frac{(-2)(-3)}{2}\frac{16}{25}x^2+\frac{(-2)(-3)(-4)}{6}\frac{64}{125}x^3+\ldots\right)$$

$$= \frac{1}{25}\left(1-\frac{8}{5}x+\frac{48}{25}x^2-\frac{256}{125}x^3+\ldots\right)$$

$$= \frac{1}{25}-\frac{8}{125}x+\frac{48}{625}x^2-\frac{256}{3125}x^3+\ldots$$

3 a $$\sqrt{(4-x)} = (4-x)^{\frac{1}{2}}$$

$$= \left[4\left(1-\frac{x}{4}\right)\right]^{\frac{1}{2}}$$

$$= 4^{\frac{1}{2}}\left(1-\frac{x}{4}\right)^{\frac{1}{2}}$$

$$= 2\left[1+\left(\frac{1}{2}\right)\left(-\frac{x}{4}\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(-\frac{x}{4}\right)^2+\ldots\right]$$

$$= 2\left(1-\frac{x}{8}-\frac{x^2}{128}+\ldots\right)$$

$$= 2-\frac{x}{4}-\frac{x^2}{64}+\ldots$$

Valid for $\left|-\frac{x}{4}\right| < 1 \Rightarrow |x| < 4$

b Substitute $x = \frac{1}{9}$ into both sides of the expansion:

$$\sqrt{\left(4-\frac{1}{9}\right)} \simeq 2-\frac{\frac{1}{9}}{4}-\frac{\left(\frac{1}{9}\right)^2}{64}-\frac{\left(\frac{1}{9}\right)^3}{512}$$

$$\sqrt{\frac{35}{9}} \simeq 2-\frac{1}{36}-\frac{1}{5184}-\frac{1}{373248}$$

$$\frac{\sqrt{35}}{3} \simeq \frac{736055}{373248}$$

3 c $\mathrm{m}(x) \approx 2 - \frac{1}{4}x - \frac{1}{64}x^2$

$$\mathrm{m}\left(\frac{1}{9}\right) = \frac{\sqrt{35}}{3}$$

$$\sqrt{35} = 3\mathrm{m}\left(\frac{1}{9}\right)$$

$$\approx 3\left(2 - \frac{1}{4}\left(\frac{1}{9}\right) - \frac{1}{64}\left(\frac{1}{9}\right)^2\right)$$

$$\approx 3\left(2 - \frac{1}{36} - \frac{1}{5184}\right)$$

$$\approx 5.916087963$$

$$\sqrt{35} = 5.916079783$$

Percentage error $= \frac{5.916087963 - 5.916079783}{5.916079783} \times 100 = 0.000138\%$

4 a $\frac{1}{\sqrt{a+bx}} = (a+bx)^{-\frac{1}{2}} = \left(a\left(1+\frac{b}{a}x\right)\right)^{-\frac{1}{2}} = a^{-\frac{1}{2}}\left(1+\frac{b}{a}x\right)^{-\frac{1}{2}} = \frac{1}{a^{\frac{1}{2}}}\left(1+\frac{b}{a}x\right)^{-\frac{1}{2}}$

$$= \frac{1}{a^{\frac{1}{2}}}\left(1 + \left(-\frac{1}{2}\right)\left(\frac{b}{a}x\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(\frac{b}{a}x\right)^2 + \ldots\right)$$

$$= \frac{1}{a^{\frac{1}{2}}}\left(1 + \left(-\frac{1}{2}\right)\left(\frac{b}{a}x\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\frac{b^2}{a^2}x^2 + \ldots\right)$$

$$= \frac{1}{a^{\frac{1}{2}}}\left(1 - \frac{b}{2a}x + \frac{3b^2}{8a^2}x^2 + \ldots\right)$$

$$= \frac{1}{a^{\frac{1}{2}}} - \frac{b}{2a^{\frac{3}{2}}}x + \frac{3b^2}{8a^{\frac{5}{2}}}x^2 + \ldots = 3 + \frac{1}{3}x + \frac{1}{18}x^2 + \ldots$$

Equating coefficients gives $\frac{1}{a^{\frac{1}{2}}} = 3$, so $a = \frac{1}{9}$

and $-\frac{b}{2\left(\frac{1}{9}\right)^{\frac{3}{2}}} = \frac{1}{3}$

$$-\frac{b}{\frac{2}{27}} = \frac{1}{3}$$

$$b = -\frac{2}{81}$$

$$a = \frac{1}{9},\ b = -\frac{2}{81}$$

4 b x^3 term of $3\left(1-\frac{2}{9}x\right)^{-\frac{1}{2}} = 3\left(\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}\left(-\frac{2}{9}x\right)^3\right)$

$$= -3\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{6}\frac{8}{729}x^3$$

$$= \frac{5}{486}x^3$$

Coefficient = $\frac{5}{486}$

5 $\frac{3+2x-x^2}{4-x} \equiv \left(3+2x-x^2\right)(4-x)^{-1}$ Write $\frac{1}{4-x}$ as $(4-x)^{-1}$

$= \left(3+2x-x^2\right)\left(4\left(1-\frac{x}{4}\right)\right)^{-1}$ Take out a factor of 4

$= \left(3+2x-x^2\right)\frac{1}{4}\left(1-\frac{x}{4}\right)^{-1}$ Expand $\left(1-\frac{x}{4}\right)^{-1}$ using the binomial expansion

$= \left(3+2x-x^2\right)\frac{1}{4}\left(1+(-1)\left(-\frac{x}{4}\right)+\frac{(-1)(-2)}{2!}\left(-\frac{x}{4}\right)^2+\ldots\right)$ Ignore terms higher than x^2

$= \left(3+2x-x^2\right)\frac{1}{4}\left(1+\frac{x}{4}+\frac{x^2}{16}+\ldots\right)$ Multiply expansion by $\frac{1}{4}$

$= \left(3+2x-x^2\right)\left(\frac{1}{4}+\frac{x}{16}+\frac{x^2}{64}+\ldots\right)$ Multiply result by $\left(3+2x-x^2\right)$

$$= 3\left(\frac{1}{4}+\frac{x}{16}+\frac{x^2}{64}+\ldots\right)+2x\left(\frac{1}{4}+\frac{x}{16}+\frac{x^2}{64}+\ldots\right)-x^2\left(\frac{1}{4}+\frac{x}{16}+\frac{x^2}{64}+\ldots\right)$$

$= \frac{3}{4}+\frac{3}{16}x+\frac{3}{64}x^2+\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{4}x^2+\ldots$ Ignore any terms bigger than x^2

$$= \frac{3}{4}+\frac{11}{16}x-\frac{5}{64}x^2$$

Expansion is valid if $\left|\frac{-x}{4}\right| < 1 \Rightarrow |x| < 4$

6 a $\frac{1}{\sqrt{5+2x}}=(5+2x)^{-\frac{1}{2}}=5^{-\frac{1}{2}}\left(1+\frac{2}{5}x\right)^{-\frac{1}{2}}=\frac{1}{\sqrt{5}}\left(1+\frac{2}{5}x\right)^{-\frac{1}{2}}$

$$=\frac{1}{\sqrt{5}}\left(1+\left(-\frac{1}{2}\right)\left(\frac{2}{5}x\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(\frac{2}{5}x\right)^2+\ldots\right)$$

$$=\frac{1}{\sqrt{5}}\left(1+\left(-\frac{1}{2}\right)\left(\frac{2}{5}x\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\frac{4}{25}x^2+\ldots\right)$$

$$=\frac{1}{\sqrt{5}}\left(1-\frac{1}{5}x+\frac{3}{50}x^2+\ldots\right)$$

$$=\frac{1}{\sqrt{5}}-\frac{1}{5\sqrt{5}}x+\frac{3}{50\sqrt{5}}x^2+\ldots$$

b $\frac{2x-1}{\sqrt{5+2x}}=\frac{2x-1}{\sqrt{5}}\left(1+\frac{2}{5}x\right)^{-\frac{1}{2}}$

$$=(2x-1)\left(\frac{1}{\sqrt{5}}-\frac{1}{5\sqrt{5}}x+\frac{3}{50\sqrt{5}}x^2+\ldots\right)$$

$$=\frac{2}{\sqrt{5}}x-\frac{2}{5\sqrt{5}}x^2-\frac{1}{\sqrt{5}}+\frac{1}{5\sqrt{5}}x-\frac{3}{50\sqrt{5}}x^2+\ldots$$

$$=-\frac{1}{\sqrt{5}}+\frac{11}{5\sqrt{5}}x-\frac{23}{50\sqrt{5}}x^2+\ldots$$

7 a $(16-3x)^{\frac{1}{4}}=\left(16\left(1-\frac{3}{16}x\right)\right)^{\frac{1}{4}}=2\left(1-\frac{3}{16}x\right)^{\frac{1}{4}}$

$$=2\left(1+\left(\frac{1}{4}\right)\left(-\frac{3}{16}x\right)+\frac{\left(\frac{1}{4}\right)\left(\frac{1}{4}-1\right)}{2!}\left(-\frac{3}{16}x\right)^2+\ldots\right)$$

$$=2\left(1+\left(\frac{1}{4}\right)\left(-\frac{3}{16}x\right)+\frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)}{2}\frac{9}{256}x^2+\ldots\right)$$

$$=2\left(1-\frac{3}{64}x-\frac{27}{8192}x^2+\ldots\right)$$

$$=2-\frac{3}{32}x-\frac{27}{4096}x^2+\ldots$$

7 b Let $x = 0.1$

$$\sqrt[4]{15.7} \approx 2 - \frac{3}{32}(0.1) - \frac{27}{4096}(0.1)^2$$
$$= 1.991$$

8 a
$$\frac{3}{4-2x} = 3(4-2x)^{-1} = 3\left(4\left(1-\frac{1}{2}x\right)\right)^{-1} = \frac{3}{4}\left(1-\frac{1}{2}x\right)^{-1}$$
$$= \frac{3}{4}\left(1+(-1)\left(-\frac{1}{2}x\right)+\frac{(-1)(-2)}{2!}\left(-\frac{1}{2}x\right)^2+\ldots\right)$$
$$= \frac{3}{4}\left(1-\left(-\frac{1}{2}x\right)+\frac{1}{4}x^2+\ldots\right)$$
$$= \frac{3}{4}+\frac{3}{8}x+\frac{3}{16}x^2+\ldots$$
$$\frac{2}{3+5x} = 2(3+5x)^{-1} = 2\left(3\left(1+\frac{5}{3}x\right)\right)^{-1} = \frac{2}{3}\left(1+\frac{5}{3}x\right)^{-1}$$
$$= \frac{2}{3}\left(1+(-1)\left(\frac{5}{3}x\right)+\frac{(-1)(-2)}{2!}\left(\frac{5}{3}x\right)^2+\ldots\right)$$
$$= \frac{2}{3}\left(1-\frac{5}{3}x+\frac{25}{9}x^2+\ldots\right)$$
$$= \frac{2}{3}-\frac{10}{9}x+\frac{50}{27}x^2+\ldots$$
$$\frac{3}{4-2x}-\frac{2}{3+5x}$$
$$= \frac{3}{4}+\frac{3}{8}x+\frac{3}{16}x^2+\ldots-\left(\frac{2}{3}-\frac{10}{9}x+\frac{50}{27}x^2+\ldots\right)$$
$$= \frac{1}{12}+\frac{107}{72}x-\frac{719}{432}x^2+\ldots$$

b $$g(0.01) = \frac{3}{4-2(0.01)}-\frac{2}{3+5(0.01)} = 0.0980311$$

c Using the series expansion:

$$g(0.01) \approx \frac{1}{12}+\frac{107}{72}(0.01)-\frac{719}{432}(0.01)^2 = 0.098028009$$

$$\text{Percentage error} = \frac{0.0980311-0.098028009}{0.098028009}\times 100 = 0.0032\%$$

Binomial expansion 4C

1 a Let $\frac{8x+4}{(1-x)(2+x)} \equiv \frac{A}{(1-x)} + \frac{B}{(2+x)}$

$$\equiv \frac{A(2+x)+B(1-x)}{(1-x)(2+x)}$$

Set the numerators equal:

$8x + 4 \equiv A(2 + x) + B(1 - x)$

Substitute $x = 1$:

$8 \times 1 + 4 = A \times 3 + B \times 0$
$\Rightarrow 12 = 3A$
$\Rightarrow A = 4$

Substitute $x = -2$:

$8 \times (-2) + 4 = A \times 0 + B \times 3$
$\Rightarrow -12 = 3B$
$\Rightarrow B = -4$

Hence $\frac{8x+4}{(1-x)(2+x)} \equiv \frac{4}{(1-x)} - \frac{4}{(2+x)}$

1 b $$\frac{4}{(1-x)}=4(1-x)^{-1}$$

$$=4\left(1+(-1)(-x)+\frac{(-1)(-2)(-x)^2}{2!}+\ldots\right)$$

$$=4(1+x+x^2+\ldots)$$

$$=4+4x+4x^2+\ldots$$

$$\frac{4}{(2+x)}=4(2+x)^{-1}$$

$$=4\left(2\left(1+\frac{x}{2}\right)\right)^{-1}$$

$$=4\times2^{-1}\left(1+\frac{x}{2}\right)^{-1}$$

$$=4\times\frac{1}{2}\times\left(1+(-1)\left(\frac{x}{2}\right)+\frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2+\ldots\right)$$

$$=2\left(1-\frac{x}{2}+\frac{x^2}{4}+\ldots\right)$$

$$=2-x+\frac{1}{2}x^2+\ldots$$

Therefore

$$\frac{8x+4}{(1-x)(2+x)}\equiv\frac{4}{(1-x)}-\frac{4}{(2+x)}$$

$$=(4+4x+4x^2+\ldots)-\left(2-x+\frac{1}{2}x^2+\ldots\right)$$

$$=2+5x+\frac{7x^2}{2}+\ldots$$

c $\frac{4}{(1-x)}$ is valid for $|x| < 1$

$\frac{4}{(2+x)}$ is valid for $|x| < 2$

Both are valid when $|x| < 1$.

2 **a** Let $\frac{-2x}{(2+x)^2} \equiv \frac{A}{(2+x)} + \frac{B}{(2+x)^2}$

$$\equiv \frac{A(2+x)+B}{(2+x)^2}$$

Set the numerators equal:

$-2x \equiv A(2 + x) + B$

Substitute $x = -2$:

$4 = A \times 0 + B \Rightarrow B = 4$

Equate terms in x:

$-2 = A \Rightarrow A = -2$

Hence $\frac{-2x}{(2+x)} \equiv \frac{-2}{(2+x)} + \frac{4}{(2+x)^2}$

2 b

$$\begin{aligned}\frac{-2}{2+x} &= -2(2+x)^{-1}\\
&= -2\left(2\left(1+\frac{x}{2}\right)\right)^{-1}\\
&= -2\times 2^{-1}\times\left(1+\frac{x}{2}\right)^{-1}\\
&= -1\times\left(1+(-1)\left(\frac{x}{2}\right)+\frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3+\ldots\right)\\
&= -1\times\left(1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\ldots\right)\\
&= -1+\frac{x}{2}-\frac{x^2}{4}+\frac{x^3}{8}+\ldots\end{aligned}$$

$$\begin{aligned}\frac{4}{(2+x)^2} &= 4(2+x)^{-2}\\
&= 4\left(2\left(1+\frac{x}{2}\right)\right)^{-2}\\
&= 4\times 2^{-2}\times\left(1+\frac{x}{2}\right)^{-2}\\
&= 1\times\left(1+(-2)\left(\frac{x}{2}\right)+\frac{(-2)(-3)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{x}{2}\right)^3+\ldots\right)\\
&= 1\times\left(1-x+\frac{3x^2}{4}-\frac{x^3}{2}+\ldots\right)\\
&= 1-x+\frac{3x^2}{4}-\frac{x^3}{2}+\ldots\end{aligned}$$

Hence

$$\begin{aligned}\frac{-2x}{(2+x)^2} &\equiv \frac{-2}{(2+x)}+\frac{4}{(2+x)^2}\\
&= -1+\frac{x}{2}-\frac{x^2}{4}+\frac{x^3}{8}+\ldots+1-x+\frac{3x^2}{4}-\frac{x^3}{2}+\ldots\\
&= 0-\frac{1}{2}x+\frac{1}{2}x^2-\frac{3}{8}x^3+\ldots\end{aligned}$$

Hence $B=\frac{1}{2}$ (coefficient of x^2) and $C=-\frac{3}{8}$ (coefficient of x^3).

2 c $\frac{-2}{(2+x)}$ is valid for $|x| < 2$

$\frac{4}{(2+x)^2}$ is valid for $|x| < 2$

Hence whole expression is valid $|x| < 2$.

3 a Let $$\frac{6+7x+5x^2}{(1+x)(1-x)(2+x)} \equiv \frac{A}{(1+x)} + \frac{B}{(1-x)} + \frac{C}{(2+x)}$$
$$\equiv \frac{A(1-x)(2+x)+B(1+x)(2+x)+C(1+x)(1-x)}{(1+x)(1-x)(2+x)}$$

Set the numerators equal

$6 + 7x + 5x^2 \equiv A(1 - x)(2 + x) + B(1 + x)(2 + x) + C(1 + x)(1 - x)$

Substitute $x = 1$:

$6 + 7 + 5 = A \times 0 + B \times 2 \times 3 + C \times 0$
$\Rightarrow 18 = 6B$
$\Rightarrow B = 3$

Substitute $x = -1$:

$6 - 7 + 5 = A \times 2 \times 1 + B \times 0 + C \times 0$
$\Rightarrow 4 = 2A$
$\Rightarrow A = 2$

Substitute $x = -2$:

$6 - 14 + 20 = A \times 0 + B \times 0 + C \times (-1) \times 3$
$\Rightarrow 12 = -3C$
$\Rightarrow C = -4$

Hence $$\frac{6+7x+5x^2}{(1+x)(1-x)(2+x)} \equiv \frac{2}{(1+x)} + \frac{3}{(1-x)} - \frac{4}{(2+x)}$$

3 b $$\frac{2}{1+x}=2(1+x)^{-1}$$

$$=2\left(1+(-1)(x)+\frac{(-1)(-2)(x)^2}{2!}+\frac{(-1)(-2)(-3)(x)^3}{3!}+\ldots\right)$$

$$=2(1-x+x^2-x^3+\ldots)$$

$$\approx 2-2x+2x^2-2x^3 \quad \text{Valid for } |x|<1$$

$$\frac{3}{1-x}=3(1-x)^{-1}$$

$$=3\left(1+(-1)(-x)+\frac{(-1)(-2)(-x)^2}{2!}+\frac{(-1)(-2)(-3)(-x)^3}{3!}+\ldots\right)$$

$$=3(1+x+x^2+x^3+\ldots)$$

$$\approx 3+3x+3x^2+3x^3 \quad \text{Valid for } |x|<1$$

$$\frac{4}{2+x}=4(2+x)^{-1}$$

$$=4\left(2\left(1+\frac{x}{2}\right)\right)^{-1}$$

$$=4\times 2^{-1}\times\left(1+\frac{x}{2}\right)^{-1}$$

$$=2\left(1+(-1)\left(\frac{x}{2}\right)+\frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3+\ldots\right)$$

$$=2\left(1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\ldots\right)$$

$$\approx 2-x+\frac{x^2}{2}-\frac{x^3}{4} \quad \text{Valid for } |x|<2$$

Hence $$\frac{6+7x+5x^2}{(1+x)(1-x)(2+x)}\equiv\frac{2}{(1+x)}+\frac{3}{(1-x)}-\frac{4}{(2+x)}$$

$$=(2-2x+2x^2-2x^3)+(3+3x+3x^2+3x^3)-\left(2-x+\frac{x^2}{2}-\frac{x^3}{4}\right)+\ldots$$

$$=2+3-2-2x+3x+x+2x^2+3x^2-\frac{x^2}{2}-2x^3+3x^3+\frac{x^3}{4}+\ldots$$

$$=3+2x+\frac{9}{2}x^2+\frac{5}{4}x^3+\ldots$$

c All expansions are valid when $|x|<1$.

4 a $$\frac{12x-1}{(1+2x)(1-3x)} \equiv \frac{A}{1+2x}+\frac{B}{1-3x} \equiv \frac{A(1-3x)+B(1+2x)}{(1+2x)(1-3x)}$$

So $12x-1 \equiv A(1-3x)+B(1+2x)$

Let $x=-\frac{1}{2}$:

$-6-1=A\times\frac{5}{2}+0$

$-7=\frac{5}{2}A$

$A=-\frac{14}{5}$

Let $x=\frac{1}{3}$:

$4-1=0+B\times\frac{5}{3}$

$3=\frac{5}{3}B$

$B=\frac{9}{5}$

$A=-\frac{14}{5},\ B=\frac{9}{5}$

b $$\frac{12x-1}{(1+2x)(1-3x)} \equiv \frac{-14}{5(1+2x)}+\frac{9}{5(1-3x)}$$

$$\frac{-14}{5(1+2x)}=-\frac{14}{5}(1+2x)^{-1}$$

$$=-\frac{14}{5}\left(1+(-1)(2x)+\frac{(-1)(-2)}{2!}(2x)^2+\ldots\right)$$

$$=-\frac{14}{5}\left(1-2x+4x^2+\ldots\right)$$

$$=-\frac{14}{5}+\frac{28}{5}x-\frac{56}{5}x^2+\ldots$$

$$\frac{9}{5(1-3x)}=\frac{9}{5}(1-3x)^{-1}$$

$$=\frac{9}{5}\left(1+(-1)(-3x)+\frac{(-1)(-2)}{2!}(-3x)^2+\ldots\right)$$

$$=\frac{9}{5}\left(1+3x+9x^2+\ldots\right)$$

$$=\frac{9}{5}+\frac{27}{5}x+\frac{81}{5}x^2+\ldots$$

$$\frac{-14}{5(1+2x)}+\frac{9}{5(1-3x)}=-\frac{14}{5}+\frac{28}{5}x-\frac{56}{5}x^2+\frac{9}{5}+\frac{27}{5}x+\frac{81}{5}x^2+\ldots$$

$$=-1+11x+5x^2+\ldots$$

5 **a** $$\frac{2x^2+7x-6}{(x+5)(x-4)} \equiv A+\frac{B}{x+5}+\frac{C}{x-4}$$

$$\begin{array}{r} 2 \\ x^2+x-20\overline{\big)2x^2+7x-6} \\ \underline{2x^2+2x-40} \\ 5x+34 \end{array}$$

$A=2$

$$\frac{2x^2+7x-6}{(x+5)(x-4)} \equiv 2+\frac{5x+34}{(x+5)(x-4)}$$

$$\frac{5x+34}{(x+5)(x-4)} \equiv \frac{B}{x+5}+\frac{C}{x-4}=\frac{B(x-4)+C(x+5)}{(x+5)(x-4)}$$

$5x+34=B(x-4)+C(x+5)$

Let $x=-5$:

$-25+34=B\times(-9)+0$

$9=-9B$

$B=-1$

Let $x=4$:

$20+34=0+C\times 9$:

$54=9C$

$C=6$

$$\frac{2x^2+7x-6}{(x+5)(x-4)} \equiv 2-\frac{1}{x+5}+\frac{6}{x-4}$$

b $$2-\frac{1}{x+5}+\frac{6}{x-4}=2-(5+x)^{-1}+6(-4+x)^{-1}=2-\frac{1}{5}\left(1+\frac{1}{5}x\right)^{-1}-\frac{3}{2}\left(1-\frac{1}{4}x\right)^{-1}$$

$$\frac{1}{5}\left(1+\frac{1}{5}x\right)^{-1}=\frac{1}{5}\left(1+(-1)\left(\frac{1}{5}x\right)+\frac{(-1)(-2)}{2!}\left(\frac{1}{5}x\right)^2+\ldots\right)$$

$$=\frac{1}{5}\left(1-\frac{1}{5}x+\frac{1}{25}x^2+\ldots\right)$$

$$=\frac{1}{5}-\frac{1}{25}x+\frac{1}{125}x^2+\ldots$$

$$\frac{3}{2}\left(1-\frac{1}{4}x\right)^{-1}=\frac{3}{2}\left(1+(-1)\left(-\frac{1}{4}x\right)+\frac{(-1)(-2)}{2!}\left(-\frac{1}{4}x\right)^2+\ldots\right)$$

$$=\frac{3}{2}\left(1+\frac{1}{4}x+\frac{1}{16}x^2+\ldots\right)$$

$$=\frac{3}{2}+\frac{3}{8}x+\frac{3}{32}x^2+\ldots$$

$$2-\frac{1}{x+5}+\frac{6}{x-4}=2-\left(\frac{1}{5}-\frac{1}{25}x+\frac{1}{125}x^2+\ldots\right)-\left(\frac{3}{2}+\frac{3}{8}x+\frac{3}{32}x^2+\ldots\right)$$

$$=\frac{3}{10}-\frac{67}{200}x+\frac{407}{4000}x^2+\ldots$$

5 c $|x| < 4$

6 a
$$\begin{array}{r} 3 \\ x^2+x-6\overline{)\,3x^2+4x-5} \\ \underline{3x^2+3x-18} \\ x+13 \end{array}$$

$A = 3$

$$\frac{3x^2+4x-5}{(x+3)(x-2)} \equiv 3+\frac{x+13}{(x+3)(x-2)}$$

$$\frac{x+13}{(x+3)(x-2)} \equiv \frac{B}{x+3}+\frac{C}{x-2}=\frac{B(x-2)+C(x+3)}{(x+3)(x-2)}$$

$x+13 = B(x-2)+C(x+3)$

Let $x = -3$

$-3+13 = B\times(-5)+0$

$10 = -5B$

$B = -2$

Let $x = 2$:

$2+13 = 0+C\times 5$:

$15 = 5C$

$C = 3$

$A = 3$, $B = -2$ and $C = 3$

6 b $\frac{3x^2+4x-5}{(x+3)(x-2)} \equiv 3-\frac{2}{x+3}+\frac{3}{x-2}$

$$3-\frac{2}{x+3}+\frac{3}{x-2}=3-2(3+x)^{-1}+3(-2+x)^{-1}=3-\frac{2}{3}\left(1+\frac{1}{3}x\right)^{-1}-\frac{3}{2}\left(1-\frac{1}{2}x\right)^{-1}$$

$$\frac{2}{3}\left(1+\frac{1}{3}x\right)^{-1}=\frac{2}{3}\left(1+(-1)\left(\frac{1}{3}x\right)+\frac{(-1)(-2)}{2!}\left(\frac{1}{3}x\right)^2+\ldots\right)$$

$$=\frac{2}{3}\left(1-\frac{1}{3}x+\frac{1}{9}x^2+\ldots\right)$$

$$=\frac{2}{3}-\frac{2}{9}x+\frac{2}{27}x^2+\ldots$$

$$\frac{3}{2}\left(1-\frac{1}{2}x\right)^{-1}=\frac{3}{2}\left(1+(-1)\left(-\frac{1}{2}x\right)+\frac{(-1)(-2)}{2!}\left(-\frac{1}{2}x\right)^2+\ldots\right)$$

$$=\frac{3}{2}\left(1+\frac{1}{2}x+\frac{1}{4}x^2+\ldots\right)$$

$$=\frac{3}{2}+\frac{3}{4}x+\frac{3}{8}x^2+\ldots$$

$$3-\frac{2}{x+3}+\frac{3}{x-2}=3-\left(\frac{2}{3}-\frac{2}{9}x+\frac{2}{27}x^2+\ldots\right)-\left(\frac{3}{2}+\frac{3}{4}x+\frac{3}{8}x^2+\ldots\right)$$

$$=\frac{5}{6}-\frac{19}{36}x-\frac{97}{216}x^2+\ldots$$

7 a $$\frac{2x^2+5x+11}{(2x-1)^2(x+1)} \equiv \frac{A}{2x-1}+\frac{B}{(2x-1)^2}+\frac{C}{x+1}$$

$$\equiv \frac{A(2x-1)(x+1)+B(x+1)+C(2x-1)^2}{(2x-1)^2(x+1)}$$

$$2x^2+5x+11 \equiv A(2x-1)(x+1)+B(x+1)+C(2x-1)^2$$

Let $x=\frac{1}{2}$:

$$\frac{1}{2}+\frac{5}{2}+11=0+B\times\frac{3}{2}+0$$

$$14=\frac{3}{2}B$$

$$B=\frac{28}{3}$$

Let $x=-1$:

$2-5+11=0+0+C\times 9$

$8=9C$

$$C=\frac{8}{9}$$

Equating coefficients of x^2 gives:

$2=2A+4C$

$$2=2A+\frac{32}{9}$$

$$A=-\frac{7}{9}$$

$A=-\frac{7}{9}$, $B=\frac{28}{3}$ and $C=\frac{8}{9}$

7 b $$\frac{2x^2+5x+11}{(2x-1)^2(x+1)} \equiv \frac{-7}{9(2x-1)} + \frac{28}{3(2x-1)^2} + \frac{8}{9(x+1)}$$

$$-\frac{7}{9}(-1+2x)^{-1} + \frac{28}{3}(-1+2x)^{-2} + \frac{8}{9}(1+x)^{-1}$$

$$=\frac{7}{9}(1-2x)^{-1} + \frac{28}{3}(1-2x)^{-2} + \frac{8}{9}(1+x)^{-1}$$

$$\frac{7}{9}(1-2x)^{-1} = \frac{7}{9}\left(1+(-1)(-2x)+\frac{(-1)(-2)}{2!}(-2x)^2+\ldots\right)$$

$$=\frac{7}{9}\left(1+2x+4x^2+\ldots\right)$$

$$=\frac{7}{9}+\frac{14}{9}x+\frac{28}{9}x^2+\ldots$$

$$\frac{28}{3}(1-2x)^{-2} = \frac{28}{3}\left(1+(-2)(-2x)+\frac{(-2)(-3)}{2!}(-2x)^2+\ldots\right)$$

$$=\frac{28}{3}\left(1+4x+12x^2+\ldots\right)$$

$$=\frac{28}{3}+\frac{112}{3}x+112x^2+\ldots$$

$$\frac{8}{9}(1+x)^{-1} = \frac{8}{9}\left(1+(-1)x+\frac{(-1)(-2)}{2!}x^2+\ldots\right)$$

$$=\frac{8}{9}\left(1-x+x^2+\ldots\right)$$

$$=\frac{8}{9}-\frac{8}{9}x+\frac{8}{9}x^2+\ldots$$

$$\frac{2x^2+5x+11}{(2x-1)^2(x+1)} = \frac{7}{9}+\frac{14}{9}x+\frac{28}{9}x^2+\ldots+\frac{28}{3}+\frac{112}{3}x+112x^2+\ldots+\frac{8}{9}-\frac{8}{9}x+\frac{8}{9}x^2+\ldots$$

$$=11+38x+116x^2+\ldots$$

c $$\mathrm{f}(0.05) = \frac{2(0.05)^2+5(0.05)+11}{(2(0.05)-1)^2(0.05+1)} = 13.23339212$$

Using the expansion:

$$\mathrm{f}(0.05) \approx 11+38(0.05)+116(0.05)^2 = 13.19$$

Percentage error = $\frac{13.23339212-13.19}{13.23339212}\times 100 = 0.33\%$

Binomial expansion Mixed exercise 4

1 a i $(1-4x)^3$ Use binomial expansion with $n = 3$ and $x = -4x$

$$=1+(3)(-4x)+\frac{(3)(2)(-4x)^2}{2!}+\frac{(3)(2)(1)(-4x)^3}{3!}$$ As $n=3,$ expansion is finite

and exact

$$=1-12x+48x^2-64x^3$$

ii Valid for all x

b i $\sqrt{16+x}$ Write in index form

$$=(16+x)^{\frac{1}{2}}$$ Take out a factor of 16

$$=\left(16\left(1+\frac{x}{16}\right)\right)^{\frac{1}{2}}$$

$$=16^{\frac{1}{2}}\left(1+\frac{x}{16}\right)^{\frac{1}{2}}$$ Use binomial expansion with $n=\frac{1}{2}$ and $x=\frac{x}{16}$

$$=4\left(1+\frac{1}{2}\left(\frac{x}{16}\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{x}{16}\right)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(\frac{x}{16}\right)^3+\ldots\right)$$

$$=4\left(1+\frac{x}{32}-\frac{x^2}{2048}+\frac{x^3}{65\,536}+\ldots\right)$$ Multiply by 4

$$=4+\frac{x}{8}-\frac{x^2}{512}+\frac{x^3}{16\,384}+\ldots$$

ii Valid for $\left|\frac{x}{16}\right|<1 \Rightarrow |x|<16$

c i $\frac{1}{1-2x}$ Write in index form

$$=(1-2x)^{-1}$$ Use binomial expansion with $n=-1$ and $x=-2x$

$$=1+(-1)(-2x)+\frac{(-1)(-2)(-2x)^2}{2!}+\frac{(-1)(-2)(-3)(-2x)^3}{3!}+\ldots$$

$$=1+2x+4x^2+8x^3+\ldots$$

ii Valid for $|2x|<1 \Rightarrow |x|<\frac{1}{2}$

1 d i $\frac{4}{2+3x}$ Write in index form

$= 4(2+3x)^{-1}$ Take out a factor of 2

$= 4\left(2\left(1+\frac{3x}{2}\right)\right)^{-1}$

$= 4\times 2^{-1}\times\left(1+\frac{3x}{2}\right)^{-1}$ Use binomial expansion with $n=-1$ and $x=\frac{3x}{2}$

$$= 2\left(1+(-1)\left(\frac{3x}{2}\right)+\frac{(-1)(-2)}{2!}\left(\frac{3x}{2}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{3x}{2}\right)^3+\ldots\right)$$

$= 2\left(1-\frac{3x}{2}+\frac{9x^2}{4}-\frac{27x^3}{8}+\ldots\right)$ Multiply by 2

$$= 2-3x+\frac{9x^2}{2}-\frac{27x^3}{4}+\ldots$$

ii Valid for $\left|\frac{3x}{2}\right|<1 \Rightarrow |x|<\frac{2}{3}$

e i $\frac{4}{\sqrt{4-x}} = 4(\sqrt{4-x})^{-1}$ Write in index form

$= 4(4-x)^{-\frac{1}{2}}$ Take out a factor of 4

$= 4\left(4\left(1-\frac{x}{4}\right)\right)^{-\frac{1}{2}}$

$= 4\times 4^{-\frac{1}{2}}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}$ Use binomial expansion with $n=-\frac{1}{2}$ and $x=-\frac{x}{4}$

$$= 4^{\frac{1}{2}}\left(1+\left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right)+\frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(-\frac{x}{4}\right)^2+\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}\left(-\frac{x}{4}\right)^3+\ldots\right)$$

$= 2\left(1+\frac{x}{8}+\frac{3}{128}x^2-\frac{5}{1024}x^3+\ldots\right)$ Multiply by 2

$$= 2+\frac{x}{4}+\frac{3}{64}x^2+\frac{5}{512}x^3+\ldots$$

ii Valid $\left|-\frac{x}{4}\right|<1 \Rightarrow |x|<4$

1 f i $\frac{1+x}{1+3x}=(1+x)(1+3x)^{-1}$ Write $\frac{1}{1+3x}$ in index form then expand

$=(1+x)\left(1+(-1)(3x)+\frac{(-1)(-2)(3x)^2}{2!}+\frac{(-1)(-2)(-3)(3x)^3}{3!}+\ldots\right)$

$=(1+x)(1-3x+9x^2-27x^3+\ldots)$ Multiply out

$=1-3x+9x^2-27x^3+x-3x^2+9x^3+\ldots$ Collect like terms

$=1-2x+6x^2-18x^3+\ldots$

ii Valid for $|3x|<1\Rightarrow|x|<\frac{1}{3}$

g i $\left(\frac{1+x}{1-x}\right)^2=\frac{(1+x)^2}{(1-x)^2}$ Write in index form

$=(1+x)^2(1-x)^{-2}$ Expand $(1-x)^{-2}$ using binomial expansion

$=(1+2x+x^2)\left(1+(-2)(-x)+\frac{(-2)(-3)(-x)^2}{2!}+\frac{(-2)(-3)(-4)(-x)^3}{3!}+\ldots\right)$

$=(1+2x+x^2)(1+2x+3x^2+4x^3+\ldots)$ Multiply out brackets

$=1+2x+3x^2+4x^3+2x+4x^2+6x^3+x^2+2x^3+\ldots$ Collect like terms

$=1+4x+8x^2+12x^3+\ldots$

ii Valid for $|x|<1$

1 h i Let $\frac{x-3}{(1-x)(1-2x)} \equiv \frac{A}{(1-x)} + \frac{B}{(1-2x)}$ Put in partial fraction form

$\equiv \frac{A(1-2x)+B(1-x)}{(1-x)(1-2x)}$ Add fractions.

Set the numerators equal:

$x - 3 = A(1 - 2x) + B(1 - x)$

Substitute $x = 1$:

$1 - 3 = A \times - 1 + B \times 0$

$\Rightarrow -2 = -1A$

$\Rightarrow A = 2$

Substitute $x = \frac{1}{2}$: $\frac{1}{2} - 3 = A \times 0 + B \times \frac{1}{2}$

$\Rightarrow -2\frac{1}{2} = \frac{1}{2}B$

$\Rightarrow B = -5$

Hence $\frac{x-3}{(1-x)(1-2x)} \equiv \frac{2}{(1-x)} - \frac{5}{(1-2x)}$

$$\frac{2}{(1-x)} = 2(1-x)^{-1}$$
$$= 2\left(1+(-1)(-x)+\frac{(-1)(-2)(-x)^2}{2!}+\frac{(-1)(-2)(-3)(-x)^3}{3!}+\ldots\right)$$
$$= 2(1+x+x^2+x^3+\ldots)$$
$$= 2+2x+2x^2+2x^3+\ldots$$

$$\frac{5}{(1-2x)} = 5(1-2x)^{-1}$$
$$= 5\left(1+(-1)(-2x)+\frac{(-1)(-2)(-2x)^2}{2!}+\frac{(-1)(-2)(-3)(-2x)^3}{3!}+\ldots\right)$$
$$= 5(1+2x+4x^2+8x^3+\ldots)$$
$$= 5+10x+20x^2+40x^3+\ldots$$

Hence $\frac{x-3}{(1-x)(1-2x)} \equiv \frac{2}{(1-x)} - \frac{5}{(1-2x)}$

$= (2 + 2x + 2x^2 + 2x^3 + \ldots) - (5 + 10x + 20x^2 + 40x^3 + \ldots)$

$= -3 - 8x - 18x^2 - 38x^3 + \ldots$

1 h ii $\frac{2}{1-x}$ is valid for $|x| < 1$

$\frac{5}{1-2x}$ is valid for $|2x|<1 \Rightarrow |x|<\frac{1}{2}$

Both are valid when $|x|<\frac{1}{2}$

2 $$\left(1-\frac{1}{2}x\right)^{\frac{1}{2}} = 1+\left(\frac{1}{2}\right)\left(-\frac{1}{2}x\right)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{1}{2}x\right)^2+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(-\frac{1}{2}x\right)^3+\ldots$$

$$=1+\left(\frac{1}{2}\right)\left(-\frac{1}{2}x\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}\frac{1}{4}x^2-\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}\frac{1}{8}x^3+\ldots$$

$$=1-\frac{1}{4}x-\frac{1}{32}x^2-\frac{1}{128}x^3+\ldots$$

3 a Using binomial expansion

$$(1+x)^{\frac{1}{2}} = 1+\left(\frac{1}{2}\right)(x)+\frac{(\frac{1}{2})(-\frac{1}{2})(x)^2}{2!}+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(x)^3}{3!}+\ldots$$

$$=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\ldots$$

Expansion is valid if $|x| < 1$.

b Substituting $x=\frac{1}{4}$ in both sides of the expansion gives

$$\left(1+\frac{1}{4}\right)^{\frac{1}{2}} \approx 1+\frac{1}{2}\times\frac{1}{4}-\frac{1}{8}\times\left(\frac{1}{4}\right)^2+\frac{1}{16}\times\left(\frac{1}{4}\right)^3$$

$$\left(\frac{5}{4}\right)^{\frac{1}{2}} \approx 1+\frac{1}{8}-\frac{1}{128}+\frac{1}{1024}$$

$$\sqrt{\frac{5}{4}} \approx \frac{1145}{1024}$$

$\frac{\sqrt{5}}{2} \approx \frac{1145}{1024}$ Multiply both sides by 2

$$\sqrt{5} \approx \frac{1145}{512}$$

4 **a** $$(1+9x)^{\frac{2}{3}}=1+\left(\frac{2}{3}\right)(9x)+\frac{\left(\frac{2}{3}\right)\left(\frac{2}{3}-1\right)}{2!}(9x)^2+\frac{\left(\frac{2}{3}\right)\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{3!}(9x)^3+\ldots$$

$$=1+\left(\frac{2}{3}\right)(9x)+\frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)}{2}81x^2+\frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{6}729x^3+\ldots$$

$$=1+6x-9x^2+36x^3+\ldots$$

Equating coefficients:
$c=-9$ and $d=36$

b $1+9x=1.45$
$x=0.05$

$$(1.45)^{\frac{2}{3}}\approx 1+6(0.05)-9(0.05)^2+36(0.05)^3$$
$$=1.282$$

c $(1.45)^{\frac{2}{3}}=1.28108713$

The approximation is correct to 2 decimal places.

5 **a** The x^2 term of $(1+ax)^{\frac{1}{2}}$ $=\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(ax)^2$

$$-\frac{1}{8}a^2=-2$$
$$a^2=16$$
$$a=\pm 4$$

b The x^3 term of $(1+ax)^{\frac{1}{2}}$ $=\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(ax)^3$

When $a=4$:

$$\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(ax)^3=\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}(4x)^3$$
$$=4x^3$$

When $a=-4$:

$$\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(ax)^3=\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}(-4x)^3$$
$$=-4x^3$$

The coefficient of the x^3 term is 4 or −4

6 **a** $(1 + 3x)^{-1}$ Use binomial expansion with $n = -1$ and $x = 3x$

$$= 1 + (-1)(3x) + \frac{(-1)(-2)(3x)^2}{2!} + \frac{(-1)(-2)(-3)(3x)^3}{3!} + \ldots$$

$$= 1 - 3x + 9x^2 - 27x^3 + \ldots$$

b $\dfrac{1+x}{1+3x} = (1+x)(1+3x)^{-1}$ Use expansion from part **a**

$= (1+x)(1 - 3x + 9x^2 - 27x^3 + \ldots)$ Multiply out

$= 1 - 3x + 9x^2 - 27x^3 + x - 3x^2 + 9x^3 + \ldots$ Collect like terms

$= 1 - 2x + 6x^2 - 18x^3 + \ldots$ Ignore terms greater than x^3

Hence $\dfrac{1+x}{1+3x} \approx 1 - 2x + 6x^2 - 18x^3$

c Substitute $x = 0.01$ into both sides of the above

$$\frac{1+0.01}{1+3\times0.01} \simeq 1 - 2\times0.01 + 6\times0.01^2 - 18\times0.01^3$$

$$\frac{1.01}{1.03} \approx 1 - 0.02 + 0.0006 - 0.000018, \quad \left(\frac{1.01}{1.03} = \frac{101}{103}\right)$$

$\dfrac{101}{103} \approx 0.980582$ Round to 5 d.p.

$\dfrac{101}{103} \approx 0.98058$ (5 d.p.)

7 **a** Using binomial expansion

$$(1+ax)^n = 1 + n(ax) + \frac{n(n-1)(ax)^2}{2!} + \frac{n(n-1)(n-2)(ax)^3}{3!} + \ldots$$

If coefficient of x is -6 then $na = -6$ **(1)**

If coefficient of x^2 is 27 then $\dfrac{n(n-1)a^2}{2} = 27$ **(2)**

From **(1)**, $a = \dfrac{-6}{n}$. Substitute in **(2)**:

$$\frac{n(n-1)}{2}\left(\frac{-6}{n}\right)^2 = 27$$

$$\frac{n(n-1)}{2} \times \frac{36}{n^2} = 27$$

$$\frac{(n-1)18}{n} = 27$$

$(n - 1)18 = 27n$

$18n - 18 = 27n$

$-18 = 9n$

$n = -2$

Substitute $n = -2$ back in **(1)**: $-2a = -6 \Rightarrow a = 3$

7 **b** Coefficient of x^3 is

$$\frac{n(n-1)(n-2)a^3}{3!}=\frac{(-2)\times(-3)\times(-4)\times 3^3}{3\times 2\times 1}=-108$$

c $(1+3x)^{-2}$ is valid if $|3x|<1\Rightarrow|x|<\frac{1}{3}$

8

$$\frac{3}{\sqrt{4+x}}=3(\sqrt{4+x})^{-1}$$ Write in index form

$$=3(4+x)^{-\frac{1}{2}}$$ Take out a factor of 4

$$=3\left(4\left(1+\frac{x}{4}\right)\right)^{-\frac{1}{2}}$$

$$=3\times 4^{-\frac{1}{2}}\times\left(1+\frac{x}{4}\right)^{-\frac{1}{2}}\qquad 4^{-\frac{1}{2}}=\frac{1}{4^{\frac{1}{2}}}=\frac{1}{2}$$

$$=\frac{3}{2}\times\left(1+\left(-\frac{1}{2}\right)\left(\frac{x}{4}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x}{4}\right)^2}{2!}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{x}{4}\right)^3}{3!}+\ldots\right)$$

$$=\frac{3}{2}\left(1-\frac{x}{8}+\frac{3}{128}x^2+\ldots\right)$$ Multiply by $\frac{3}{2}$

$$=\frac{3}{2}-\frac{3}{16}x+\frac{9}{256}x^2+\ldots$$

$$\approx\frac{3}{2}-\frac{3}{16}x+\frac{9}{256}x^2$$ if terms higher than x^2 are ignored.

9 **a** $$\frac{1}{\sqrt{4-x}}=(4-x)^{-\frac{1}{2}}=\left(4\left(1-\frac{1}{4}x\right)\right)^{-\frac{1}{2}}=\frac{1}{2}\left(1-\frac{1}{4}x\right)^{-\frac{1}{2}}$$

$$=\frac{1}{2}\left(1+\left(-\frac{1}{2}\right)\left(-\frac{1}{4}x\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(-\frac{1}{4}x\right)^2+\ldots\right)$$

$$=\frac{1}{2}\left(1+\left(-\frac{1}{2}\right)\left(-\frac{1}{4}x\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\frac{1}{16}x^2+\ldots\right)$$

$$=\frac{1}{2}\left(1+\frac{1}{8}x+\frac{3}{128}x^2+\ldots\right)$$

$$=\frac{1}{2}+\frac{1}{16}x+\frac{3}{256}x^2+\ldots$$

9 **b** $\frac{1+2x}{\sqrt{4-x}}=(1+2x)\left(\frac{1}{2}+\frac{1}{16}x+\frac{3}{256}x^2+\ldots\right)$

$$=\frac{1}{2}+\frac{1}{16}x+\frac{3}{256}x^2+x+\frac{1}{8}x^2+\ldots$$

$$=\frac{1}{2}+\frac{17}{16}x+\frac{35}{256}x^2+\ldots$$

10 **a** $(2+3x)^{-1}$ Take out factor of 2

$$=\left(2\left(1+\frac{3x}{2}\right)\right)^{-1}$$

$=2^{-1}\left(1+\frac{3x}{2}\right)^{-1}$ Use binomial expansion with $n=-1$ and $x=\frac{3x}{2}$

$$=\frac{1}{2}\left(1+(-1)\left(\frac{3x}{2}\right)+\frac{(-1)(-2)}{2!}\left(\frac{3x}{2}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{3x}{2}\right)^3+\ldots\right)$$

$=\frac{1}{2}\left(1-\frac{3}{2}x+\frac{9}{4}x^2-\frac{27}{8}x^3+\ldots\right)$ Multiply by $\frac{1}{2}$

$$=\frac{1}{2}-\frac{3}{4}x+\frac{9}{8}x^2-\frac{27}{16}x^3+\ldots$$

Valid for $\left|\frac{3x}{2}\right|<1\Rightarrow|x|<\frac{2}{3}$

b $\frac{1+x}{2+3x}$ Put in index form

$=(1+x)(2+3x)^{-1}$ Use expansion from part **a**

$=(1+x)\left(\frac{1}{2}-\frac{3}{4}x+\frac{9}{8}x^2-\frac{27}{16}x^3+\ldots\right)$ Multiply out

$=\frac{1}{2}-\frac{3}{4}x+\frac{9}{8}x^2-\frac{27}{16}x^3+\frac{1}{2}x-\frac{3}{4}x^2+\frac{9}{8}x^3+\ldots$ Collect like terms

$$=\frac{1}{2}-\frac{1}{4}x+\frac{3}{8}x^2-\frac{9}{16}x^3+\ldots$$

Valid for $\left|\frac{3x}{2}\right|<1\Rightarrow|x|<\frac{2}{3}$

11 a $(4+x)^{-\frac{1}{2}}=\left(4\left(1+\frac{x}{4}\right)\right)^{-\frac{1}{2}}$ Take out factor of 4

$=4^{-\frac{1}{2}}\left(1+\frac{x}{4}\right)^{-\frac{1}{2}}$ $\quad 4^{-\frac{1}{2}}=\frac{1}{4^{\frac{1}{2}}}=\frac{1}{2}$

$=\frac{1}{2}\left(1+\frac{x}{4}\right)^{-\frac{1}{2}}$ Use binomial expansion with $n=-\frac{1}{2}$ and $x=\frac{x}{4}$

$$=\frac{1}{2}\left(1+\left(-\frac{1}{2}\right)\left(\frac{x}{4}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x}{4}\right)^2}{2!}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{x}{4}\right)^3}{3!}+\ldots\right)$$

$$=\frac{1}{2}\left(1-\frac{1}{8}x+\frac{3}{128}x^2-\frac{5}{1024}x^3+\ldots\right)$$

$$=\frac{1}{2}-\frac{1}{16}x+\frac{3}{256}x^2-\frac{5}{2048}x^3+\ldots$$

Valid for $\left|\frac{x}{4}\right|<1\Rightarrow|x|<4$

b i $(4+x)^{-\frac{1}{2}}\approx\frac{1}{2}-\frac{x}{16}+\frac{3x^2}{256}-\frac{5x^3}{2048}$

When $x=-2$

$$\frac{1}{\sqrt{4+(-2)}}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$$

So $\frac{\sqrt{2}}{2}\approx\frac{1}{2}-\frac{(-2)}{16}+\frac{3(-2)^2}{256}-\frac{5(-2)^3}{2048}$

$\approx\frac{1}{2}+\frac{2}{16}+\frac{12}{256}+\frac{40}{2048}$

$\approx\frac{177}{256}$

So $\sqrt{2}\approx 2\times\frac{177}{256}=1.3828$ (4 d.p.)

b ii $(4+x)^{-\frac{1}{2}} \approx \frac{1}{2}-\frac{x}{16}+\frac{3x^2}{256}-\frac{5x^3}{2048}$

When $x=\frac{1}{2}$

$$\frac{1}{\sqrt{4+\frac{1}{2}}}=\frac{1}{\sqrt{\frac{9}{2}}}=\frac{\sqrt{2}}{3}$$

$$\text{So } \frac{\sqrt{2}}{3} \approx \frac{1}{2}-\frac{\left(\frac{1}{2}\right)}{16}+\frac{3\left(\frac{1}{2}\right)^2}{256}-\frac{5\left(\frac{1}{2}\right)^3}{2048}$$

$$\approx \frac{1}{2}-\frac{1}{32}+\frac{3}{1024}-\frac{5}{16384}$$

$$\approx \frac{7723}{16384}$$

So $\sqrt{2} \approx 3\times\frac{7723}{16384}=1.4141$ (4 d.p.)

c $x=\frac{1}{2}$ because it is closer to 0.

12 $(3+4x)^{-3}=\left(3\left(1+\frac{4}{3}x\right)\right)^{-3}=\frac{1}{27}\left(1+\frac{4}{3}x\right)^{-3}$

$$=\frac{1}{27}\left(1+(-3)\left(\frac{4}{3}x\right)+\frac{(-3)(-4)}{2!}\left(\frac{4}{3}x\right)^2+\ldots\right)$$

$$=\frac{1}{27}\left(1-4x+\frac{32}{3}x^2+\ldots\right)$$

$$=\frac{1}{27}-\frac{4}{27}x+\frac{32}{81}x^2+\ldots$$

13 a $$\frac{39x+12}{(x+1)(x+4)(x-8)} \equiv \frac{A}{x+1}+\frac{B}{x+4}+\frac{C}{x-8}$$

$$\equiv \frac{A(x+4)(x-8)+B(x+1)(x-8)+C(x+1)(x+4)}{(x+1)(x+4)(x-8)}$$

$$39x+12 \equiv A(x+4)(x-8)+B(x+1)(x-8)+C(x+1)(x+4)$$

Let $x=-1$:

$-39+12 = A\times 3\times(-9)+0+0$

$-27=-27A$

$A=1$

Let $x=-4$:

$-156+12=0+B\times(-3)\times(-12)+0$

$-144=36B$

$B=-4$

Let $x=8$:

$312+12=0+0+C\times 9\times 12$

$324=108C$

$C=3$

$A = 1$, $B = -4$ and $C = 3$

13 b $$\frac{39x+12}{(x+1)(x+4)(x-8)} \equiv \frac{1}{x+1}-\frac{4}{x+4}+\frac{3}{x-8}$$

$$\frac{1}{x+1}-\frac{4}{x+4}+\frac{3}{x-8}=(1+x)^{-1}-4(4+x)^{-1}+3(-8+x)^{-1}$$

$$=(1+x)^{-1}-4\left(4\left(1+\frac{1}{4}x\right)\right)^{-1}+3\left(-8\left(1-\frac{1}{8}x\right)\right)^{-1}$$

$$=(1+x)^{-1}-\left(1+\frac{1}{4}x\right)^{-1}-\frac{3}{8}\left(1-\frac{1}{8}x\right)^{-1}$$

$$(1+x)^{-1}=1+(-1)x+\frac{(-1)(-2)}{2!}x^2+\ldots$$

$$=1-x+x^2+\ldots$$

$$\left(1+\frac{1}{4}x\right)^{-1}=1+(-1)\left(\frac{1}{4}x\right)+\frac{(-1)(-2)}{2!}\left(\frac{1}{4}x\right)^2+\ldots$$

$$=1-\frac{1}{4}x+\frac{1}{16}x^2+\ldots$$

$$\frac{3}{8}\left(1-\frac{1}{8}x\right)^{-1}=\frac{3}{8}\left(1+(-1)\left(-\frac{1}{8}x\right)+\frac{(-1)(-2)}{2!}\left(-\frac{1}{8}x\right)^2+\ldots\right)$$

$$=\frac{3}{8}+\frac{3}{64}x+\frac{3}{512}x^2+\ldots$$

$$\frac{39x+12}{(x+1)(x+4)(x-8)}=\left(1-x+x^2+\ldots-\left(1-\frac{1}{4}x+\frac{1}{16}x^2+\ldots\right)-\left(\frac{3}{8}+\frac{3}{64}x+\frac{3}{512}x^2+\ldots\right)\right)$$

$$=-\frac{3}{8}-\frac{51}{64}x+\frac{477}{512}x^2+\ldots$$

14 a $$\frac{12x+5}{(1+4x)^2}\equiv\frac{A}{1+4x}+\frac{B}{(1+4x)^2}$$

$$\equiv\frac{A(1+4x)+B}{(1+4x)^2}$$

$$12x+5\equiv A(1+4x)+B$$

Let $x=-\frac{1}{4}$:

$-3+5=0+B$

$B=2$

Let $x=0$:

$5=A\times1+B$

$5=A+2$

$A=3$

$A=3,\ B=2$

14 b $\dfrac{12x+5}{(1+4x)^2} \equiv \dfrac{3}{1+4x} + \dfrac{2}{(1+4x)^2}$

$= 3(1+4x)^{-1} + 2(1+4x)^{-2}$

$3(1+4x)^{-1} = 3\left(1 + (-1)(4x) + \dfrac{(-1)(-2)}{2!}(4x)^2 + \ldots\right)$

$= 3\left(1 - 4x + 16x^2 + \ldots\right)$

$= 3 - 12x + 48x^2 + \ldots$

$2(1+4x)^{-2} = 2\left(1 + (-2)(4x) + \dfrac{(-2)(-3)}{2!}(4x)^2 + \ldots\right)$

$= 2\left(1 - 8x + 48x^2 + \ldots\right)$

$= 2 - 16x + 96x^2 + \ldots$

$\dfrac{12x+5}{(1+4x)^2} = 3 - 12x + 48x^2 + 2 - 16x + 96x^2 + \ldots$

$= 5 - 28x + 144x^2 + \ldots$

15 a $\dfrac{9x^2+26x+20}{(1+x)(2+x)} \equiv A + \dfrac{B}{1+x} + \dfrac{C}{2+x}$

$$\begin{array}{r l}
 & 9 \\
x^2+3x+2 \big) & 9x^2+26x+20 \\
 & \underline{9x^2+27x+18} \\
 & -x+2
\end{array}$$

$A = 9$

$\dfrac{9x^2+26x+20}{(1+x)(2+x)} \equiv 9 + \dfrac{-x+2}{(1+x)(2+x)}$

$\dfrac{-x+2}{(1+x)(2+x)} \equiv \dfrac{B}{1+x} + \dfrac{C}{2+x}$

$= \dfrac{B(2+x)+C(1+x)}{(1+x)(2+x)}$

$-x+2 \equiv B(2+x) + C(1+x)$

Let $x = -1$:

$1+2 = B\times 1 + 0$:

$B = 3$

Let $x = -2$:

$2+2 = 0 + C\times(-1)$:

$C = -4$

15 a (continued)

$$\frac{9x^2+26x+20}{(1+x)(2+x)} \equiv 9+\frac{3}{1+x}-\frac{4}{2+x}$$

$$=9+3(1+x)^{-1}-4(2+x)^{-1}$$

$$=9+3\left(1+x\right)^{-1}-4\left(2\left(1+\frac{1}{2}x\right)\right)^{-1}$$

$$=9+3\left(1+x\right)^{-1}-2\left(1+\frac{1}{2}x\right)^{-1}$$

$$3\left(1+x\right)^{-1}=3\left(1+(-1)x+\frac{(-1)(-2)}{2!}x^2+\frac{(-1)(-2)(-3)}{3!}x^3+\ldots\right)$$

$$=3\left(1-x+x^2-x^3+\ldots\right)$$

$$=3-3x+3x^2-3x^3+\ldots$$

$$2\left(1+\frac{1}{2}x\right)^{-1}=2\left(1+(-1)\left(\frac{1}{2}x\right)+\frac{(-1)(-2)}{2!}\left(\frac{1}{2}x\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{1}{2}x\right)^3+\ldots\right)$$

$$=2\left(1-\frac{1}{2}x+\frac{1}{4}x^2-\frac{1}{8}x^3+\ldots\right)$$

$$=2-x+\frac{1}{2}x^2-\frac{1}{4}x^3+\ldots$$

$$9+\frac{3}{1+x}-\frac{4}{2+x}=9+3-3x+3x^2-3x^3+\ldots-\left(2-x+\frac{1}{2}x^2-\frac{1}{4}x^3+\ldots\right)$$

$$=10-2x+\frac{5}{2}x^2-\frac{11}{4}x^3+\ldots$$

Equating coefficients gives:

$$B=\frac{5}{2}, C=-\frac{11}{4}$$

15 b $q(0.1)=\dfrac{9(0.1)^2+26(0.1)+20}{(1+0.1)(2+0.1)}=9.822510823$

Using the expansion:

$$q(0.1)\approx 10-2(0.1)+\frac{5}{2}(0.1)^2-\frac{11}{4}(0.1)^3=9.82225$$

$$\text{Percentage error}=\frac{9.822510823-9.82225}{9.822510823}\times 100=0.0027\%$$

Challenge

$$f(x)=\frac{1}{\sqrt{1+3x^2}}=(\sqrt{1+3x^2})^{-1}$$

$=(1+3x^2)^{-\frac{1}{2}}$ Use binomial expansion with $n=-\dfrac{1}{2}$ and $x=3x^2$

$$=1+\left(-\frac{1}{2}\right)(3x^2)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(3x^2)^2}{2!}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(3x^2)^3}{3!}+\ldots$$

$$=1-\frac{3x^2}{2}+\frac{27x^4}{8}-\frac{135x^6}{16}+\ldots$$

Valid for $|3x^2|<1$

Review Exercise 1

1 Assumption: there are a finite number of prime numbers, p_1, p_2, p_3 up to p_n.

Let $X = (p_1 \times p_2 \times p_3 \times \ldots \times p_n) + 1$

None of the prime numbers $p_1, p_2, p_3 \ldots p_n$ can be a factor of X as they all leave a remainder of 1 when X is divided by them. But X must have at least one prime factor. This is a contradiction. So there are infinitely many prime numbers.

2 Assumption: $x = \frac{a}{b}$ is a solution to the equation, $x^2 - 2 = 0$, where a and b are integers with no common factors.

$$\left(\frac{a}{b}\right)^2 - 2 = 0 \Rightarrow \frac{a^2}{b^2} = 2 \Rightarrow a^2 = 2b^2$$

So a^2 is even, which implies that a is even.
Write $a = 2n$ for some integer n.

$$(2n)^2 = 2b^2 \Rightarrow 4n^2 = 2b^2 \Rightarrow 2n^2 = b^2$$

So b^2 is even, which implies that b is even.
This contradicts the assumption that a and b have no common factor. Hence there are no rational solutions to the equation.

3
$$\frac{4x}{x^2-2x-3} + \frac{1}{x^2+x} = \frac{4x}{(x-3)(x+1)} + \frac{1}{x(x+1)}$$
$$= \frac{4x(x)+1(x-3)}{x(x+1)(x-3)} = \frac{4x^2+x-3}{x(x+1)(x-3)}$$
$$= \frac{(x+1)(4x-3)}{(x+1)x(x-3)} = \frac{4x-3}{x(x-3)}$$

4 **a**
$$\mathrm{f}(x) = 1 - \frac{3}{x+2} + \frac{3}{(x+2)^2}$$
$$= \frac{(x+2)^2 - 3(x+2) + 3}{(x+2)^2}$$
$$= \frac{x^2+4x+4-3x-6+3}{(x+2)^2}$$
$$= \frac{x^2+x+1}{(x+2)^2}$$

4 b $x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}$

Use the method of completing the square

$$\geqslant\frac{3}{4}$$

As $\left(x+\frac{1}{2}\right)^2\geqslant 0$

$$>0$$

for all values of x, $x\neq 2$

c $\mathrm{f}(x)=\frac{x^2+x+1}{(x+2)^2}$ from (a)

$$\frac{x^2+x+1}{(x+2)^2}>0$$

as $x^2+x+1>0$ from (b)

and $(x+2)^2>0$, for $x\neq -2$

So $\mathrm{f}(x)>0$, for $x\neq 2$

5 $\frac{2x-1}{(x-1)(2x-3)}=\frac{A}{x-1}+\frac{B}{2x-3}$

$\Rightarrow 2x-1=A(2x-3)+B(x-1)$

Set $x=1$: $2(1)-1=1=A(2(1)-3)=-A$

$\Rightarrow A=-1$

Set $\boldsymbol{x}=\frac{\mathbf{3}}{\mathbf{2}}$: $2\left(\frac{3}{2}\right)-1=2=B\left(\frac{3}{2}-1\right)=\frac{1}{2}B$

$\Rightarrow B=4$

So $\frac{2x-1}{(x-1)(2x-3)}=\frac{-1}{x-1}+\frac{4}{2x-3}$

6 $\frac{3x+7}{(x+1)(x+2)(x+3)}=\frac{P}{x+1}+\frac{Q}{x+2}+\frac{R}{x+3}$

$\Rightarrow 3x+7=P(x+2)(x+3)Q(x+1)(x+3)+R(x+1)(x+2)$

Set $x=-1$: $3(-1)+7=4=P((-1)+2)((-1)+3)=2P$

$\Rightarrow P=2$

Set $x=-2$: $3(-2)+7=1=Q((-2)+1)((-2)+3)=-Q$

$\Rightarrow Q=-1$

Set $x=-3$: $3(-3)+7=-2=R((-3)+1)((-3)+2)=2R$

$\Rightarrow R=-1$

So $P=2, Q=-1, R=-1$

7 $$\frac{2}{(2-x)(1+x)^2}=\frac{A}{2-x}+\frac{B}{1+x}+\frac{C}{(1+x)^2}$$

$$\Rightarrow 2=A(1+x)^2+B(1+x)(2-x)+C(2-x)$$

Set $x=2$: $2=A(1+2)^2=9A$ so $A=\frac{2}{9}$

Set $x=-1$: $2=C[2-(-1)]=3C$ so $C=\frac{2}{3}$

Compare coefficients of x^2: $0=A-B$

$\Rightarrow B=A=\frac{2}{9}$

Solution: $A=\frac{2}{9}, B=\frac{2}{9}, C=\frac{2}{3}$

8 $$\frac{14x^2+13x+2}{(x+1)(2x+1)^2}\equiv\frac{A}{x+1}+\frac{B}{2x+1}+\frac{C}{(2x+1)^2}$$

You need denominators of $(x+1)$, $(2x+1)$ and $(2x+1)^2$

$$\equiv\frac{A(2x+1)^2+B(x+1)(2x+1)+C(x+1)}{(x+1)(2x+1)^2}$$

Add the three fractions

Compare numerators of fractions

$14x^2+13x+2=A(2x+1)^2+B(x+1)(2x+1)+C(x+1)$

Set the numerators equal

Put $x=-1$

$3=A+0+0\Rightarrow A=3$

To find A set $x=-1$

Put $x=-\frac{1}{2}$

$\frac{14}{4}-\frac{13}{2}+2=\frac{1}{2}C\Rightarrow C=-2$

To find C set $x=-\frac{1}{2}$

So $14x^2+13x+2=3(2x+1)^2+B(x+1)(2x+1)-2(x+1)$

Compare coefficients of x^2:

Equate terms in x^2

$14x^2=3(2x)^2+2Bx^2$

$14=12+2B\Rightarrow B=1$

Solve equation to find B

Check constant term

$2=3+1-2$

So $$\frac{14x^2+13x+2}{(x+1)(2x+1)^2}\equiv\frac{3}{x+1}+\frac{1}{2x+1}-\frac{2}{(2x+1)^2}$$

9 $\dfrac{3x^2+6x-2}{x^2+4} \equiv d + \dfrac{ex+f}{x^2+4}$

$\Rightarrow 3x^2+6x-2 = d(x^2+4)+ex+f$

Compare coefficients of x^2: $3 = d$

Compare coefficients of x: $6 = e$

Compare constant terms: $-2 = 4d + f$

So $f = -2-4d = -2-4(3) = -14$

Solution: $d = 3$, $e = 6$, $f = -14$

10 $\mathrm{p}(x) = \dfrac{9-3x-12x^2}{(1-x)(1+2x)} = A + \dfrac{B}{1-x} + \dfrac{C}{1+2x}$

$\Rightarrow 9-3x-12x^2 = A(1-x)(1+2x)+B(1+2x)+C(1-x)$

Set $x = 1$: $9 - 3(1) - 12(1)^2 = -6 = B(1 + 2(1)) = 3B$

$\Rightarrow B = -2$

Set $\boldsymbol{x} = -\dfrac{\mathbf{1}}{\mathbf{2}}$: $9-3\left(-\dfrac{1}{2}\right)-12\left(-\dfrac{1}{2}\right)^2 = \dfrac{15}{2} = C\left(1-\left(-\dfrac{1}{2}\right)\right) = \dfrac{3}{2}C$

$\Rightarrow C = 5$

Compare coefficients of x^2: $-12 = -2A$

$\Rightarrow A = 6$

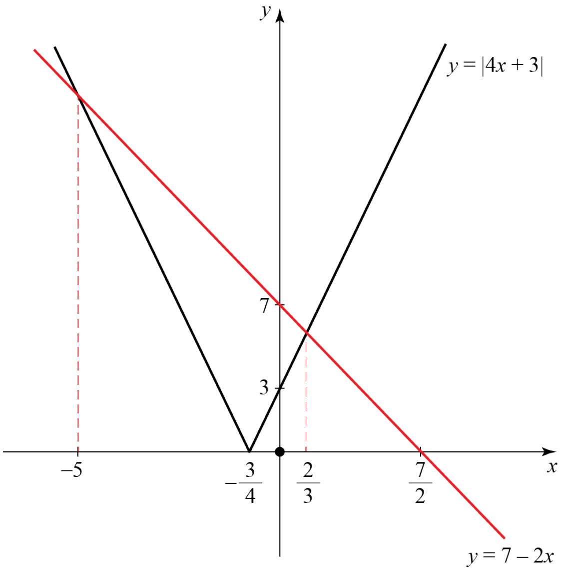
Solution: $A = 6$, $B = -2$, $C = 5$

11 First solve $|4x-3| = 7-2x$

$x > -\dfrac{3}{4}$: $4x+3 = 7-2x \Rightarrow x = \dfrac{2}{3}$

$x < -\dfrac{3}{4}$: $-(4x+3) = 7-2x \Rightarrow x = -5$

Now draw the lines $y = |4x+3|$ and $y = 7-2x$



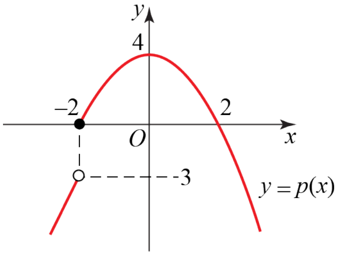
From the graph, we see that $|4x+3| > 7-2x$ when $x < -5$ or $x > \dfrac{2}{3}$

12 a For $x < -2$, p(x) is a straight line with gradient 4.

At $x = -2$, there is a discontinuity. $\mathrm{p}(-2) = 0$ so draw an open dot at $(-2, -3)$ where the line section ends and a solid dot at $(-2,\ 0)$ where p(x) is defined.

For $x > -2$, $\mathrm{p}(x) = 4 - x^2$. There is a maximum at (0, 4) since $x^2 \geqslant 0$, and the curve intersects the x-axis at (2, 0) since $4 - x^2 = 0 \Rightarrow x = \pm 2$

From the diagram, the range is $\mathrm{p}(x) \leqslant 4$



b $\mathrm{p}(a) = -20$

Check both sections of the domain for solutions.

$x < -2$: $4x + 5 = -20 \Rightarrow x = -\frac{25}{4}$

This is less than –2 so it is a solution.

$x \geqslant -2$: $4 - x^2 = -20 \Rightarrow x = \pm 2\sqrt{6}$

But $-2\sqrt{6} < -2$ so discard this possibility; $a = 2\sqrt{6} \geqslant 2$ so is a solution

Solutions are $a = -\frac{25}{4}, a = 2\sqrt{6}$

13 a
$$\begin{aligned}\mathrm{qp}(x) &= 2\left(\frac{1}{x+4}\right) - 5\\ &= \frac{2}{x+4} - \frac{5(x+4)}{x+4}\\ &= \frac{2 - 5x - 20}{x+4}\\ &= \frac{-5x - 18}{x+4}\end{aligned}$$

So $\mathrm{qp}(x) = \frac{-5x-18}{x+4}, x \in \mathbb{R}, x \neq -4$

Solutions are: $a = -5, b = -18, c = 1, d = 4$

b $\mathrm{qp}(x) = 15$

$$\begin{aligned}&\Rightarrow \frac{-5x-18}{x+4} = 15\\ &-5x - 18 = 15(x+4) = 15x + 60\\ &-5x - 18 = 15x + 60\\ &20x = -78\\ &x = -\frac{39}{10}\end{aligned}$$

13 c Let $y = \mathrm{r}(x)$

$$y = \frac{-5x-18}{x+4}$$

$$y(x+4) = -5x-18$$

$$x(y+5) = -4y-18$$

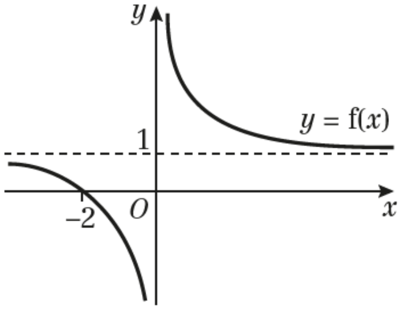
$$x = \frac{-4y-18}{y+5}$$

So $\mathrm{r}^{-1}(x) = \frac{-4x-18}{x+5}, \quad x \in \mathbb{R}, x \neq -5$

14 a $\frac{x+2}{x} = 1 + \frac{2}{x}$

Sketch $y = \frac{1}{x}$, stretch by a factor of 2

in the y-direction, translate by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



14 b $\mathrm{f}^2(x) = \mathrm{f}\left(\frac{x+2}{x}\right)$

$$= \frac{\frac{x+2}{2}+2}{\frac{x+2}{x}} \longleftarrow \boxed{\frac{\frac{x+2+2x}{x}}{\frac{x+2}{x}}}$$

$$= \frac{(3x+2)}{x} \times \frac{x}{(x+2)}$$

$$= \frac{3x+2}{x+2}$$

So $\mathrm{f}^2(x) = \frac{3x+2}{x+2}, x \in \mathbb{R}, x \neq 0, x \neq -2$

c $\mathrm{gf}\left(\frac{1}{4}\right) = \mathrm{g}\left(\frac{2\frac{1}{4}}{\frac{1}{4}}\right) = \mathrm{g}(9)$

$$= \ln(18-5)$$

$$= \ln 13$$

14 d Let $y = \ln(2x-5)$

$$e^y = 2x-5$$

$$\Rightarrow x = \frac{e^y+5}{2}$$

$$g^{-1}(x) = \frac{e^x+5}{2}, \quad x \in \mathbb{R}$$

The range of g(x) is $x \in \mathbb{R}$ so the domain of $g^{-1}(x)$ is $x \in \mathbb{R}$

15 a $pq(x) = 3(1-2x)+b = 3+b-6x$

$qp(x) = 1-2(3x+b) = 1-2b-6x$

As $pq(x) = qp(x)$

$\Rightarrow 3+b-6x = 1-2b-6x$

$$\Rightarrow b = -\frac{2}{3}$$

b Let $y = p(x)$

$$y = 3x - \frac{2}{3}$$

$$\Rightarrow x = \frac{2+3y}{9}$$

$$p^{-1}(x) = \frac{3x+2}{9}, x \in \mathbb{R}$$

Let $z = q(x)$

$z = 1-2x$

$$\Rightarrow x = \frac{1-z}{2}$$

$$q^{-1}(x) = \frac{1-x}{2}, x \in \mathbb{R}$$

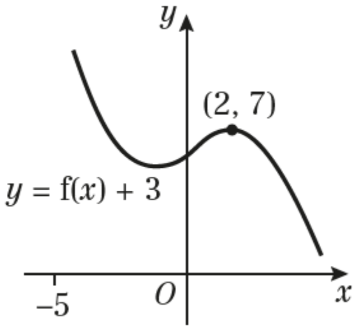
c $$p^{-1}q^{-1}(x) = \frac{2+3\left(\frac{1-x}{2}\right)}{9} = \frac{-3x+7}{18}, x \in \mathbb{R}$$

$$q^{-1}p^{-1}(x) = \frac{1-\frac{2+3x}{9}}{2} = \frac{-3x+7}{18}, x \in \mathbb{R}$$

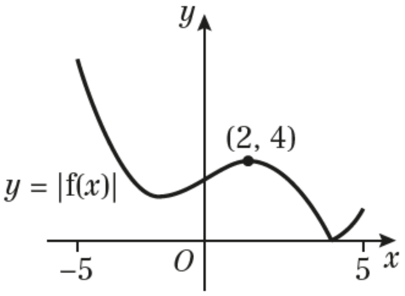
So $p^{-1}q^{-1}(x) = q^{-1}p^{-1}(x)$

And $a = -3, b = 7, c = 18$

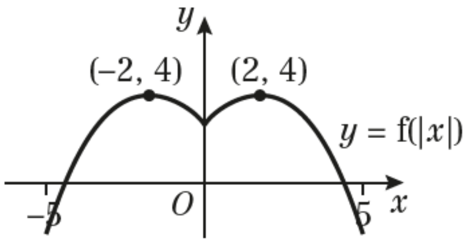
16 a Translation of $+3$ in the y direction. The maximum turning point is (2, 7).



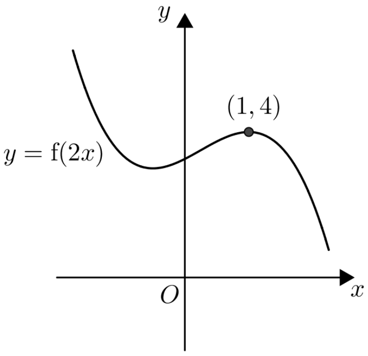
b For $y \geq 0$, curve is $y = \mathrm{f}(x)$
For $y < 0$, reflect in x-axis.
The maximum turning point is (2, 4)



c For $x < 0$, $\mathrm{f}|x| = \mathrm{f}(-x)$, so draw $y = \mathrm{f}(x)$ for $x \geqslant 0$, and then reflect this in $x = 0$
The maximum turning points are (–2, 4) and (2, 4)



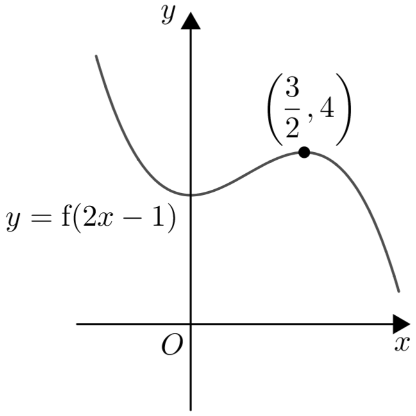
d $y = \mathrm{f}(2x-1)$ can be written as $y = \mathrm{f}(2(x-\frac{1}{2}))$
$y = \mathrm{f}(2x)$
Horizontal stretch, scale factor $\frac{1}{2}$.



16 d (continued)

$y = \mathrm{f}(2(x - \frac{1}{2}))$

Horizontal translation of $+\frac{1}{2}$



17 a To find intersections with the x-axis, solve $\mathrm{h}(x) = 0$

$2(x+3)^2 - 8 = 0$

$\Rightarrow (x+3)^2 = 4$

$\Rightarrow x = -3 \pm 2$

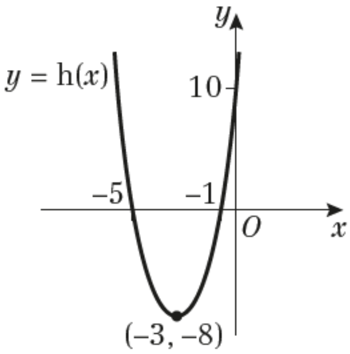
So there are intersections at (–5, 0) and (–1, 0)

To find intersections with the y-axis, find h(0)

$\mathrm{h}(0) = 2(3)^2 - 8 = 10$

So there is an intersection at (0, 10)

Since $(x + 3)2 \geqslant 0$, there is a turning point (minimum) at (–3, –8)



17 b i $y = 3\mathrm{h}(x+2)$

$\Rightarrow y = 3\left(2\left(x+2+3\right)^2 - 8\right)$

$\Rightarrow y = 6(x+5)^2 - 24$

This has a turning point when $x = -5$ at $(-5, -24)$

ii $y = \mathrm{h}(-x)$

$\Rightarrow y = 2(-x+3)^2 - 8$

$\Rightarrow y = 2(3-x)^2 - 8$

This has a turning point when $x = 3$ at $(3, -8)$

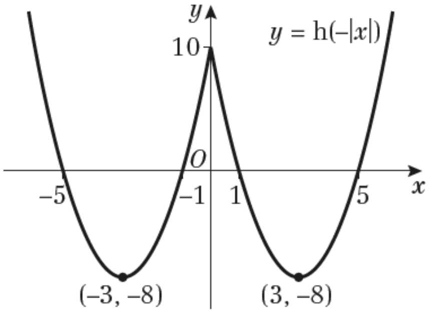
iii The modulus of $\mathrm{h}(x)$ is the curve in part (a), with the section for $-5 < x < -1$ reflected in the x-axis. The turning point is $(-3, 8)$

c On one graph, reflect $\mathrm{h}(x)$ in the y-axis to see what $\mathrm{h}(-x)$ looks like.

Now to obtain the sketch of $\mathrm{h}(-|x|)$, start a new graph,

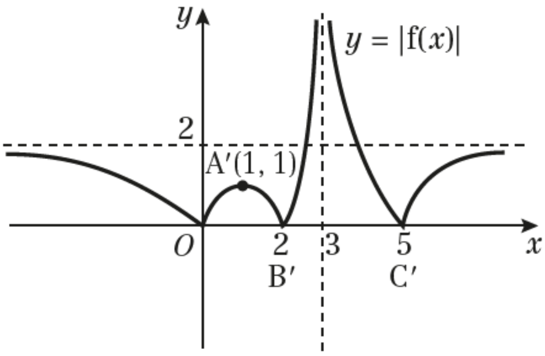
copy $\mathrm{h}(-x)$ for $x \geqslant 0$, then reflect the result in the y-axis.

The x-intercepts are $(-5, 0)$, $(-1, 0)$, $(1, 0)$, $(5, 0)$; the y-intercept is $(0, 10)$ and there are minimum turning points at $(-3, -8)$ and $(3, -8)$.



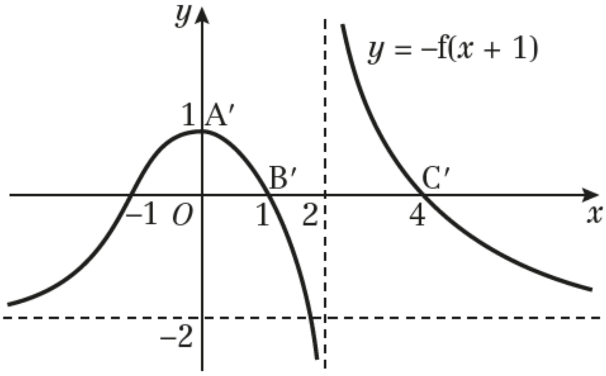
18 a i All parts of curve $y = \mathrm{f}(x)$ below the x-axis are reflected in x-axis.

$A \rightarrow (1, 1)$, B and C do not move.



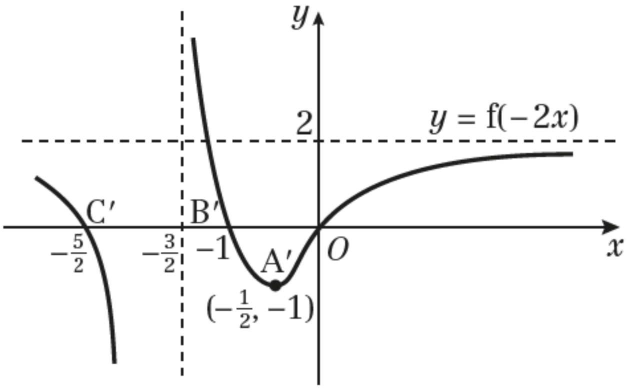
18 a ii Translate by -1 in the x direction and reflect in the x-axis.

$A \to (0, 1)$, $B \to (1, 0)$, $C \to (4, 0)$



iii Stretch in the x direction with scale factor $\frac{1}{2}$ and reflect in the y-axis.

$A \to (-\frac{1}{2}, -1)$, $B \to (-1, 0)$, $C \to (-\frac{5}{2}, 0)$



b i $3|f(x)| = 2 \Rightarrow |f(x)| = \frac{2}{3}$

Number of solutions is 6

ii $2|f(x)| = 3 \Rightarrow |f(x)| = \frac{3}{2}$

Number of solutions is 4

Consider graph **a i**

i How many times does the line $y = \frac{2}{3}$ cross the curve?

Line is below A'

ii Draw the line $y = \frac{3}{2}$

19 a $\mathrm{q}(x) = \frac{1}{2}|x+b| - 3$

$\mathrm{q}(0) = \frac{|b|}{2} - 3 = \frac{3}{2} \Rightarrow |b| = 9$

$b < 0$ so $b = -9$

b A is $(9, -3)$

To find B:

$x > 9$ so solve $\frac{1}{2}(x-9) - 3 = 0$

$\Rightarrow x = 15$

So B is (15, 0)

c $\mathrm{q}(x) = \frac{1}{2}|x-9| - 3 = -\frac{x}{3} + 5$

$$x < 9: \quad \frac{9-x}{2} - 3 = -\frac{x}{3} + 5$$
$$3(9-x) - 18 = -2x + 30$$
$$27 - 18 - 30 = x$$
$$x = -21$$

$$x > 9: \quad \frac{x-9}{2} - 3 = -\frac{x}{3} + 5$$
$$3(x-9) - 18 = -2x + 30$$
$$5x = 27 + 18 + 30$$
$$5x = 75$$
$$x = 15$$

Solution set; –21, 15

20 a $-\frac{5}{3}|x+4| \leqslant 0 \Rightarrow$ range is $\mathrm{f}(x) \leqslant 8$

b Over the whole domain, f(x) is not a one-one function so it cannot have an inverse.

20 c First solve $-\frac{5}{3}|x+4|+8=\frac{2}{3}x+4$

$$x<4:\ \frac{5}{3}(x+4)+8=\frac{2}{3}x+4$$

$$5(x+4)+24=2x+12$$

$$3x=12-24-20$$

$$x=-\frac{32}{3}$$

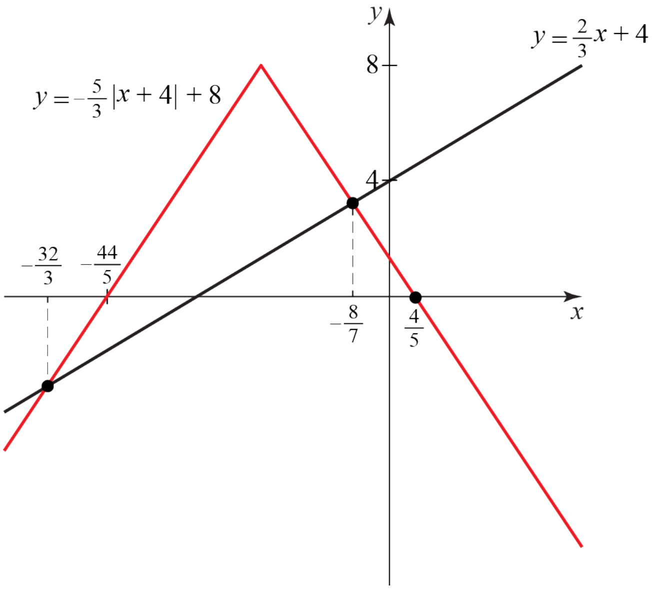
$$x>4:\ -\frac{5}{3}(x+4)+8=\frac{2}{3}x+4$$

$$-5(x+4)+24=2x+12$$

$$7x=-20+24-12$$

$$x=-\frac{8}{7}$$

Now sketch the lines $y=-\frac{5}{3}|x+4|+8$ and $y=\frac{2}{3}x+4$



From the graph we see that the inequality is satisfied in the region

$$-\frac{32}{3}<x<-\frac{8}{7}$$

d From the sketch drawn from part (c), the equation will have no solutions if the line lies above the apex of f(x) at (–4, 8)

$$\Rightarrow \frac{5}{3}(-4)+k>8$$

$$\Rightarrow k>8+\frac{20}{3}$$

$$\Rightarrow k>\frac{44}{3}$$

21 a $12-7k+d=3k^2 \Rightarrow 3k^2+7k-12=d$

$3k^2+d=k^2-10k \Rightarrow -2k^2-10k=d$

Subtracting the second equation from the first gives

$5k^2+17k-12=(5k-3)(k+4)=0$

So $k=\frac{3}{5}=0.6$ or $k=-4$

b Since the sequence contains only integer terms, $k=-4$.

$u_4=12-7(-4)=40$ **,** $u_5=3(-4)^2=48$

So common difference d is $d=u_5-u_4=48-40=8$

The first term a satisfies

$a+3d=u_4 \Rightarrow a=40-3(8)=16$

So $a=16$**,** $d=8$

22 a First find the common difference and first term.

$u_4=a+3d=72 \quad (1)$

$u_{11}=a+10d=51 \quad (2)$

$(1)-(2)$: $\ -7d=21 \Rightarrow d=-3$

Into (1): $\ a=72-3(-3)=81$

Now, using $S_n=\frac{n}{2}(2a+(n-1)d)$

$S_n=\frac{n}{2}(2(81)+(n-1)(-3))=1125$

$\Rightarrow n(162-3n+3)=2250$

$\Rightarrow -3n^2+165n=2250$

$\Rightarrow 3n^2-165n+2250=0$

b $3n^2-165n+2250=0$

$\Rightarrow n^2-55n+750=0$

$\Rightarrow (n-25)(n-30)=0$

$\Rightarrow n=25, n=30$

23 a $a = 19p - 18$

$d = u_2 - a = (17p - 8) - (19p - 18) = 10 - 2p$

So $u_{30} = a + 29d = (19p - 18) + 29(-2p + 10)$

$u_{30} = 272 - 39p$

b $S_{31} = \frac{31}{2}(2a + (31-1)d) = 0$

$\Rightarrow 2a + 30d = 0$

So $2(19p - 18) + 30(10 - 2p) = 0$

$(38 - 60)p - 36 + 300 = 0$

$22p = 264$

$p = 12$

24 a $u_2 = ar = 256, u_8 = ar^7 = 900$

$\frac{ar^7}{ar} = \frac{900}{256}$

$\Rightarrow r^6 = \frac{225}{64}$

$\Rightarrow \ln r^6 = \ln\left(\frac{225}{64}\right)$

$\Rightarrow 6\ln r - \ln\left(\frac{225}{64}\right) = 0$ (as $\ln x^k = k\ln x$)

$\Rightarrow 6\ln r + \ln\left(\frac{64}{225}\right) = 0$ (as $\ln x^{-1} = -\ln x$)

b Noting $r > 1$, so r is positive

$r = \left(\frac{225}{64}\right)^{\frac{1}{6}} = 1.2331060\ldots = 1.23$ (3 s.f.)

25 a $r = \frac{ar}{a} = \frac{u_2}{u_1} = \frac{ar}{r}$

So $r = \frac{\frac{50}{6}}{10} = \frac{5}{6}$

$\therefore$ As $|r| < 1, S_\infty = \frac{a}{1-r} = \frac{10}{1-\frac{5}{6}} = 60$

25 b $a = 10, r = \frac{5}{6}$

$$S_k = \frac{10\left(1-\left(\frac{5}{6}\right)^k\right)}{1-\frac{5}{6}}$$

As $S_k > 55 \Rightarrow 1-\left(\frac{5}{6}\right)^k > \frac{55}{60}$

$$\Rightarrow 1-\left(\frac{5}{6}\right)^k > \frac{11}{12}$$

$$\Rightarrow \frac{1}{12} > \left(\frac{5}{6}\right)^k \Rightarrow \log\left(\frac{1}{12}\right) > \log\left(\frac{5}{6}\right)^k$$

$$\Rightarrow \log\left(\frac{1}{12}\right) > k\log\left(\frac{5}{6}\right)$$

$$\Rightarrow k > \frac{\log\left(\frac{1}{12}\right)}{\log\left(\frac{5}{6}\right)}$$

(the inequality reverses direction in the final step because $\ln\frac{5}{6} < 0$)

c k must be a positive integer.

$$\frac{\ln\frac{1}{12}}{\ln\frac{5}{6}} = 13.629 \text{ (3 d.p.)}$$

So the minimum value of k is 14.

26 a $4,\ 4r,\ 4r^2,\ldots$

$4+4r+4r^2 = 7$

$4r^2+4r-3=0$ (as required)

Use ar^{n-1} to write down expressions for the first three terms. Here $a = 4$ and $n = 1, 2, 3$

b $4r^2+4r-3=0$

$(2r-1)(2r+3)=0$

$r=\frac{1}{2},\ r=-\frac{3}{2}$

Factorise $4r^2+4r-3 = -12$

$(-2)+(+6) = +4,$ so

$$4r^2-2r+6r-3 = 2r(2r-1)+3(2r-1)$$
$$= (2r-1)(2r+3)$$

c $r=\frac{1}{2}$

$$\frac{a}{1-r} = \frac{4}{1-\frac{1}{2}}$$
$$= \frac{4}{\frac{1}{2}}$$
$$= 8$$

Use $S_\infty = \frac{a}{1-r}$

Here $a = 4$ and $r = \frac{1}{2}$

27 a $ar^3 = x,\ ar^4 = 3,\ ar^5 = x + 8$

$$\frac{ar^5}{ar^4} = \frac{ar^4}{ar^3}$$

$$\text{so } \frac{x+8}{3} = \frac{3}{x}$$

$$x(x+8) = 9$$

$$x^2 + 8x - 9 = 0$$

$$(x+9)(x-1) = 0$$

$$x = 1,\ x = -9$$

$\frac{ar^5}{ar^4} = r$ and $\frac{ar^4}{ar^3} = r$ so $\frac{ar^5}{ar^4} = \frac{ar^4}{ar^3}$

Clear the fractions. Multiply each side by $3x$

so that $3x \times \frac{x+8}{3} = x(x+8)$ and $3x \times \frac{3}{x} = 9$

$$r = \frac{ar^4}{ar^3} = \frac{3}{x}$$

When $x = 1,\ r = 3$

When $x = -9,\ r = -\frac{1}{3}$

Find r. Substitute $x = 1$, then $x = -9$, into $r = \frac{ar^4}{ar^3} = \frac{3}{x}$

b $$r = -\frac{1}{3}$$

$$ar^4 = 3$$

$$a\left(-\frac{1}{3}\right)^4 = 3$$

$$a = 243$$

Remember $S_\infty = \frac{a}{1-r}$ for $|r| < 1$, so $r = -\frac{1}{3}$

c $S_\infty = \frac{a}{1-r} = \frac{243}{1+\frac{1}{3}} = 182.25$

28 a $a_{n+1} = 3a_n + 5$

$n = 1: a_2 = 3a_1 + 5$

$a_2 = 3k + 5$

Use the given formula with $n = 1$

b $n = 2:\ a_3 = 3a_2 + 5$

$= 3(3k + 5) + 5$

$= 9k + 15 + 5$

$= 9k + 20$

c i $\sum_{r=1}^{4} a_r = a_1 + a_2 + a_3 + a_4$

$n = 3: a_4 = 3a_3 + 5$

$= 3(9k + 20) + 5$

$= 27k + 65$

This is *not* an arithmetic series.

You cannot use a standard formula, so work out each separate term and then add them together to find the required sum.

$\sum_{r=1}^{4} a_r = k + (3k + 5) + (9k + 20) + (27k + 65)$

$= 40k + 90$

28 c ii $\sum_{r=1}^{4} a_r = 10(4k+9)$

There is a factor of 10, so the sum is divisible by 10.

Give a conclusion.

29 a $a = 2400,\ r = 1.06$

After 4 years,

$2400(1.06)^3 = 2858.44\ldots = 2860$ to the nearest 10.

b $2400 \times 1.06^{N-1} > 6000 \Rightarrow 1.06^{N-1} > 2.5$

$\Rightarrow \log 1.06^{N-1} > \log 2.5 \Rightarrow (N-1)\log 1.06 > \log 2.5$

c Rearranging the inequality

$$N > \frac{\ln 2.5}{\ln 1.06} + 1 = 16.7 \text{ (1 d.p.)}$$

So $N = 17$

d The total amount raised is $5(S_{10})$

$$S_{10} = \frac{a\left(1-r^{10}\right)}{1-r} = \frac{2400\left(1-1.06^{10}\right)}{1.106} = 31633.90 \text{ (2 d.p.)}$$

Therefore the total amount raised is 5 × 31633.9, which to the nearest £1000 is £158,000

30 a Common ratio is $r = -4x$

Condition for the convergence of infinite sum is

$|r| < 1 \Rightarrow |-4x| < 1$

$\Rightarrow |x| < \frac{1}{4}$

b $\sum_{r=1}^{\infty} 6 \times (-4x)^{r-1} = S_\infty = \frac{24}{5}$

Another equation for S_∞ is $S_\infty = \frac{a}{1-r} = \frac{6}{1+4x}$

So $\frac{6}{1+4x} = \frac{24}{5}$

$\Rightarrow 30 = 24 + 96x$

$\Rightarrow x = \frac{6}{96} = \frac{1}{16}$

31 a Using the binomial expansion

$$\begin{aligned} g(x) &= (1-x)^{-\frac{1}{2}} \\ &= 1+\left(-\tfrac{1}{2}\right)(-x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x)^3+\ldots \\ &= 1+\frac{x}{2}+\frac{3x^2}{8}+\frac{5x^3}{16}+\ldots \end{aligned}$$

b $|x|<1$

32 a $(1+ax)^n \equiv 1+nax+\dfrac{n(n-1)}{2}a^2x^2+\ldots$

$na=-6 \qquad (1)$

$\dfrac{n(n-1)}{2}a^2=45 \qquad (2)$

Set coefficient of x, from binomial expansion, equal to –6 and set coefficient of x^2 equal to 45

From equation (1) $\quad a=-\dfrac{6}{n}$

Substitute into equation (2)

$$\frac{n(n-1)}{2}\times\frac{36}{n^2}=45$$

Eliminate a from the simultaneous equations to obtain an equation in one variable n

$$\begin{aligned} 36n^2-36n &= 90n^2 \\ -36n &= 54n^2 \end{aligned}$$

Solve to find non-zero value for n

$\Rightarrow n=0$ or $n=-\dfrac{36}{54}=-\dfrac{2}{3}$

Substitute into equation (1) to give $a=9$

Check solutions in equation (2)

b Coefficient of x^3

$$\begin{aligned} &= \frac{n(n-1)(n-2)}{3!}a^3 \\ &= \frac{-\frac{2}{3}\times-\frac{5}{3}\times-\frac{8}{3}\times 9^3}{3!} \\ &= \frac{-80\times 27}{6} \\ &= -360 \end{aligned}$$

Substitute values found for n and a into the binomial expansion to give the coefficient of x^3

c The expansion is valid if $|9x|<1$

So $-\dfrac{1}{9}<x<\dfrac{1}{9}$

The terms in the expansion are $(9x)$, $(9x)^2$, $(9x)^3$… and so $|9x|<1$

33 a Using the binomial expansion

$$(1+4x)^{\frac{3}{2}} = 1+\left(\frac{3}{2}\right)(4x)+\frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{2!}(4x)^2+\frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{3!}(4x)^3+\ldots$$

$$= 1+6x+6x^2-4x^3+\ldots$$

b $$\left(1+4\left(\frac{3}{100}\right)\right)^{\frac{3}{2}} = \left(\frac{112}{100}\right)^{\frac{3}{2}}$$

$$= \left(\sqrt{\frac{112}{100}}\right)^3$$

$$= \frac{112\sqrt{122}}{1000}$$

c $$1+6\left(\frac{3}{100}\right)+6\left(\frac{3}{100}\right)^2-4\left(\frac{3}{100}\right)^3 = 1.185292$$

So $\frac{112\sqrt{112}}{1000} \approx 1.185292$

$$\Rightarrow \sqrt{112} \approx \frac{1185.292}{112} = 10.582962857\ldots = 10.58296 \text{ (5 d.p.)}$$

d Using a calculator $\sqrt{112} = 10.5830052$ (7 d.p.)

Percentage error $= \frac{10.5830052-10.5829643}{10.5830052}\times 100 = 0.00039\%$ (5 d.p.)

Note, you will get different answers if you use values rounded to 5 d.p. in calculating the percentage error.

34 Expand $(3+2x)^{-3}$ using the binomial expansion:

$$(3+2x)^{-3}=3^{-3}\left(1+\frac{2}{3}x\right)^{-3}$$

$$=\frac{1}{27}\left(1+(-3)\left(\frac{2}{3}x\right)+\frac{(-3)(-4)}{2!}\left(\frac{2}{3}x\right)^2+\frac{(-3)(-4)(-5)}{3!}\left(\frac{2}{3}x\right)^3+\ldots\right)$$

$$=\frac{1}{27}\left(1-2x+\frac{8}{3}x^2-\frac{80}{27}x^3+\ldots\right)$$

So $(1+x)(3+2x)^{-3}=\frac{1}{27}(1+x)\left(1-2x+\frac{8}{3}x^2-\frac{80}{27}x^3+\ldots\right)$

$$=\frac{1}{27}\left(1+(-2+1)x+\left(\frac{8}{3}-2\right)x^2+\left(-\frac{80}{27}+\frac{8}{3}\right)x^3+\ldots\right)$$

$$=\frac{1}{27}-\frac{1}{27}x+\frac{2}{81}x^2-\frac{8}{729}x^3+\ldots$$

35 a $\mathrm{h}(x)=(4-9x)^{\frac{1}{2}}=2\left(1-\frac{9}{4}x\right)^{\frac{1}{2}}$

So using the binomial expansion

$$\mathrm{h}(x)=2\left(1+\left(\frac{1}{2}\right)\left(-\frac{9}{4}x\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(-\frac{9}{4}x\right)^2+\ldots\right)$$

$$=2\left(1-\frac{9}{8}x-\frac{81}{128}x^2+\ldots\right)$$

$$=2-\frac{9}{4}x-\frac{81}{64}x^2+\ldots$$

b $\mathrm{h}\left(\frac{1}{100}\right)=\left(4-\frac{9}{100}\right)^{\frac{1}{2}}=\left(\frac{400-9}{100}\right)^{\frac{1}{2}}=\frac{\sqrt{391}}{10}$

c $\mathrm{h}\left(\frac{1}{100}\right)\approx 2-\frac{9}{4}\left(\frac{1}{100}\right)-\frac{81}{64}\left(\frac{1}{100}\right)^2=1.97737$ (5 d.p.)

36 a $$(a+bx)^{-2}=\frac{1}{a^2}\left(1+\frac{b}{a}x\right)^{-2}$$

$$=\frac{1}{a^2}\left(1+(-2)\left(\frac{b}{a}x\right)+\frac{(-2)(-3)}{2!}\left(\frac{b}{a}x\right)^2+\ldots\right)$$

$$=\frac{1}{a^2}-\frac{2b}{a^3}x+\frac{3b^2}{a^4}x^2+\ldots$$

$$=\frac{1}{4}+\frac{1}{4}x+cx^2\ldots$$

So $\frac{1}{a^2}=\frac{1}{4}\Rightarrow a^2=4\Rightarrow a=\pm 2$

When $a=2$, comparing the x coefficient gives

$$-\frac{2b}{a^3}=\frac{1}{4}\Rightarrow b=-\frac{a^3}{8}=-1$$

Comparing the x^2 coefficient gives

$$c=\frac{3b^2}{a^4}=\frac{3}{2^4}=\frac{3}{16}$$

So one solution is $a=2,\ b=-1,\ c=\frac{3}{16}$

When $a=-2$, comparing the x coefficient gives

$$-\frac{2b}{a^3}=\frac{1}{4}\Rightarrow b=-\frac{a^3}{8}=1$$

Comparing the x^2 coefficient gives

$$c=\frac{3b^2}{a^4}=\frac{3}{2^4}=\frac{3}{16}$$

So second solution is $a=-2,\ b=1,\ c=\frac{3}{16}$

Note that the two solutions yield the same expression

$$(2-x)^{-2}=\left(-1\times(x-2)\right)^{-2}=(-1)^{-2}(x-2)^{-2}=(x-2)^{-2}$$

b Coefficient of x^3 in expansion of $(x-2)^{-2}$

$$\frac{1}{4}\frac{(-2)(-3)(-4)}{3!}\left(-\frac{1}{2}\right)^3=\frac{1}{8}$$

37 a $$\frac{3+5x}{(1+3x)(1-x)}=\frac{A}{1+3x}+\frac{B}{1-x}$$

$\Rightarrow 3+5x=A(1-x)+B(1+3x)$

Set $x=1$: $8=4B\Rightarrow B=2$

Set $x=-\frac{1}{3}$: $\frac{4}{3}=\frac{4}{3}A\Rightarrow A=1$

37 b $$\frac{3+5x}{(1+3x)(1-x)} = (1+3x)^{-1} + 2(1-x)^{-1}$$
$$= \left(1+(-1)(3x)+\frac{(-1)(-2)}{2!}(3x)^2+\ldots\right)+2\left(1+(-1)(-x)+\frac{(-1)(-2)}{2!}(-x)^2+\ldots\right)$$
$$= (1+2)+(-3x+2x)+(9x^2+2x^2)+\ldots$$
$$= 3-x+11x^2+\ldots$$

38 a $$\frac{3x-1}{(1-2x)^2} = \frac{A}{1-2x}+\frac{B}{(1-2x)^2}$$

$\Rightarrow 3x-1 = A(1-2x)+B$

Set $x=\frac{1}{2}$: gives B= $\frac{1}{2}$

Compare coefficients of x gives $3=-2A \Rightarrow A=-\frac{3}{2}$

b $$\frac{3x-1}{(1-2x)^2} = -\frac{3}{2}(1-2x)^{-1}+\frac{1}{2}(1-2x)^{-2}$$

Expand each term using the bionomial expansion

$$-\frac{3}{2}(1-2x)^{-1} = -\frac{3}{2}\left(1+(-1)(-2x)+\frac{(-1)(-2)}{2!}(-2x)^2+\frac{(-1)(-2)(-3)}{3!}(-2x)^3+\ldots\right)$$
$$\frac{1}{2}(1-2x)^{-2} = \frac{1}{2}\left(1+(-2)(-2x)+\frac{(-2)(-3)}{2!}(-2x)^2+\frac{(-2)(-3)(-4)}{3!}(-2x)^3+\ldots\right)$$

Now sum the expansions

$$-\frac{3}{2}(1-2x)^{-1}+\frac{1}{2}(1-2x)^{-2} = \left(-\frac{3}{2}+\frac{1}{2}\right)+\left(-3x+2x\right)+(-6x^2+6x^2)+(-12x^3+16x^3)+\ldots$$
$$= -1-x+4x^3+\ldots$$

39 a $$\mathrm{f}(x) = \frac{25}{(3+2x)^2(1-x)} = \frac{A}{3+2x}+\frac{B}{(3+2x)^2}+\frac{C}{1-x}$$

$\Rightarrow 25 = A(3+2x)(1-x)+B(1-x)+C(3+2x)^2$

Set $x=1$: $25=25C \Rightarrow C=1$

Set $x=-\frac{3}{2}$: $25=\frac{5}{2}B \Rightarrow B=10$

Compare the coefficients of x^2

$0=-2A+4C \Rightarrow A=2C=2$

So $A=2, B=10, C=1$

39 b From part (a) $\mathrm{f}(x)=2(3+2x)^{-1}+10(3+2x)^{-2}+(1-x)^{-1}$

$$=\frac{2}{3}\left(1+\frac{2}{3}x\right)^{-1}+\frac{10}{9}\left(1+\frac{2}{3}x\right)^{-2}+(1-x)^{-1}$$

Now expand each part of the equation using the binomial expansion

$$\mathrm{f}(x)=\frac{2}{3}\left(1+(-1)\left(\frac{2}{3}x\right)+\frac{(-1)(-2)}{2!}\left(\frac{2}{3}x\right)^{2}+\ldots\right)+\frac{10}{9}\left(1+(-2)\left(\frac{2}{3}x\right)+\frac{(-2)(-3)}{2!}\left(\frac{2}{3}x\right)^{2}+\ldots\right)$$

$$+\left(1+(-1)(-x)+\frac{(-1)(-2)}{2!}(-x)^{2}+\ldots\right)$$

$$=\left(\frac{2}{3}+\frac{10}{9}+1\right)+\left(-\frac{4}{9}x-\frac{40}{27}x+x\right)+\left(\frac{8}{27}x^{2}+\frac{40}{27}x^{2}+x^{2}\right)+\ldots$$

$$=\frac{25}{9}-\frac{25}{27}x+\frac{25}{9}x^{2}+\ldots$$

40 a $\dfrac{40x^{2}+30x+31}{(x+4)(2x+3)}=A+\dfrac{B}{x+4}+\dfrac{C}{2x+3}$

$\Rightarrow 4x^{2}+30x+31=A(x+4)(2x+3)+B(2x+3)+C(x+4)$

Set $x=-4$: $64-120+31=-25=-5B\Rightarrow B=5$

Set $x=-\dfrac{3}{2}$: $9-45+31=-5=\dfrac{5}{2}C\Rightarrow C=-2$

Compare coefficients of x^{2}

$4=2A\Rightarrow A=2$

Solution: $A=2,\ B=5,\ C=-2$

b $2+5(x+4)^{-1}-2(2x+3)^{-1}$

Rewrite as $\mathrm{f}(x)=2+\dfrac{5}{4}\left(1+\dfrac{x}{4}\right)^{-1}-\dfrac{2}{3}\left(1+\dfrac{2}{3}x\right)^{-1}$

$$\mathrm{f}(x)=2+\frac{5}{4}\left(1+(-1)\left(\frac{x}{4}\right)+\frac{(-1)(-2)}{2!}\left(\frac{x}{4}\right)^{2}+\ldots\right)-\frac{2}{3}\left(1+(-1)\left(\frac{2}{3}x\right)+\frac{(-1)(-2)}{2!}\left(\frac{2}{3}x\right)^{2}+\ldots\right)$$

$$=\left(2+\frac{5}{4}-\frac{2}{3}\right)+\left(-\frac{5}{16}x+\frac{4}{9}x\right)+\left(\frac{5}{64}x^{2}-\frac{8}{27}x^{2}\right)+\ldots$$

$$=\frac{31}{12}+\frac{19}{144}x-\frac{377}{1728}x^{2}+\ldots$$

Challenge

1 a B is located where $\mathbf{g}(\boldsymbol{x}) = -\frac{\mathbf{3}}{\mathbf{4}}\boldsymbol{x} + \frac{\mathbf{3}}{\mathbf{2}} = \mathbf{0} \Rightarrow \boldsymbol{x} = \mathbf{2}$

So B has coordinates $(2, 0)$

To find A solve f(x) = g(x) for $x < -3$

$$3(x+3)+15 = -\frac{3}{4}x + \frac{3}{2}$$
$$\Rightarrow 12x + 96 = -3x + 6$$
$$\Rightarrow 15x = -90$$
$$\Rightarrow x = -6$$
$$\text{g}(-6) = \text{f}(-6) = 6$$

So A has coordinates $(-6, 6)$

M is the midpoint of A and so has coordinates $\left(\frac{-6+2}{2}, \frac{6+0}{2}\right) = (-2, 3)$

To find the radius of the circle, use Pythagoras' theorem to find the length of MA:

$$|MA| = \sqrt{(2-(-2))^2 + (3-0)^2} = \sqrt{25} = 5$$

Therefore the equation of the circle is

$$(x+2)^2 + (y-3)^2 = 25$$

b For $x < -3$, $\text{f}(x) = 3(x+3)+15 = 3x+24$

Substituting $y = 3x + 24$ into the equation of the circle

$$(x+2)^2 + (3x+21)^2 = (x+2)^2 + 9(x+7)^2 = 25$$
$$\Rightarrow 10x^2 + 130x + 420 = 0$$
$$\Rightarrow x^2 + 13x + 42 = 0$$
$$\Rightarrow (x+7)(x+6) = 0$$

Solutions $x = -7$, $x = -7$

From the diagram, at P $x = -7$, and $\text{f}(x) = -12 + 15 = 3$

So P has coordinates $(-7, 3)$

Angle $\angle APB = 90^o$ by circle theorems so the area of the triangle is $\frac{1}{2}|AP||PB|$

$$|AP| = \sqrt{1^2 + 3^2} = \sqrt{10}$$
$$|PB| = \sqrt{9^2 + 3^2} = \sqrt{90} = 3\sqrt{10}$$
$$\text{Area} = \frac{1}{2}(\sqrt{10})(3\sqrt{10}) = 15$$

2 The general term of the sequence is

$$a_m = m + (m-1)k$$

$$\Rightarrow \sum_{i=6}^{11} a_i = 6m + (5+6+7+8+9+10)k = 6m + 45k$$

$$\Rightarrow \sum_{i=12}^{15} a_i = 4m + (11+12+13+14)k = 4m + 50k$$

So $6m + 45k = 4m + 50k \Rightarrow m = \frac{5}{2}k$

3 $\mathrm{p}(x) = |x^2 - 8x + 12| = |(x-6)(x-2)|$

$\mathrm{q}(x) = |x^2 - 11x + 28| = |(x-4)(x-7)|$

To find the x-coordinate of A solve

$$-x^2 + 8x - 12 = x^2 - 11x + 28$$

$$\Rightarrow 2x^2 - 19x + 40 = 0$$

$$\Rightarrow x = \frac{19 - \sqrt{361 - 4(2)(40)}}{2(2)} = \frac{19 - \sqrt{41}}{4}$$

Using the quadratic formula, and from the graph we know to take the negative square root.

To find the x-coordinate of B solve

$$-x^2 + 8x - 12 = -x^2 + 11x - 28$$

$$\Rightarrow x = \frac{16}{3}$$

To find the x-coordinate of C solve

$$x^2 - 8x + 12 = -x^2 + 11x - 28$$

$$\Rightarrow 2x^2 - 19x + 40 = 0$$

$$\Rightarrow x = \frac{19 + \sqrt{41}}{4}$$

Taking the positive square root this time.

Solution is $A : \frac{19 - \sqrt{41}}{4}, B : \frac{16}{3}, C : \frac{19 + \sqrt{41}}{4}$

4 $$\sum_{r=1}^{40} \log_3\left(\frac{2n+1}{2n-1}\right) = \log_3\left(\frac{3}{1}\right) + \log_3\left(\frac{5}{3}\right) + \ldots + \log_3\left(\frac{79}{77}\right) + \log_3\left(\frac{81}{79}\right)$$

$$= \log_3\left(\frac{3}{1} \times \frac{5}{3} \times \ldots \times \frac{79}{77} \times \frac{81}{79}\right)$$

$$= \log_3 81$$

$$= 4$$

5 $y = \mathrm{f}(ax+b)$ is a stretch by horizontal scale factor $\frac{1}{a}$ followed by a translation $\begin{pmatrix} -\frac{b}{a} \\ 0 \end{pmatrix}$.

Point (x, y) maps to point $\left(\frac{x}{a} - \frac{b}{a}, y\right)$.

So (x, y) invariant implies that: $\frac{x}{a} - \frac{b}{a} = x \Rightarrow x = \frac{b}{1-a}$

Radians 5A

1 a $\frac{\pi}{20} \times \frac{180}{\pi} = 9°$

b $\frac{\pi}{15} \times \frac{180}{\pi} = 12°$

c $\frac{5\pi}{12} \times \frac{180}{\pi} = 75°$

d $\frac{5\pi}{4} \times \frac{180}{\pi} = 225°$

e $\frac{3\pi}{2} \times \frac{180}{\pi} = 270°$

f $3\pi \times \frac{180}{\pi} = 540°$

2 a $0.46 \times \frac{180}{\pi} = 26.4°$

b $1 \times \frac{180}{\pi} = 57.3°$

c $1.135 \times \frac{180}{\pi} = 65.0°$

d $\sqrt{3} \times \frac{180}{\pi} = 99.2°$

3 a $\sin(0.5\,\text{rad}) = 0.479$

b $\cos(\sqrt{2}\ \text{rad}) = 0.156$

c $\tan(1.05\,\text{rad}) = 1.74$

d $\sin(2\,\text{rad}) = 0.909$

e $\sin(3.6\,\text{rad}) = -0.443$

4 a $8 \times \frac{\pi}{180} = \frac{2\pi}{45}$

b $10 \times \frac{\pi}{180} = \frac{\pi}{18}$

c $22.5 \times \frac{\pi}{180} = \frac{\pi}{8}$

d $30 \times \frac{\pi}{180} = \frac{\pi}{6}$

e $112.5 \times \frac{\pi}{180} = \frac{5\pi}{8}$

f $240 \times \frac{\pi}{180} = \frac{4\pi}{3}$

g $270 \times \frac{\pi}{180} = \frac{3\pi}{2}$

h $315 \times \frac{\pi}{180} = \frac{7\pi}{4}$

i $330 \times \frac{\pi}{180} = \frac{11\pi}{6}$

5 a $50 \times \frac{\pi}{180} = 0.873\,\text{rad}$

b $75 \times \frac{\pi}{180} = 1.31\,\text{rad}$

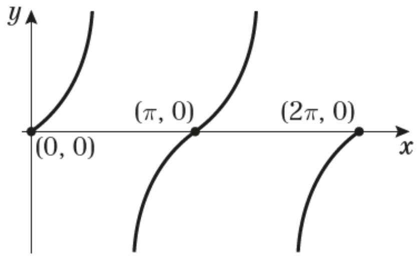
c $100 \times \frac{\pi}{180} = 1.75\,\text{rad}$

d $160 \times \frac{\pi}{180} = 2.79\,\text{rad}$

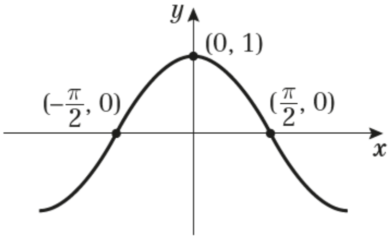
e $230 \times \frac{\pi}{180} = 4.01\,\text{rad}$

f $320 \times \frac{\pi}{180} = 5.59\,\text{rad}$

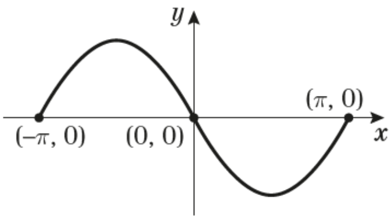
6 a



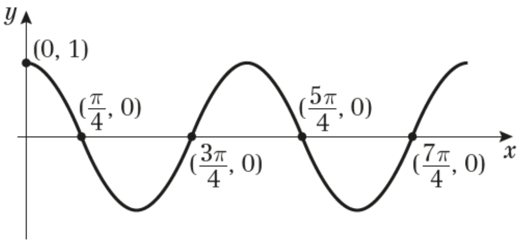
6 b



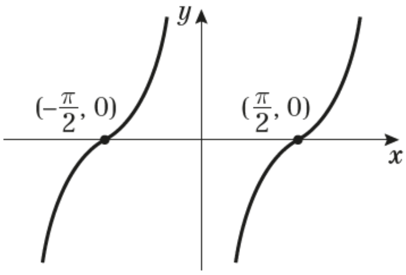
7 a



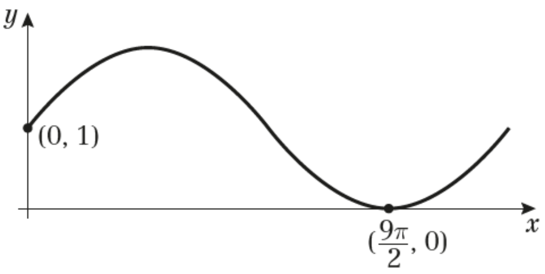
b



c



d



8 $(0, -0.5)$

$$\left(-\frac{11\pi}{6}, 0\right), \left(-\frac{5\pi}{6}, 0\right), \left(\frac{\pi}{6}, 0\right), \left(\frac{7\pi}{6}, 0\right)$$

Challenge

a $\cos\theta = 1$

$$\theta = 0, 2\pi, 4\pi, 6\pi, \ldots$$

$$\theta = 2n\pi,\ n \in \mathbb{Z}$$

b $\sin\theta = -1$

$$\theta = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \ldots$$

$$\theta = \frac{3\pi}{2} + 2n\pi,\ n \in \mathbb{Z}$$

c $\tan\theta$ undefined

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

$$\theta = \frac{\pi}{2} + n\pi,\ n \in \mathbb{Z}$$

Radians 5B

1 a $\sin\frac{3\pi}{4} = \sin\left(\pi - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

b $\sin\left(-\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

c $\sin\frac{11\pi}{6} = \sin\left(2\pi - \frac{\pi}{6}\right) = -\frac{1}{2}$

d $\cos\frac{2\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = -\frac{1}{2}$

e $\cos\frac{5\pi}{3} = \cos\left(2\pi - \frac{\pi}{3}\right) = \frac{1}{2}$

f $\cos\frac{5\pi}{4} = \cos\left(\pi + \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

g $\tan\frac{3\pi}{4} = \tan\left(\pi - \frac{\pi}{4}\right) = -1$

h $\tan\left(-\frac{5\pi}{4}\right) = -\tan\left(\pi + \frac{\pi}{4}\right) = -1$

i $\tan\frac{7\pi}{6} = \tan\left(\pi + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$

2 a $\sin\frac{7\pi}{3} = \sin\left(2\pi + \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

b $\sin\left(-\frac{5\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

c $\cos\left(-\frac{7\pi}{6}\right) = \cos\frac{5\pi}{6} = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$

d $\cos\frac{11\pi}{4} = \cos\frac{3\pi}{4} = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$

e $\tan\frac{5\pi}{3} = -\tan\frac{\pi}{3} = -\sqrt{3}$

f $\tan\left(-\frac{2\pi}{3}\right) = \tan\frac{\pi}{3} = \sqrt{3}$

3 $AC = \frac{2}{\sin\frac{\pi}{3}} = \frac{2}{\frac{\sqrt{3}}{2}} = \frac{4\sqrt{3}}{3}$

$$DC^2 = AD^2 + AC^2$$
$$= \left(\frac{2\sqrt{6}}{3}\right)^2 + \left(\frac{4\sqrt{3}}{3}\right)^2$$
$$= \frac{24}{9} + \frac{48}{9} = \frac{72}{9} = 8$$
$$DC = \sqrt{8} = 2\sqrt{2} = k\sqrt{2}$$

So $k = 2$

Radians 5C

1 **a** Using $l = r\theta$:

i $l = 6 \times 0.45 = 2.7$

ii $l = 4.5 \times 0.45 = 2.025$

iii $l = 20 \times \frac{3}{8}\pi = 7.5\pi$ (23.6 to 3 s.f.)

b Using $r = \frac{l}{\theta}$:

i $r = \frac{10}{0.6} = \frac{50}{3}$

ii $r = \frac{1.26}{0.7} = 1.8$

iii $r = \frac{1.5\pi}{\frac{5}{12}\pi} = 1.5 \times \frac{12}{5} = \frac{18}{5} = 3.6$

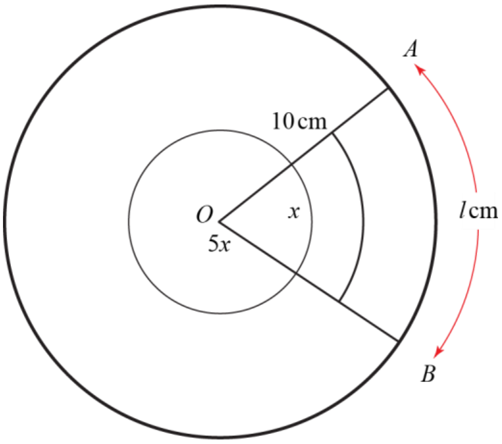
c Using $\theta = \frac{l}{r}$:

i $\theta = \frac{10}{7.5} = \frac{4}{3}$

ii $\theta = \frac{4.5}{5.625} = 0.8$

iii $\theta = \frac{\sqrt{12}}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3}} = 2$

2



The total angle at the centre is $6x$ so

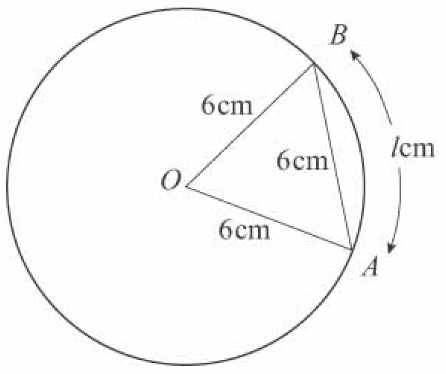
$6x = 2\pi$

$x = \frac{\pi}{3}$

Using $l = r\theta$ to find the minor arc AB:

$l = 10 \times \frac{\pi}{3} = \frac{10\pi}{3}$ cm

3



Triangle OAB is equilateral, so $\angle AOB = \frac{\pi}{3}$

Using $l = r\theta$:

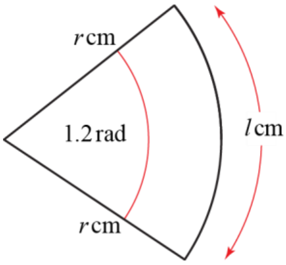
$l = 6 \times \frac{\pi}{3} = 2\pi$

4 $r = \sqrt{10}$ cm and $\theta = \sqrt{5}$ rad
Using $l = r\theta$:
$l = \sqrt{10} \times \sqrt{5} = \sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2}$

5 **a** Using $l = r\theta$:
length of shorter arc $= 3 \times 0.8 = 2.4$ cm
length of longer arc $= (3+2) \times 0.8 = 4$ cm
Perimeter $= 2.4\text{ cm} + 2\text{ cm} + 4\text{ cm} + 2\text{ cm}$
$= 10.4$ cm

b Length of shorter arc $= 3\theta$ cm
Length of longer arc $= 5\theta$ cm
So perimeter $= (3\theta + 5\theta + 2 + 2)$ cm
As the perimeter $= 14$ cm,
$8\theta + 4 = 14$
$8\theta = 10$
$\theta = \frac{10}{8} = 1.25$ rad

6

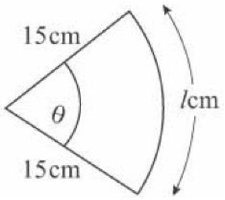


Using $l = r\theta$, the arc length $= 1.2r$ cm.
The area of the square $= 36\text{ cm}^2$, so each side $= 6$ cm and the perimeter is, therefore, 24 cm.

The perimeter of the sector
$=$ arc length $+ 2r$ cm
$= (1.2r + 2r)$ cm $= 3.2r$ cm

Perimeter of square $=$ perimeter of sector, so
$24 = 3.2r$
$r = \frac{24}{3.2} = 7.5$

7



Using $l = r\theta$:
the arc length of the sector $= 15\theta$ cm
So the perimeter $= (15\theta + 30)$ cm
As the perimeter $= 42$ cm,
$15\theta + 30 = 42$
$15\theta = 12$
$\theta = \frac{12}{15} = 0.8$

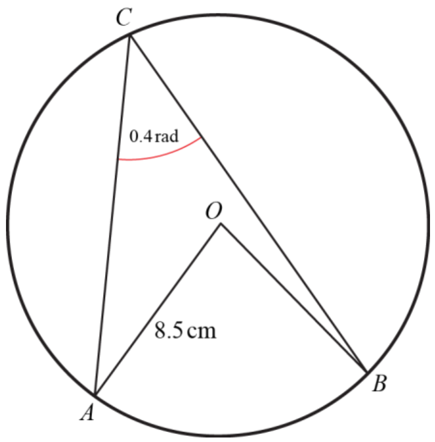
8 **a** $\angle COA = \pi - \frac{2}{3}\pi = \frac{\pi}{3}$

b The perimeter of the brooch
$= AB +$ arc $BC +$ chord AC
$AB = 4$ cm
$l = r\theta$ with $r = 2$ cm and $\theta = \frac{2}{3}\pi$

So length of arc $BC = 2 \times \frac{2}{3}\pi = \frac{4}{3}\pi$ cm

As $\angle COA = \frac{\pi}{3}$ (60°), triangle COA is equilateral.
So length of chord $AC = 2$ cm
So perimeter $= 4\text{ cm} + \frac{4}{3}\pi\text{ cm} + 2\text{ cm}$
$= \left(6 + \frac{4}{3}\pi\right)$ cm

9



Using the circle theorem, that angle subtended at the centre of a circle = 2 × angle subtended at the circumference:

$\angle AOB = 2\angle ACB = 0.8\text{ rad}$

Using $l = r\theta$:

length of minor arc $AB = 8.5 \times 0.8\text{ cm}$

$= 6.8\text{ cm}$

10 a $OC = R - r$

b

$$\begin{aligned} OC &= R - r \\ \sin\theta &= \frac{r}{R-r} \\ (R-r)\sin\theta &= r \\ R\sin\theta - r\sin\theta &= r \\ R\sin\theta &= r + r\sin\theta \\ &= r(1+\sin\theta) \end{aligned}$$

c $R\sin\theta = r(1+\sin\theta)$

$$\begin{aligned} \frac{3}{4}R &= r\left(1+\frac{3}{4}\right) \\ r &= \frac{3}{7}R \\ \sin\theta = \frac{3}{4} &\Rightarrow \theta = 0.848... \\ 2R + 2R\theta &= 21 \\ 2R + 1.696R &= 21 \\ 3.696R &= 21 \\ R &= 5.681\text{ cm} \\ r &= \frac{3}{7} \times R = 2.43\text{ cm} \end{aligned}$$

11 Length of arc $= r\theta$

$$\begin{aligned} \text{Perimeter} &= 2r + r\theta \\ 2r + r\theta &= 2r\theta \\ 2r &= r\theta \\ \theta &= 2\text{ rad} \end{aligned}$$

12 a $\theta = \frac{2\pi}{24} = \frac{\pi}{12}$

$$\begin{aligned} r\theta &= \frac{3\pi}{2} \\ r &= \frac{3\pi}{2} \div \frac{\pi}{12} = 18\text{ m} \\ d &= 36\text{ m} \end{aligned}$$

b $C = \pi d = 36\pi$

$$\begin{aligned} \text{Speed} &= \frac{36\pi \times 60 \times 60}{30 \times 1000} \\ &= 13.6\text{ km/h} \end{aligned}$$

13 a $SR = 7 \times 0.5 = 3.5\text{ m}$

b Using the cosine rule:

$$\begin{aligned} &QR^2 = 7^2 + 12^2 - 2 \times 7 \times 12 \times \cos 0.5 \\ &QR = 6.75\text{ m} \\ &SQ = PQ - PS = 12 - 7 = 5\text{ m} \\ &\text{Perimeter} = 6.75 + 5 + 3.5 \\ &= 15.3\text{ m (3 s.f.)} \end{aligned}$$

14 a $\angle XOZ = \frac{2\pi - 1.1}{2} = 2.59\text{ rad}$

b Using the cosine rule:

$$\begin{aligned} &XZ^2 = 5^2 + 15^2 - 2 \times 5 \times 15 \times \cos 2.59 \\ &XZ = 19.44\text{ mm} \\ &\text{Arc length } YZ = 5 \times 1.1 = 5.5\text{ mm} \\ &\text{Perimeter} = 19.44 \times 2 + 5.5 \approx 44\text{ mm} \end{aligned}$$

Radians 5D

1 **a** Area of shaded sector

$$= \frac{1}{2} \times 8^2 \times 0.6 = 19.2\,\text{cm}^2$$

b Area of shaded sector

$$= \frac{1}{2} \times 9^2 \times \frac{\pi}{6} = \frac{27\pi}{4} = 6.75\pi\,\text{cm}^2$$

c Angle subtended at C by major arc

$$= 2\pi - \frac{\pi}{5} = \frac{9\pi}{5}$$

Area of shaded sector

$$= \frac{1}{2} \times 1.2^2 \times \frac{9\pi}{5} = \frac{162\pi}{125} = 1.296\pi\,\text{cm}^2$$

d Area of shaded segment

$$= \frac{1}{2} \times 10^2 (1.5 - \sin 1.5) = 25.1\,\text{cm}^2$$

e Area of shaded segment

$$= \frac{1}{2} \times 6^2 \left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right)$$

$$= \left(6\pi - 9\sqrt{3}\right)\text{cm}^2 = 3.26\,\text{cm}^2$$

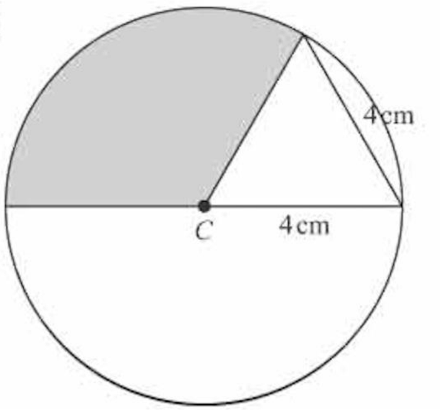
f Area of shaded segment

$$= \pi \times 6^2 - \left(\frac{1}{2} \times 6^2 \left(\frac{\pi}{4} - \sin\frac{\pi}{4}\right)\right)$$

$$= 36\pi - \frac{36\pi}{8} + \frac{36}{2} \times \frac{\sqrt{2}}{2}$$

$$= \left(\frac{63\pi}{2} + 9\sqrt{2}\right)\text{cm}^2 = 111.7\,\text{cm}^2$$

2 **a**



The triangle is equilateral, so the angle at C in the triangle is $\frac{\pi}{3}$

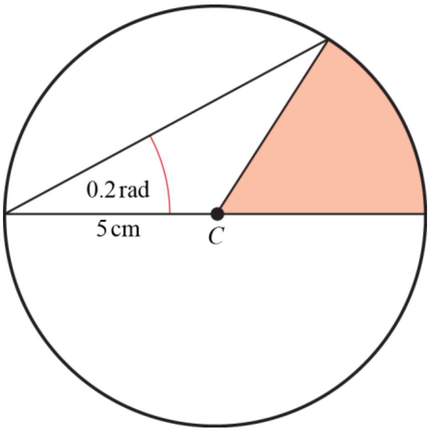
Angle subtended at C by shaded sector

$$= \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

Area of shaded sector

$$= \frac{1}{2} \times 4^2 \times \frac{2\pi}{3} = \frac{16}{3}\pi\,\text{cm}^2$$

b

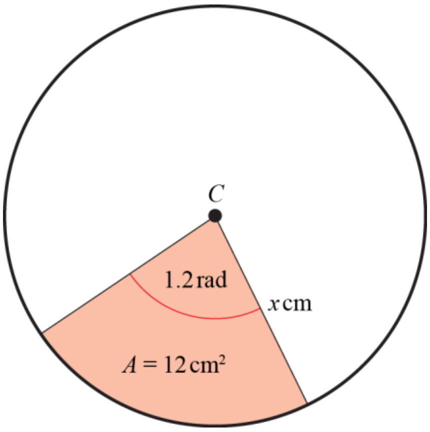


The triangle is isosceles, so the angle at C in the shaded sector is 0.4 rad.

Area of shaded sector

$$= \frac{1}{2} \times 5^2 \times 0.4 = 5\,\text{cm}^2$$

3 a



Area of shaded sector

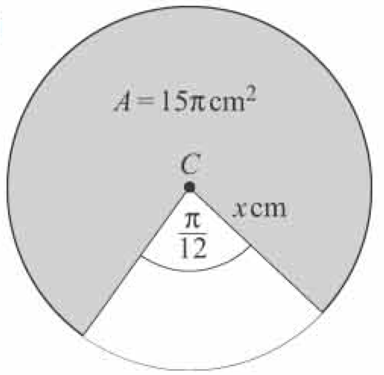
$$= \frac{1}{2} \times x^2 \times 1.2 = 0.6x^2 \text{ cm}^2$$

So $0.6x^2 = 12$

$x^2 = 20$

$x = 4.47$ (3 s.f.)

b



Area of shaded sector

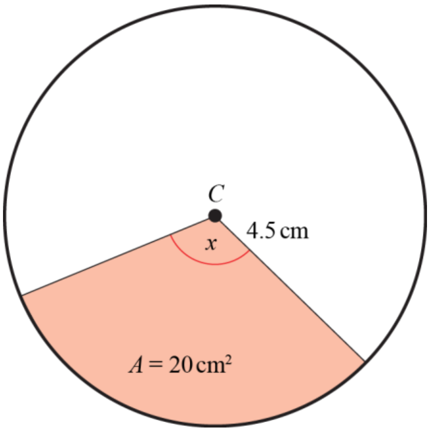
$$= \frac{1}{2} \times x^2 \times \left(2\pi - \frac{\pi}{12}\right) = \frac{1}{2} x^2 \times \frac{23\pi}{12} \text{ cm}^2$$

So $15\pi = \frac{23}{24}\pi x^2$

$x^2 = \frac{24 \times 15}{23}$

$x = 3.96$ (3 s.f.)

c



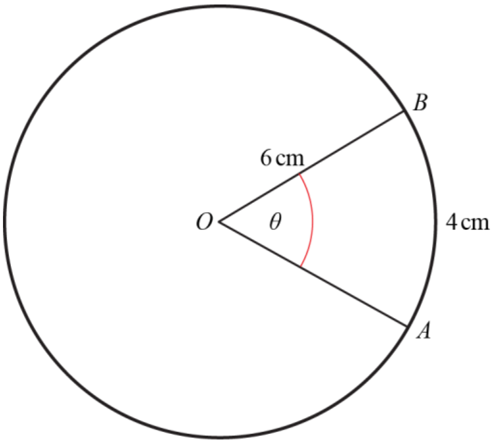
Area of shaded sector

$$= \frac{1}{2} \times 4.5^2 \times x \text{ cm}^2$$

So $2 = \frac{1}{2} \times 4.5^2 x$

$x = \frac{40}{4.5^2} = 1.98$ (3 s.f.)

4



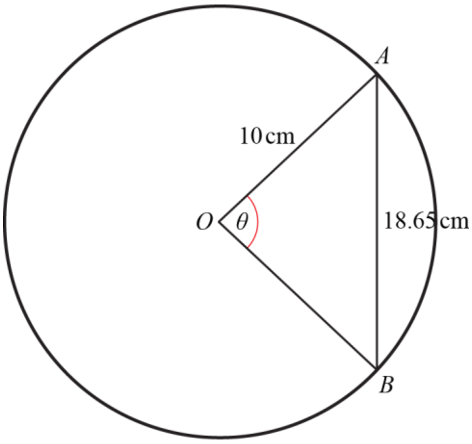
Using $l = r\theta$:

$4 = 6\theta$

$\theta = \frac{2}{3}$

So area of sector $= \frac{1}{2} \times 6^2 \times \frac{2}{3} = 12 \text{ cm}^2$

5



a $\cos\theta = \dfrac{10^2 + 10^2 - 18.65^2}{2 \times 10 \times 10}$

$= -0.739$ (3 s.f.)

b $\cos\theta = -0.739... \Rightarrow \theta = 2.4025...$

Area $= \dfrac{1}{2} \times 10^2 \times 2.4025...$

$= 120\,\text{cm}^2$ (3 s.f.)

6 Using area of sector $= \dfrac{1}{2}r^2\theta$:

$100 = \dfrac{1}{2} \times 12^2\theta$

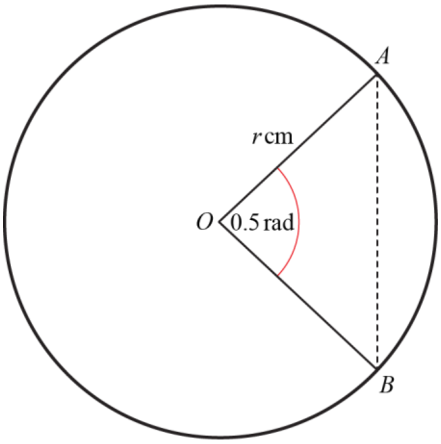
$\Rightarrow \theta = \dfrac{100}{72} = \dfrac{25}{18}$ rad

The perimeter of the sector

$= 12 + 12 + 12\theta = 12(2 + \theta)$

$= 12 \times \dfrac{61}{18} = \dfrac{122}{3} = 40\dfrac{2}{3}$ cm

7



a The perimeter of minor sector *AOB*

$= r + r + 0.5r = 2.5r$ cm

So $30 = 2.5r$

$\Rightarrow r = \dfrac{30}{2.5} = 12$

b Area of minor sector $AOB = \dfrac{1}{2}r^2\theta$

$= \dfrac{1}{2} \times 12^2 \times 0.5 = 36\,\text{cm}^2$

c Area of segment

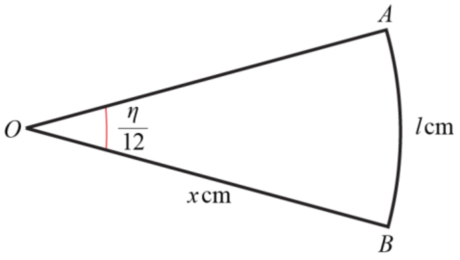
$= \dfrac{1}{2}r^2(\theta - \sin\theta)$

$= \dfrac{1}{2} \times 12^2(0.5 - \sin 0.5)$

$= 72(0.5 - \sin 0.5)$

$= 1.48\,\text{cm}^2$ (3 s.f.)

8



a $l = r\theta \Rightarrow l = x \times \frac{\pi}{12} \Rightarrow x = \frac{12l}{\pi}$

Area of sector $= \frac{1}{2}r^2\theta$

$$= \frac{1}{2} \times \left(\frac{12l}{\pi}\right)^2 \times \frac{\pi}{12}$$

$$= \frac{1}{2} \times \frac{12l^2}{\pi}$$

$$= \frac{6l^2}{\pi}$$

b $\frac{6l^2}{\pi} \times 24 = 3600\pi$

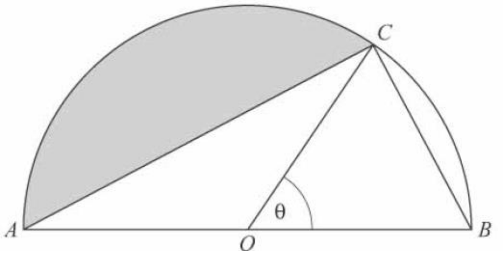
$$l^2 = 25\pi^2$$

$$l = 5\pi$$

The arc length of AB is 5π cm.

c $x = \frac{12l}{\pi} = \frac{12}{\pi} \times 5\pi = 60$

9



Using the formula,

area of a triangle $= \frac{1}{2}ab\sin C$:

area of triangle $COB = \frac{1}{2}r^2\sin\theta$ **(1)**

$\angle AOC = \pi - \theta,$ so area of shaded segment

$= \frac{1}{2}r^2\left((\pi - \theta) - \sin(\pi - \theta)\right)$ **(2)**

As **(1)** and **(2)** are equal:

$$\frac{1}{2}r^2\sin\theta = \frac{1}{2}r^2\left(\pi - \theta - \sin\left(\pi - \theta\right)\right)$$

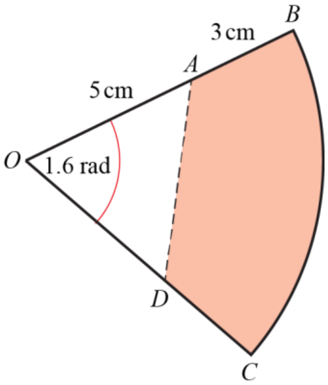
$\sin\theta = \pi - \theta - \sin(\pi - \theta)$

But $\sin(\pi - \theta) = \sin\theta,$

so $\sin\theta = \pi - \theta - \sin\theta$

Hence $\theta + 2\sin\theta = \pi$

10



Area of sector $OBC = \frac{1}{2}r^2\theta$

with $r = 8\,\text{cm}$ and $\theta = 1.6\,\text{rad}$

So area of sector OBC

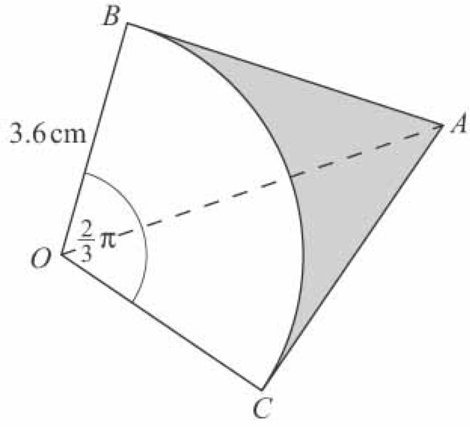
$= \frac{1}{2} \times 8^2 \times 1.6 = 51.2\,\text{cm}^2$

Using area of triangle formula:
area of triangle OAD

$= \frac{1}{2} \times 5 \times 5 \times \sin 1.6 = 12.49...\text{cm}^2$

So area of shaded region
$= 51.2 - 12.49... = 38.7\,\text{cm}^2$ (3 s.f.)

11



In right-angled triangle OBA:

$\tan\frac{\pi}{3} = \frac{AB}{3.6} \Rightarrow AB = 3.6 \times \tan\frac{\pi}{3}$

So area of triangle OBA

$= \frac{1}{2} \times 3.6 \times 3.6 \times \tan\frac{\pi}{3}$

and area of quadrilateral $OBAC$

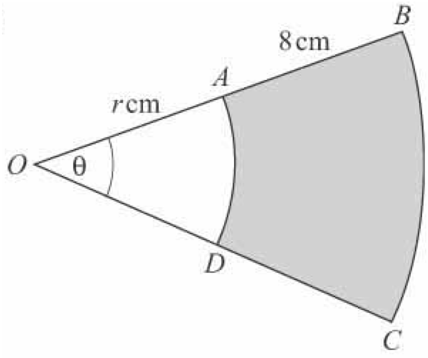
$= 3.6^2 \times \tan\frac{\pi}{3} = 22.447...\text{cm}^2$

Area of sector

$= \frac{1}{2} \times 3.6^2 \times \frac{2}{3}\pi = 13.57...\text{cm}^2$

So area of shaded region
$= 22.447... - 13.57... = 8.88\,\text{cm}^2$ (3 s.f.)

12



a Area of sector $OBC = \frac{1}{2}(r+8)^2\,\theta\,\text{cm}^2$

Area of sector $OAD = \frac{1}{2}r^2\theta\,\text{cm}^2$

So area of shaded region $ABCD$

$$= \left(\frac{1}{2}(r+8)^2\theta - \frac{1}{2}r^2\theta\right)\text{cm}^2 = 48\,\text{cm}^2$$

$$\begin{aligned}\theta\left((r^2+16r+64)-r^2\right) &= 96\\ \theta(16r+64) &= 96\\ \theta(r+4) &= 6\\ r\theta+4\theta &= 6 \qquad \textbf{(1)}\\ r\theta &= 6-4\theta\\ r &= \frac{6}{\theta}-4\end{aligned}$$

b Substituting $r = 10\theta$ in equation **(1)**:

$6 = 10\theta^2 + 4\theta$

Rearranging:

$5\theta^2 + 2\theta - 3 = 0$

$(5\theta-3)(\theta+1) = 0$

So $\theta = \frac{3}{5}$ and $r = 10\times\frac{3}{5} = 6$

Perimeter of shaded region

$= \left(r\theta + 8 + (r+8)\theta + 8\right)\text{cm}$

$= \frac{18}{5} + 8 + \frac{42}{5} + 8 = 28\,\text{cm}$

13 Area of sector $= A\,\text{cm}^2 = \frac{1}{2}\times 28^2\times\theta$

Perimeter of sector $= P$ cm

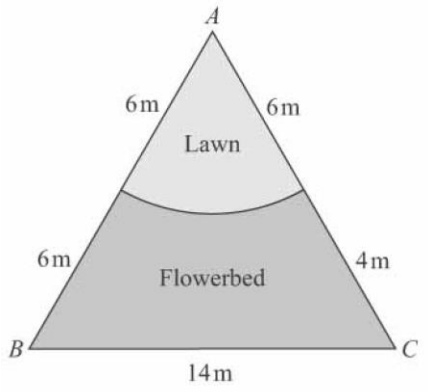
$= r\theta + 2r = (28\theta + 56)\,\text{cm}$

As $A = 4P$:

$$\begin{aligned}392\theta &= 4(28\theta+56)\\ 98\theta &= 28\theta+56\\ 70\theta &= 56\\ \theta &= \frac{56}{70} = 0.8\end{aligned}$$

So $P = 28\theta + 56 = 28\times 0.8 + 56 = 78.4$

14



a Using the cosine rule:

$\cos BAC = \frac{10^2 + 12^2 - 14^2}{2\times 10\times 12} = 0.2$

$\angle BAC = \cos^{-1} 0.2$

$\angle BAC = 1.369... = 1.37$ rad (3 s.f.)

b Area of triangle ABC

$= \frac{1}{2}\times 12\times 10\times \sin A = 58.787...\,\text{m}^2$

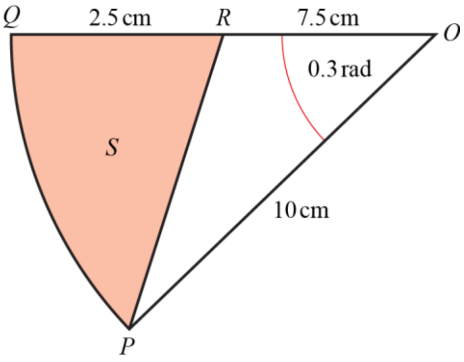
Area of sector (lawn)

$= \frac{1}{2}\times 6^2\times A = 24.649...\ \text{m}^2$

So area of flowerbed

$= 58.787... - 24.649... = 34.1\,\text{m}^2$ (3 s.f.)

15



a $RP^2 = 2.5^2 + 10^2 - 2 \times 10 \times 2.5 \times \cos 0.3$

$= 58.48...$

$RP = 7.65\,\text{cm}$

$QP = 10 \times 0.3 = 3\,\text{cm}$

So perimeter of S

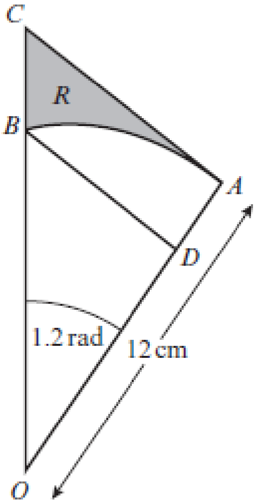
$= 3 + 7.5 + 7.65 = 18.1\,\text{cm}$ (3 s.f.)

b Area of S

$= \frac{1}{2} \times 10^2 \times 0.3 - \frac{1}{2} \times 2.5 \times 10 \times \sin 0.3$

$= 11.3\,\text{cm}^2$ (3 s.f.)

16 a



$AC = 12 \times \tan 1.2 = 30.865...$ cm

Area of triangle AOC

$= \frac{1}{2} \times 12 \times 30.865... = 185.194...\ \text{cm}^2$

So area of R

$= 185.194... - \frac{1}{2} \times 12^2 \times 1.2 = 98.794...$

$= 98.79\,\text{cm}^2$ (2 d.p.)

b Length of arc $AB = 12 \times 1.2 = 14.4$ cm

$OD = 12 \times \cos 1.2 = 4.348...\,\text{cm}$

$BD = 12 \times \sin 1.2 = 11.184...\,\text{cm}$

$AD = 12 - 4.348... = 7.651...\,\text{cm}$

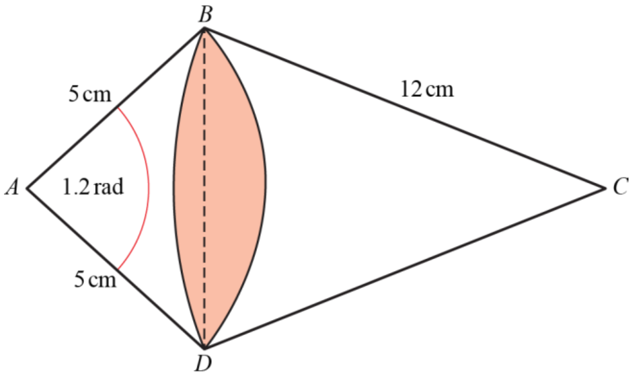
Perimeter of DAB

$= AB + AD + BD$

$= 14.4 + 7.651... + 11.184... = 33.236...$

$= 33.24\,\text{cm}$ (2 d.p.)

17



$BE = 5 \times \sin 0.6 = 2.823...$

so $\sin BCE = \dfrac{2.823...}{12}$

hence $\angle BCE = 0.237...$

and $\angle BCD = 0.474...$

Shaded area to left of BD

$= \frac{1}{2} \times 12^2 \times (0.474... - \sin 0.474...)$

$= 1.271...$

Shaded area to right of BD

$= \frac{1}{2} \times 5^2 \times (1.2 - \sin 1.2)$

$= 3.349...$

So total shaded area

$= 1.271... + 3.349... = 4.620...$

$= 4.62\ \text{cm}^2$ (3 s.f.)

Challenge

Arc length $l = r\theta \Rightarrow \theta = \frac{l}{r}$

So area $= \frac{1}{2}r^2\theta = \frac{1}{2}r^2\left(\frac{l}{r}\right) = \frac{1}{2}rl$

Radians 5E

1 a $\cos\theta = 0.7$, $0 \leqslant \theta \leqslant 2\pi$

$\cos^{-1} 0.7 = 0.795$

$2\pi - 0.795 = 5.49$

$\theta = 0.795, 5.49$

b $\sin\theta = -0.2$, $0 \leqslant \theta \leqslant 2\pi$

$\sin^{-1}(-0.2) = -0.201$

$\pi + 0.201 = 3.34$

$2\pi - 0.201 = 6.08$

$\theta = 3.34, 6.08$

c $\tan\theta = 5$, $0 \leqslant \theta \leqslant 2\pi$

$\tan^{-1} 5 = 1.37$

$\pi + 1.37 = 4.51$

$\theta = 1.37, 4.51$

d $\cos\theta = -1$, $0 \leqslant \theta \leqslant 2\pi$

$\cos^{-1}(-1) = \pi$

$\theta = \pi$

2 a $4 \sin\theta = 3$, $0 \leqslant \theta \leqslant 2\pi$

$\sin\theta = \frac{3}{4}$

$\sin^{-1}\frac{3}{4} = 0.848$

$\pi - 0.848 = 2.29$

$\theta = 0.848, 2.29$

b $7 \tan\theta = 1$, $0 \leqslant \theta \leqslant 2\pi$

$\tan\theta = \frac{1}{7}$

$\tan^{-1}\frac{1}{7} = 0.142$

$\pi + 0.142 = 3.28$

$\theta = 0.142, 3.28$

c $8 \tan\theta = 15$, $0 \leqslant \theta \leqslant 2\pi$

$\tan\theta = \frac{15}{8}$

$\tan^{-1}\frac{15}{8} = 1.08$

$\pi + 1.08 = 4.22$

$\theta = 1.08, 4.22$

d $\sqrt{5}\cos\theta = \sqrt{2}$, $0 \leqslant \theta \leqslant 2\pi$

$\cos\theta = \frac{\sqrt{2}}{\sqrt{5}}$

$\cos^{-1}\frac{\sqrt{2}}{\sqrt{5}} = 0.886$

$2\pi - 0.886 = 5.40$

$\theta = 0.886, 5.40$

3 a $5\cos\theta + 1 = 3$, $0 \leqslant \theta \leqslant 2\pi$

$\cos\theta = \frac{2}{5}$

$\cos^{-1}\frac{2}{5} = 1.16$

$2\pi - 1.16 = 5.12$

$\theta = 1.16, 5.12$

b $\sqrt{5}\sin\theta + 2 = 1$, $0 \leqslant \theta \leqslant 2\pi$

$\sin\theta = -\frac{1}{\sqrt{5}}$

$\sin^{-1}\left(-\frac{1}{\sqrt{5}}\right) = -0.464$

$\pi + 0.464 = 3.61$

$2\pi - 0.464 = 5.82$

$\theta = 3.61, 5.82$

3 c $8\tan\theta - 5 = 5,\ 0 \leqslant \theta \leqslant 2\pi$

$\tan\theta = \frac{10}{8}$

$\tan^{-1}\frac{10}{8} = 0.896$

$\pi + 0.896 = 4.04$

$\theta = 0.896, 4.04$

d $\sqrt{7}\cos\theta - 1 = \sqrt{2},\ 0 \leqslant \theta \leqslant 2\pi$

$\cos\theta = \frac{\sqrt{2}+1}{\sqrt{7}}$

$\cos^{-1}\frac{\sqrt{2}+1}{\sqrt{7}} = 0.421$

$2\pi - 0.421 = 5.86$

$\theta = 0.421, 5.86$

4 a $\sqrt{3}\tan\theta - 1 = 0,\ -\pi \leqslant \theta \leqslant \pi$

$\tan\theta = \frac{1}{\sqrt{3}}$

$\tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

$\frac{\pi}{6} - \pi = -\frac{5\pi}{6}$

$\theta = -\frac{5\pi}{6}, \frac{\pi}{6}$

b $5\sin\theta = 1,\ -\pi \leqslant \theta \leqslant 2\pi$

$\sin\theta = \frac{1}{5}$

$\sin^{-1}\frac{1}{5} = 0.201$

$\theta = 0.201, 2.94$

c $8\cos\theta = 5,\ -2\pi \leqslant \theta \leqslant 2\pi$

$\cos\theta = \frac{5}{8}$

$\cos^{-1}\frac{5}{8} = 0.896$

$\theta = -0.896, -5.39, 0.896, 5.39$

d $3\cos\theta - 1 = 0.02 - \pi \leqslant \theta \leqslant 3\pi$

$\cos\theta = \frac{1.02}{3} = 0.34$

$\cos^{-1} 0.34 = 1.22$

$\theta = -1.22, 1.22, 5.06, 7.51$

e $0.4\tan\theta - 5 = -7,\ 0 \leqslant \theta \leqslant 4\pi$

$\tan\theta = -\frac{2}{0.4} = -5$

$\tan^{-1}(-5) = -1.37$

$\theta = 1.77, 4.91, 8.05, 11.2$

f $\cos\theta - 1 = -0.82,\ \frac{\pi}{2} \leqslant \theta \leqslant \frac{7\pi}{3}$

$\cos\theta = 0.18$

$\cos^{-1} 0.18 = 1.39$ (not in given interval)

$\theta = 4.89$

5 a $5\cos 2\theta = 4,\ 0 \leqslant \theta \leqslant 2\pi$

Let $X = 2\theta$

$5\cos X = 4,\ 0 \leqslant X \leqslant 4\pi$

$\cos X = \frac{4}{5}$

$X = 0.64, 5.64, 6.92, 11.92$

$\theta = 0.322, 2.82, 3.46, 5.96$

b $5\sin 3\theta + 3 = 1,\ 0 \leqslant \theta \leqslant 2\pi$

Let $X = 3\theta$

$5\sin X + 3 = 1,\ 0 \leqslant X \leqslant 6\pi$

$\sin X = -\frac{2}{5}$

$X = (-0.412), 3.55, 5.87, 9.83, 12.2,$

$16.1,\ 18.4$

$\theta = 1.18, 1.96, 3.28, 4.05, 5.37, 6.15$

5 **c** $\sqrt{3}\tan 4\theta - 5 = -4,\ 0 \leqslant \theta \leqslant 2\pi$

Let $X = 4\theta$

$\sqrt{3}\tan X - 5 = -4,\ 0 \leqslant X \leqslant 8\pi$

$$\tan X = \frac{1}{\sqrt{3}}$$

$$X = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{19\pi}{6}, \frac{25\pi}{6}, \frac{31\pi}{6}, \frac{37\pi}{6}, \frac{43\pi}{6}$$

$$\theta = \frac{\pi}{24}, \frac{7\pi}{24}, \frac{13\pi}{24}, \frac{19\pi}{24}, \frac{25\pi}{24}, \frac{31\pi}{24}, \frac{37\pi}{24}, \frac{43\pi}{24}$$

d $\sqrt{10}\cos 2\theta + \sqrt{2} = 3\sqrt{2},\ 0 \leqslant \theta \leqslant 2\pi$

Let $X = 2\theta$

$\sqrt{10}\cos X + \sqrt{2} = 3\sqrt{2},\ 0 \leqslant X \leqslant 4\pi$

$$\cos X = \frac{2\sqrt{2}}{\sqrt{10}} = \frac{2\sqrt{5}}{5}$$

$$X = 0.464, 5.82, 6.75, 12.1$$

$$\theta = 0.232, 2.91, 3.37, 6.05$$

6 **a** $\sqrt{2}\sin 3\theta - 1 = 0,\ -\pi \leqslant \theta \leqslant \pi$

Let $X = 3\theta$

$\sqrt{2}\sin X - 1 = 0,\ -3\pi \leqslant X \leqslant 3\pi$

$$\sin X = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$X = -\frac{7\pi}{4}, -\frac{5\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$$

$$\theta = -\frac{7\pi}{12}, -\frac{5\pi}{12}, \frac{\pi}{12}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{11\pi}{12}$$

b $2\cos 4\theta = -1,\ -\pi \leqslant \theta \leqslant 2\pi$

Let $X = 4\theta$

$2\cos X = -1,\ -4\pi \leqslant X \leqslant 8\pi$

$$\cos X = -\frac{1}{2}$$

$$X = -\frac{10\pi}{3}, -\frac{8\pi}{3}, -\frac{4\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{14\pi}{3}, \frac{16\pi}{3}, \frac{20\pi}{3}, \frac{22\pi}{3}$$

$$\theta = -\frac{5\pi}{6}, -\frac{2\pi}{3}, -\frac{\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6}$$

c $8\tan 2\theta = 7,\ -2\pi \leqslant \theta \leqslant 2\pi$

Let $X = 2\theta$

$8\tan X = 7,\ -4\pi \leqslant X \leqslant 4\pi$

$$\tan X = \frac{7}{8}$$

$$X = -11.8, -8.71, -5.56, -2.42, 0.719, 3.86, 7.00, 10.1$$

$$\theta = -5.92, -4.35, -2.78, -1.21, 0.359, 1.93, 3.50, 5.07$$

d $6\cos 2\theta - 1 = 0.2,\ -\pi \leqslant \theta \leqslant 3\pi$

Let $X = 2\theta$

$6\cos \mathrm{X} - 1 = 0.2,\ -2\pi \leqslant X \leqslant 6\pi$

$$\cos X = \frac{1.2}{6} = 0.2$$

$$X = -4.91, -1.37, 1.37, 4.91, 7.65, 11.2, 13.9, 17.5$$

$$\theta = -2.46, -0.685, 0.685, 2.46, 3.83, 5.60, 6.97, 8.74$$

7 **a** $4\cos^2\theta = 2,\ 0 \leqslant \theta \leqslant 2\pi$

$$\cos^2\theta = \frac{1}{2}$$

$$\cos\theta = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$$

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

b $3\tan^2\theta + \tan\theta = 0,\ 0 \leqslant \theta \leqslant 2\pi$

$$\tan\theta(3\tan\theta + 1) = 0$$

$$\tan\theta = 0 \Rightarrow \theta = 0, \pi, 2\pi$$

$$\text{or } \tan\theta = -\frac{1}{3} \Rightarrow \theta = 2.82, 5.96$$

c $\cos^2\theta - 2\cos\theta = 3,\ 0 \leqslant \theta \leqslant 2\pi$

$$\cos^2\theta - 2\cos\theta - 3 = 0$$

$$(\cos\theta - 3)(\cos\theta + 1) = 0$$

$$\cos\theta = 3 \text{ (no solutions)}$$

$$\text{or } \cos\theta = -1 \Rightarrow \theta = \pi$$

d $2\sin^2 2\theta - 5\cos 2\theta = -2,\ 0 \leqslant \theta \leqslant 2\pi$

$$2\sin^2 2\theta - 5\cos 2\theta + 2 = 0$$

Let $X = 2\theta$

$2\sin^2 X - 5\cos X + 2 = 0,\ 0 \leqslant X \leqslant 4\pi$

$$2(1 - \cos^2 X) - 5\cos X + 2 = 0$$

$$2 - 2\cos^2 X - 5\cos X + 2 = 0$$

$$2\cos^2 X + 5\cos X - 4 = 0$$

$$X = \frac{-5 \pm \sqrt{57}}{4}$$

$$\cos X = -3.14 \text{ (no solutions)}$$

$$\text{or } \cos X = 0.637 \Rightarrow$$

$$X = 0.880, 5.40, 7.16, 11.7$$

$$\theta = 0.440, 2.70, 3.58, 5.84$$

8 **a** $\cos\theta + 2\sin^2\theta + 1 = 0,\ 0 \leqslant \theta \leqslant 2\pi$

$$\cos\theta + 2(1 - \cos^2\theta) + 1 = 0$$

$$\cos\theta + 2 - 2\cos^2\theta + 1 = 0$$

$$2\cos^2\theta - \cos\theta - 3 = 0$$

$$(2\cos\theta - 3)(\cos\theta + 1) = 0$$

$$\cos\theta = 1.5 \text{ (no solutions)}$$

$$\text{or } \cos\theta = -1 \Rightarrow \theta = \pi$$

b $10\sin^2\theta = 3\cos^2\theta,\ 0 \leqslant \theta \leqslant 2\pi$

$$\frac{\sin^2\theta}{\cos^2\theta} = \frac{3}{10}$$

$$\tan^2\theta = 0.3$$

$$\tan\theta = \pm\sqrt{0.3}$$

$$\theta = 0.501, 2.64, 3.64, 5.78$$

c $4\cos^2\theta + 8\sin^2\theta = 2\sin^2\theta - 2\cos^2\theta,$
$0 \leqslant \theta \leqslant 2\pi$

$$6\sin^2\theta = -6\cos^2\theta$$

$$\tan^2\theta = -1 \Rightarrow \text{no solutions}$$

d $2\sin^2\theta - 7 + 12\cos\theta = 0,\ 0 \leqslant \theta \leqslant 2\pi$

$$2(1 - \cos^2\theta) - 7 + 12\cos\theta = 0$$

$$2 - 2\cos^2\theta - 7 + 12\cos\theta = 0$$

$$2\cos^2\theta - 12\cos\theta + 5 = 0$$

$$\cos\theta = \frac{12 \pm \sqrt{104}}{4}$$

$$\cos\theta = 5.55 \text{ (no solutions)}$$

$$\text{or } \cos\theta = 0.45 \Rightarrow \theta = 1.10, 5.18$$

9 a $\cos\left(x-\frac{\pi}{12}\right)=\frac{1}{\sqrt{2}},\ 0\leqslant x<2\pi$

Let $X=x-\frac{\pi}{12}$

$\cos X=\frac{1}{\sqrt{2}},\ -\frac{\pi}{12}\leqslant X<\frac{23\pi}{12}$

$X=\frac{\pi}{4},\frac{7\pi}{4}$

$x=\frac{\pi}{3},\frac{11\pi}{6}$

b $\sin 3x=-\frac{1}{2},\ 0\leqslant x<2\pi$

Let $X=3x$

$\sin X=-\frac{1}{2},\ 0\leqslant X<6\pi$

$X=\left(-\frac{\pi}{6}\right),\frac{7\pi}{6},\frac{11\pi}{6},\frac{19\pi}{6},\frac{23\pi}{6},$
$\frac{31\pi}{6},\frac{35\pi}{6}$

$x=\frac{7\pi}{18},\frac{11\pi}{18},\frac{19\pi}{18},\frac{23\pi}{18},\frac{31\pi}{18},\frac{35\pi}{18}$

10 a $(1+\tan\theta)(5\sin\theta-2),\ -\pi\leqslant\theta<\pi$

$\tan\theta=-1\Rightarrow\theta=-\frac{\pi}{4},\frac{3\pi}{4}$

or $\sin\theta=\frac{2}{5}\Rightarrow\theta=0.412,2.73$

b $4\tan x=5\sin x,\ 0\leqslant x<2\pi$

$4\frac{\sin x}{\cos x}=5\sin x$

$4\sin x=5\sin x\cos x$

$\sin x=0\Rightarrow x=0,\ \pi$

or $\cos x=\frac{4}{5}\Rightarrow x=0.644,\ 5.64$

11 $8\cos^2 x+6\sin x-6=3,\ 0\leqslant x\leqslant 2\pi$

$8(1-\sin^2 x)+6\sin x-6-3=0$

$8-8\sin^2 x+6\sin x-9=0$

$8\sin^2 x-6\sin x+1=0$

$(4\sin x-1)(2\sin x-1)=0$

$\sin x=0.25\Rightarrow x=0.3,2.9$

or $\sin x=0.5\Rightarrow x=0.5,2.6$

12 $\cos^2 x-1=\frac{7}{2}\sin^2 x-2,0\leqslant x\leqslant 2\pi$

$(1-\sin^2 x)-1=\frac{7}{2}\sin^2 x-2$

$\frac{9}{2}\sin^2 x-2=0$

$\sin^2 x=\frac{4}{9}$

$\sin x=\frac{2}{3}\Rightarrow x=0.7,2.4$

or $\sin x=-\frac{2}{3}\Rightarrow x=3.9,5.6$

13 $8\sin^2 x+4\sin x-20=4$

$8\sin^2 x+4\sin x-24=0$

$2\sin^2 x+\sin x-6=0$

Let $Y=\sin x$

$2Y^2+Y-6=0$

$(2Y-3)(Y+2)=0$

$Y=1.5\Rightarrow\sin x=1.5$ (no solutions)

or $Y=-2\Rightarrow\sin x=-2$ (no solutions)

14 a $\tan^2 x - 2\tan x - 6 = 0$

Using the quadratic formula with $a = 1, b = -2$ and $c = -6$ (or by completing the square):

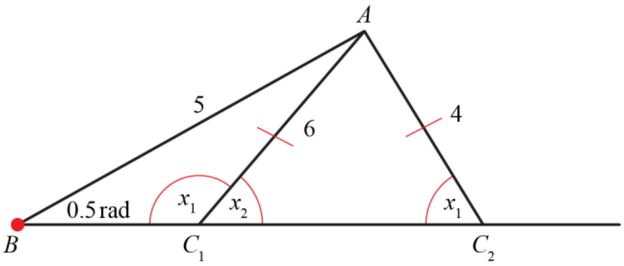
$$\tan x = \frac{2 \pm \sqrt{(-2)^2 - 4\times 1\times(-6)}}{2\times 1}$$
$$= \frac{2 \pm \sqrt{4+24}}{2}$$
$$= \frac{2 \pm \sqrt{28}}{2}$$
$$= \frac{2 \pm 2\sqrt{7}}{2}$$
$$= 1 \pm \sqrt{7}$$

So $p = 1$ and $q = 7$

b $\tan\theta = 1 + \sqrt{7} \Rightarrow \theta = 1.3, 4.4, 7.6$

$\tan\theta = 1 - \sqrt{7} \Rightarrow \theta = 2.1, 5.3, 8.4$

15 a



$$\frac{\sin x}{5} = \frac{\sin 0.5}{4}$$
$$\sin x = 5 \times \frac{\sin 0.5}{4}$$
$$= 0.599 \text{ (3 d.p.)}$$

b $x_1 = 0.643\ldots \approx 0.64$ (2 d.p.)

$x_2 = \pi - 0.643 = 2.498\ldots \approx 2.50$ (2 d.p.)

Radians 5F

1 a $$\frac{\sin 4\theta - \tan 2\theta}{3\theta} \approx \frac{4\theta - 2\theta}{3\theta} = \frac{2}{3}$$

b $$\frac{1-\cos 2\theta}{\tan 2\theta \sin\theta} \approx \frac{1-\left(1-\frac{(2\theta)^2}{2}\right)}{2\theta^2} = \frac{\frac{4\theta^2}{2}}{2\theta^2} = \frac{2\theta^2}{2\theta^2} = 1$$

c $$\frac{3\tan\theta - \theta}{\sin 2\theta} \approx \frac{3\theta - \theta}{2\theta} = \frac{2\theta}{2\theta} = 1$$

2 a $$\frac{\sin 3\theta}{\theta \sin 4\theta} \approx \frac{3\theta}{4\theta^2} = \frac{3}{4\theta}$$

b $$\frac{\cos\theta - 1}{\tan 2\theta} \approx \frac{\left(1-\frac{\theta^2}{2}\right)-1}{2\theta} = \frac{-\frac{\theta^2}{2}}{2\theta} = -\frac{\theta}{4}$$

c $$\frac{\tan 4\theta + \theta^2}{3\theta - \sin 2\theta} \approx \frac{4\theta + \theta^2}{3\theta - 2\theta} = \frac{4\theta + \theta^2}{\theta} = 4 + \theta$$

3 a $\cos 0.244 = 0.970\,379$ (6 d.p.)

b $$\cos 0.244 \approx 1 - \frac{0.244^2}{2} = 0.970\,232$$

c $$\frac{0.970\,232 - 0.970\,379}{0.970\,379} \times 100 = -0.015\%$$

d $\cos 0.75 = 0.731689$ (6 d.p.)

$$\cos 0.75 \approx 1 - \frac{0.75^2}{2} = 0.718\,75$$

$$\frac{0.718\,75 - 0.731689}{0.731689} \times 100 = -1.77\%$$

e The larger the value of θ, the less accurate the approximation is.

4 $$\frac{\theta - \sin\theta}{\sin\theta} \times 100 = 1$$

$$(\theta - \sin\theta) \times 100 = \sin\theta$$

$$100\theta - 100\sin\theta = \sin\theta$$

$$100\theta = 101\sin\theta$$

5 a

$$\frac{4\cos 3\theta - 2 + 5\sin\theta}{1-\sin 2\theta}$$

$$\approx \frac{4\left(1-\frac{(3\theta)^2}{2}\right)-2+5\theta}{1-2\theta}$$

$$= \frac{4\left(1-\frac{9\theta^2}{2}\right)-2+5\theta}{1-2\theta}$$

$$= \frac{4-18\theta^2-2+5\theta}{1-2\theta}$$

$$= \frac{2+5\theta-18\theta^2}{1-2\theta}$$

$$= \frac{(1-2\theta)(2+9\theta)}{1-2\theta}$$

$$= 9\theta + 2$$

b When θ is small, 9θ is also small, so

$$\frac{4\cos 3\theta - 2 + 5\sin\theta}{1-\sin 2\theta} \approx 2$$

Challenge

1 a $CD = r\theta = AC \times \theta$

b In the right-anged triangle ABC:

$$\sin\theta = \frac{BC}{AB} \approx \frac{CD}{AC} = \frac{AC\times\theta}{AC} = \theta$$

$$\tan\theta = \frac{BC}{AC} \approx \frac{CD}{AC} = \frac{AC\times\theta}{AC} = \theta$$

2 a For $|x| < 1$,

$$\sqrt{1-x^2}$$

$$= (1-x^2)^{\frac{1}{2}}$$

$$= 1 + \frac{1}{2}(-x^2) + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2}(-x^2)^2 + \ldots$$

$$= 1 - \frac{x^2}{2} - \frac{x^4}{8} + \ldots$$

$$\approx 1 - \frac{x^2}{2}$$

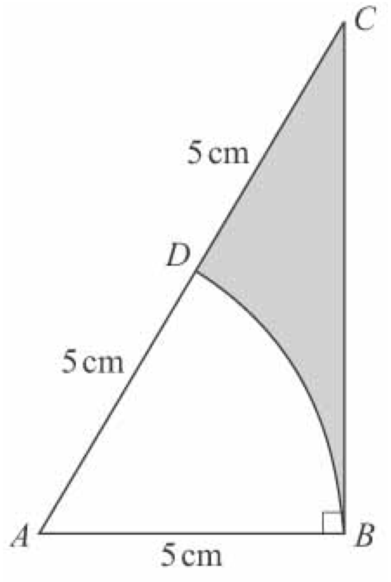
b $\cos\theta = \sqrt{1-\sin^2\theta}$

$$\approx 1 - \frac{\sin^2\theta}{2} \quad \text{since } |\sin\theta| < 1$$

$$\approx 1 - \frac{\theta^2}{2} \quad \text{since } \sin\theta \approx \theta$$

Radians Mixed exercise

1



a In the right-angled triangle ABC :

$$\cos \angle BAC = \frac{BA}{AC} = \frac{5}{10} = \frac{1}{2}$$

so $\angle BAC = \frac{\pi}{3}$

b Area of triangle ABC

$= \frac{1}{2} \times AB \times AC \times \sin \angle BAC$

$= \frac{1}{2} \times 5 \times 10 \times \sin \frac{\pi}{3} = 21.650... \text{cm}^2$

Area of sector DAB

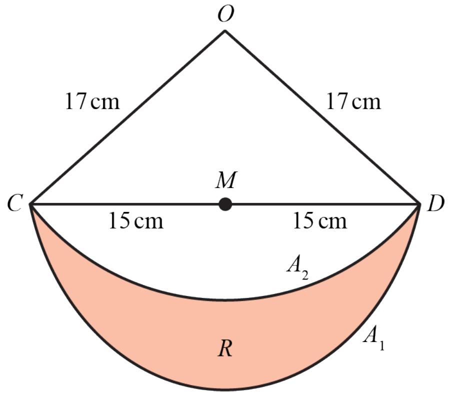
$= \frac{1}{2} \times 5^2 \times \frac{\pi}{3} = 13.089... \text{cm}^2$

Area of shaded region

$=$ area of $\triangle\, ABC$ $-$ area of sector DAB

$= 21.650... - 13.089... = 8.56 \text{cm}^2$ (3 s.f.)

2



a Using Pythagoras' theorem to find OM :

$OM^2 = 17^2 - 15^2 = 64 \Rightarrow OM = 8 \text{cm}$

Area of $\triangle OCD = \frac{1}{2} \times CD \times OM$

$= \frac{1}{2} \times 30 \times 8 = 120 \text{cm}^2$

b Area of shaded region R

$=$ area of semicircle CDA_1

$-$ area of segment CDA_2

Area of semicircle CDA_1

$= \frac{1}{2} \times \pi \times 15^2 = 353.429... \text{cm}^2$

Area of segment CDA_2

$=$ area of sector OCD

$-$ area of triangle OCD

$= \frac{1}{2} \times 17^2 \times \angle COD - 120$

In right-angled triangle COM:

$\sin \angle COM = \frac{CM}{OC} = \frac{15}{17}$

so $\angle COM = 1.0808...$

hence $\angle COD = 2.1616...$

So area of segment CDA_2

$= \frac{1}{2} \times 17^2 \times 2.1616... - 120$

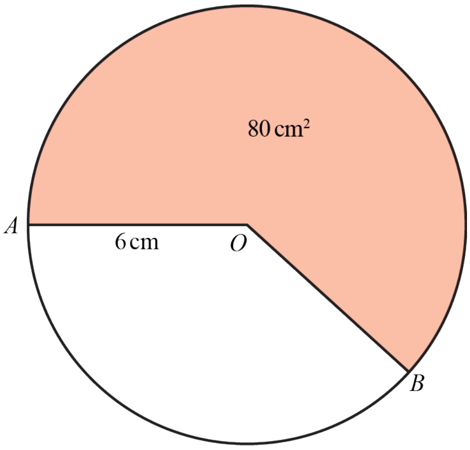
$= 192.362... \text{cm}^2$

So area of shaded region R

$= 353.429... - 192.362...$

$= 161.07 \text{cm}^2$ (2 d.p.)

3



a Reflex angle $AOB = (2\pi - \theta)$ rad

Area of shaded sector

$$= \frac{1}{2} \times 6^2 \times (2\pi - \theta)$$

$$= (36\pi - 18\theta)\text{ cm}^2$$

So $80 = 36\pi - 18\theta$

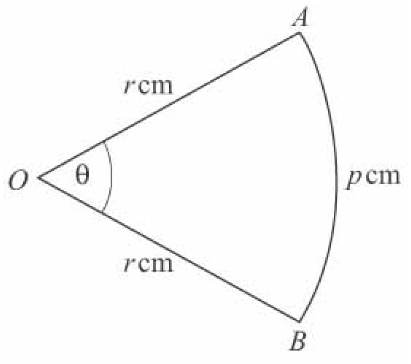
$\Rightarrow 18\theta = 36\pi - 80$

$\Rightarrow \theta = \dfrac{36\pi - 80}{18} = 1.839$ (3 d.p.)

b Length of minor arc AB

$= 6\theta = 6 \times 1.8387... = 11.03\text{ cm}$ (2 d.p.)

4



a Using $l = r\theta$:

$p = r\theta \Rightarrow \theta = \dfrac{p}{r}$

b Area of sector

$$= \frac{1}{2}r^2\theta = \frac{1}{2}r^2 \times \frac{p}{r} = \frac{1}{2}pr\text{ cm}^2$$

c $4.65 \leqslant r < 4.75$, $5.25 \leqslant p < 5.35$

Least possible value for area of sector

$= \dfrac{1}{2} \times 5.25 \times 4.65 = 12.207\text{ cm}^2$ (3 d.p.)

(Note: Least possible value is 12.20625, so 12.207 should be given, not 12.206)

d Maximum possible value of θ

$= \dfrac{\max p}{\min r} = \dfrac{5.35}{4.65} = 1.1505...$

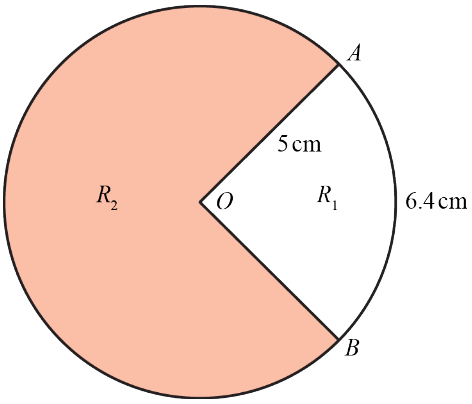
So give 1.150 (3 d.p.)

Minimum possible value of θ

$= \dfrac{\min p}{\max r} = \dfrac{5.25}{4.75} = 1.1052...$

So give 1.106 (3 d.p.)

5



a Using $l = r\theta$:

$6.4 = 5\theta \Rightarrow \theta = \dfrac{6.4}{5} = 1.28\text{ rad}$

b Using area of sector $= \dfrac{1}{2}r^2\theta$:

$R_1 = \dfrac{1}{2} \times 5^2 \times 1.28 = 16$

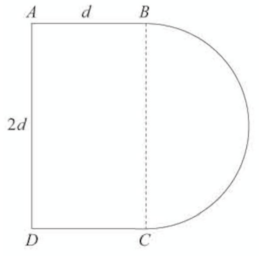
c R_2 = area of circle $- R_1$

$= \pi \times 5^2 - 16 = 62.5398...$

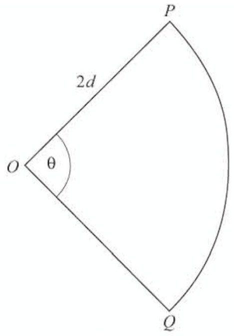
So $\dfrac{R_1}{R_2} = \dfrac{16}{62.5398...} = \dfrac{1}{3.908...} = \dfrac{1}{p}$

$\Rightarrow p = 3.91$ (3 s.f.)

6



Shape X



Shape Y

a Area of shape X

= area of rectangle + area of semicircle

$= (2d^2 + \frac{1}{2}\pi d^2)\,\text{cm}^2$

Area of shape $Y = \frac{1}{2}(2d)^2\theta = 2d^2\theta\,\text{cm}^2$

Since $X = Y$:

$2d^2 + \frac{1}{2}\pi d^2 = 2d^2\theta$

Divide by $2d^2$:

$1 + \frac{\pi}{4} = \theta$

b Perimeter of shape X

$= (d + 2d + d + \pi d)\,\text{cm}$ with $d = 3$

$= (3\pi + 12)\,\text{cm}$

c Perimeter of shape Y

$= (2d + 2d + 2d\theta)\,\text{cm}$

with $d = 3$ and $\theta = 1 + \frac{\pi}{4}$

$= 12 + 6\left(1 + \frac{\pi}{4}\right)$

$= \left(18 + \frac{3\pi}{2}\right)\text{cm}$

d Difference

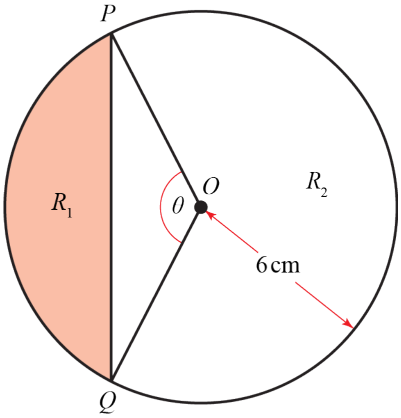
$= \left(18 + \frac{3\pi}{2}\right) - (3\pi + 12)$

$= 6 - \frac{3\pi}{2}$

$= 1.287...\,\text{cm}$

$= 12.9\,\text{mm}$ (3 s.f.)

7



a Area of segment R_1

= area of sector OPQ

− area of triangle OPQ

$\Rightarrow A_1 = \frac{1}{2} \times 6^2 \times \theta - \frac{1}{2} \times 6^2 \times \sin\theta$

$\Rightarrow A_1 = 18(\theta - \sin\theta)$

b $A_2 =$ area of circle $- A_1$

$= \pi \times 6^2 - 18(\theta - \sin\theta)$

$= 36\pi - 18(\theta - \sin\theta)$

Since $A_2 = 3A_1$:

$36\pi - 18(\theta - \sin\theta) = 3 \times 18(\theta - \sin\theta)$

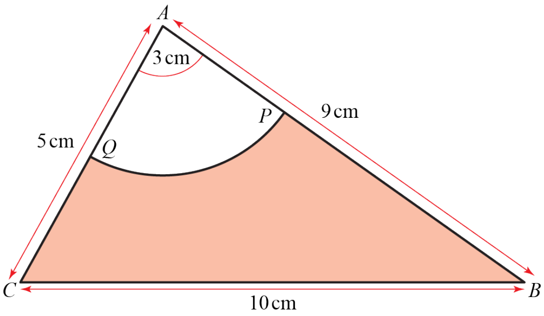
$36\pi - 18(\theta - \sin\theta) = 54(\theta - \sin\theta)$

$36\pi = 72(\theta - \sin\theta)$

$\pi = \theta - \sin\theta$

$\sin\theta = \theta - \frac{\pi}{2}$

8



a Using the cosine rule in $\triangle ABC$:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\Rightarrow \cos \angle BAC = \frac{5^2 + 9^2 - 10^2}{2 \times 5 \times 9} = 0.0\dot{6}$$

$$\Rightarrow \angle BAC = 1.50408...$$

$$= 1.504 \text{ rad (3 d.p.)}$$

b i Using the sector area formula:

$$\text{area of sector} = \frac{1}{2}r^2\theta$$

$\Rightarrow$ area of sector APQ

$$= \frac{1}{2} \times 3^2 \times 1.504 = 6.77 \text{ cm}^2 \ \text{(3 s.f.)}$$

ii Area of shaded region $BPQC$

= area of $\triangle ABC$ – area of sector APQ

$$= \frac{1}{2} \times 5 \times 9 \times \sin 1.504 - \frac{1}{2} \times 3^2 \times 1.504$$

$$= 15.681...$$

$$= 15.7 \text{ cm}^2 \ \text{(3 s.f.)}$$

iii Perimeter of shaded region $BPQC$

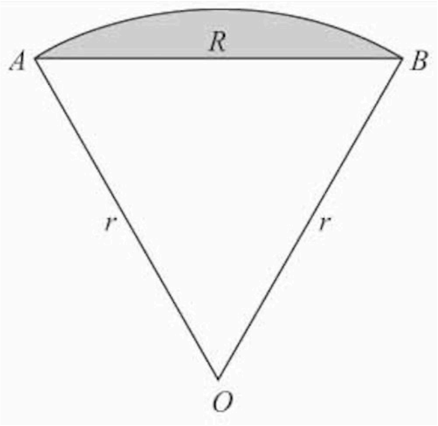
$= QC + CB + BP +$ arc length PQ

$$= 2 + 10 + 6 + (3 \times 1.504)$$

$$= 22.51...$$

$$= 22.5 \text{ cm (3 s.f.)}$$

9



a Area of sector $= \frac{1}{2}r^2\theta = \frac{1}{2}r^2 \times 1.5 \text{ cm}^2$

So $\frac{3}{4}r^2 = 15$

$$\Rightarrow r^2 = \frac{60}{3} = 20$$

$$\Rightarrow r = \sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$$

b Arc length $AB = r(1.5) = 3\sqrt{5}$ cm

Perimeter of sector

$= AO + OB +$ arc length AB

$$= 2\sqrt{5} + 2\sqrt{5} + 3\sqrt{5}$$

$$= 7\sqrt{5}$$

$$= 15.7 \text{ cm (3 s.f.)}$$

c Area of segment R

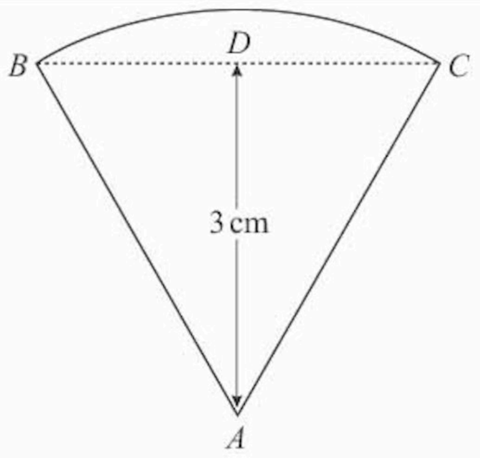
= area of sector – area of $\triangle AOB$

$$= 15 - \frac{1}{2}r^2 \sin 1.5$$

$$= 15 - 10 \sin 1.5$$

$$= 5.025 \text{ cm}^2 \text{ (3 d.p.)}$$

10



a Using the right-angled $\triangle ABD$, with

$\angle ABD = \frac{\pi}{3}$:

$$\sin\frac{\pi}{3} = \frac{3}{AB}$$

$$\Rightarrow AB = \frac{3}{\sin\frac{\pi}{3}} = \frac{3}{\frac{\sqrt{3}}{2}}$$

$$= 3 \times \frac{2}{\sqrt{3}} = 2\sqrt{3} \text{ cm}$$

b Area of badge = area of sector

$$= \frac{1}{2} \times (2\sqrt{3})^2 \theta \text{ where } \theta = \frac{\pi}{3}$$

$$= \frac{1}{2} \times 4 \times 3 \times \frac{\pi}{3}$$

$$= 2\pi \text{ cm}^2$$

c Perimeter of badge

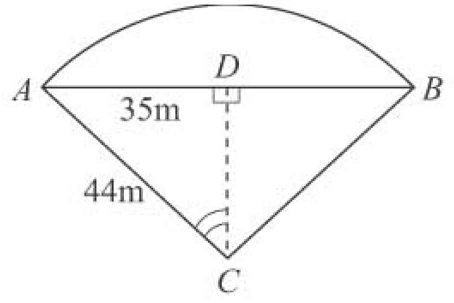
$= AB + AC +$ arc length BC

$$= 2\sqrt{3} + 2\sqrt{3} + 2\sqrt{3} \times \frac{\pi}{3}$$

$$= 2\sqrt{3}\left(2 + \frac{\pi}{3}\right)$$

$$= \frac{2\sqrt{3}}{3}(6 + \pi) \text{ cm}$$

11



a Using the right-angled $\triangle ADC$:

$$\sin \angle ACD = \frac{35}{44}$$

$$\text{So } \angle ACD = \sin^{-1}\frac{35}{44}$$

$$\text{and } \angle ACB = 2\sin^{-1}\frac{35}{44}$$

$$\Rightarrow \angle ACB = 1.8395...$$

$$= 1.84 \text{ rad (2 d.p.)}$$

b i Length of railway track

= length of arc AB

$= 44 \times 1.8395...$

$= 80.9$m (3 s.f.)

ii Shortest distance from C to AB is DC. Using Pythagoras' theorem:

$$DC^2 = 44^2 - 35^2$$

$$DC = \sqrt{44^2 - 35^2} = 26.7\text{m (3 s.f.)}$$

iii Area of region

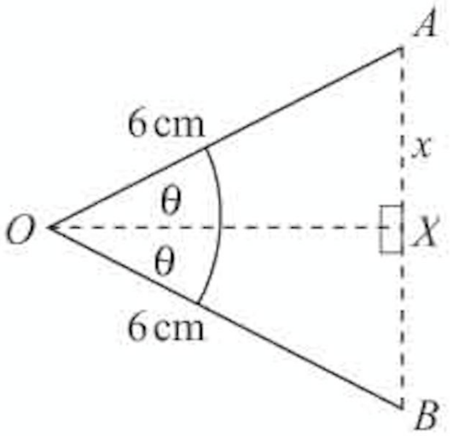
= area of segment

= area of sector ABC – area of $\triangle ABC$

$$= \frac{1}{2} \times 44^2 \times 1.8395... - \frac{1}{2} \times 70 \times DC$$

$$= 847\text{m}^2 \text{ (3 s.f.)}$$

12



a In right-angled $\triangle OAX$ (see diagram):

$$\frac{x}{6} = \sin\theta$$

$$\Rightarrow x = 6\sin\theta$$

So $AB = 2x = 12\sin\theta$ $(AB = DC)$

Perimeter of the cross-section

$= \text{arc length } AB + AD + DC + BC$

$= 6\times 2\theta + 4 + 12\sin\theta + 4\text{ cm}$

$= (8 + 12\theta + 12\sin\theta)\text{ cm}$

So $2(7+\pi) = 8 + 12\theta + 12\sin\theta$

$\Rightarrow 14 + 2\pi = 8 + 12\theta + 12\sin\theta$

$\Rightarrow 12\theta + 12\sin\theta - 6 = 2\pi$

Divide by 6:

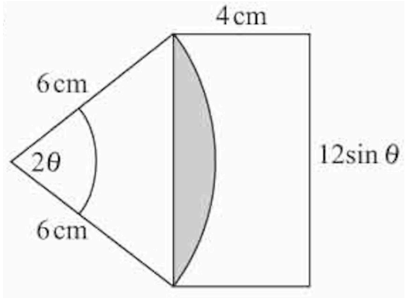
$$2\theta + 2\sin\theta - 1 = \frac{\pi}{3}$$

b When $\theta = \frac{\pi}{6}$,

$$2\theta + 2\sin\theta - 1 = \frac{\pi}{3} + \left(2\times\frac{1}{2}\right) - 1$$

$$= \frac{\pi}{3}$$

c



Area of the cross-section

$=$ area of rectangle $ABCD$

$-$ area of shaded segment

Area of rectangle $= 4\times 12\times\sin\frac{\pi}{6}$

$= 24\text{ cm}^2$

Area of shaded segment

$=$ area of sector $-$ area of triangle

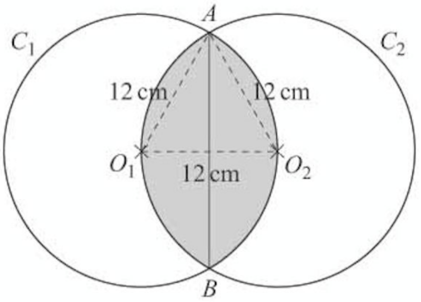
$$= \frac{1}{2}\times 6^2\times\frac{\pi}{3} - \frac{1}{2}\times 6^2\times\sin\frac{\pi}{3}$$

$= 3.261\ldots\text{cm}^2$

So area of cross-section

$= 20.7\text{ cm}^2$ (3 s.f.)

13



a $O_1A = O_2A = 12$, as they are radii of their respective circles.
$O_1O_2 = 12$, as O_2 is on the circumference of C_1 and hence is a radius (and vice versa).
Therefore $\triangle AO_1O_2$ is equilateral.

So $\angle AO_1O_2 = \frac{\pi}{3}$

and $\angle AO_1B = 2\times\angle AO_1O_2 = \frac{2\pi}{3}$

13 b Consider arc AO_2B of circle C_1.
Using arc length $= r\theta$:

arc length $AO_2B = 12 \times \frac{2\pi}{3} = 8\pi$ cm

Perimeter of R

$= $ arc length AO_2B + arc length AO_1B

$= 2 \times 8\pi = 16\pi$ cm

c Consider the segment AO_2B in circle C_1.
Area of segment AO_2B

$= $ area of sector O_1AB − area of $\triangle O_1AB$

$$= \frac{1}{2} \times 12^2 \times \frac{2\pi}{3} - \frac{1}{2} \times 12^2 \times \sin\frac{2\pi}{3}$$

$= 88.442...\text{cm}^2$

Area of region R

$= $ area of segment AO_2B

+ area of segment AO_1B

$= 2 \times 88.442...$

$= 177\text{ cm}^2$ (3 s.f.)

14 a The student has used an angle measured in degrees – it needs to be measured in radians to use that formula.

b
$$50° = \frac{50}{180} \times \pi \text{ rad}$$
$$\frac{1}{2}r^2\theta = \frac{1}{2} \times 3^2 \times \frac{5}{18}\pi$$
$$= \frac{5}{4}\pi \text{ cm}^2$$

15 a
$$\frac{\cos\theta - 1}{\theta\tan 2\theta} \approx \frac{\left(1 - \frac{\theta^2}{2}\right) - 1}{\theta \times 2\theta}$$
$$= \frac{-\frac{\theta^2}{2}}{2\theta^2}$$
$$= \frac{-\theta^2}{4\theta^2}$$
$$= -\frac{1}{4}$$

b
$$\frac{2(1-\cos\theta) - 1}{\tan\theta - 1} \approx \frac{2\left(1 - \left(1 - \frac{\theta^2}{2}\right)\right) - 1}{\theta - 1}$$
$$= \frac{2 \times \frac{\theta^2}{2} - 1}{\theta - 1}$$
$$= \frac{\theta^2 - 1}{\theta - 1}$$
$$= \frac{(\theta-1)(\theta+1)}{\theta - 1}$$
$$= \theta + 1$$

16 a
$$\frac{7 + 2\cos 2\theta}{\tan 2\theta + 3} \approx \frac{7 + 2\left(1 - \frac{(2\theta)^2}{2}\right)}{2\theta + 3}$$
$$= \frac{7 + 2\left(1 - \frac{4\theta^2}{2}\right)}{2\theta + 3}$$
$$= \frac{9 - 4\theta^2}{2\theta + 3}$$
$$= \frac{(3 + 2\theta)(3 - 2\theta)}{2\theta + 3}$$
$$= 3 - 2\theta$$

b 3

17 a When θ is small:
LHS $= 32\cos 5\theta + 203\tan 10\theta$
$$\approx 32\left(1 - \frac{(5\theta)^2}{2}\right) + 203(10\theta)$$
$$= 32 - 16(25\theta^2) + 2010\theta$$

So $32 - 400\theta^2 - 2030\theta = 182$
$$400\theta^2 + 2030\theta + 150 = 0$$
$$40\theta^2 + 203\theta + 15 = 0$$

b $40\theta^2 + 203\theta + 15 = 0$
$(40\theta - 3)(\theta - 5) = 0$
$$\theta = \frac{3}{40}, 5$$

17 c $\theta = 5$ is not a valid solution, as 5 is not 'small'. $\frac{3}{40}$ is 'small', so this solution is valid.

18
$$
\begin{aligned}
&\cos^4\theta - \sin^4\theta\\
&= (\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)\\
&= \cos^2\theta - \sin^2\theta\\
&= 1 - \sin^2\theta - \sin^2\theta\\
&= 1 - 2\sin^2\theta\\
&\approx 1 - 2\theta^2
\end{aligned}
$$

19 a $3\sin\theta = 2,\ 0 \leqslant \theta \leqslant \pi$
$$\sin\theta = \frac{2}{3}$$
$$\theta = 0.730, 2.41$$

b $\sin\theta = -\cos\theta,\ -\pi \leqslant \theta \leqslant \pi$
$$\tan\theta = -1$$
$$\theta = -\frac{\pi}{4}, \frac{3\pi}{4}$$

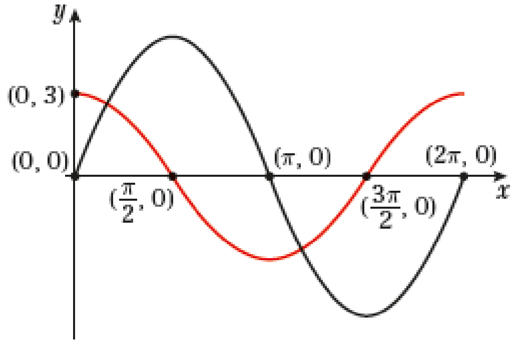
c
$$
\begin{aligned}
\tan\theta + \frac{1}{\tan\theta} &= 2, 0 \leqslant \theta \leqslant 2\pi\\
\tan^2\theta + 1 &= 2\tan\theta\\
\tan^2\theta - 2\tan\theta + 1 &= 0\\
(\tan\theta - 1)^2 &= 0\\
\tan\theta &= 1\\
\theta &= \frac{\pi}{4}, \frac{5\pi}{4}
\end{aligned}
$$

d $2\sin^2\theta - \sin\theta - 1 = \sin^2\theta,\ -\pi \leqslant \theta \leqslant \pi$
$$\sin^2\theta - \sin\theta - 1 = 0$$
$$\sin\theta = \frac{1 \pm \sqrt{5}}{2}$$
$\sin\theta = 1.618$ (no solutions)

or $\sin\theta = -0.618$

$\Rightarrow \theta = -0.666, -2.48$

20 a



b The curves intersect twice in the given range, so the equation has two solutions.

c $5\sin x = 3\cos x$
$$\tan x = \frac{3}{5}$$
$$x = 0.540, 3.68$$

21 a
$$
\begin{aligned}
4\sin\theta - \cos\left(\frac{\pi}{2} - \theta\right) &= 4\sin\theta - \sin\theta\\
&= 3\sin\theta
\end{aligned}
$$

b $4\sin\theta - \cos\left(\frac{\pi}{2} - \theta\right) = 1,\ 0 \leqslant \theta \leqslant 2\pi$
$$3\sin\theta = 1$$
$$\sin\theta = \frac{1}{3}$$
$$\theta = 0.340, 2.80$$

22 $\dfrac{\sin 2x + 0.5}{1 - \sin 2x} = 2,\ 0 < x < \dfrac{3\pi}{2}$

$\sin 2x + 0.5 = 2 - 2\sin 2x$

$3\sin 2x = 1.5$

$\sin 2x = 0.5$

Let $X = 2x$

$\sin X = 0.5,\ 0 < X < 3\pi$

$$X = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$
$$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

23 a Cosine can be negative, so do not reject $-\frac{1}{\sqrt{2}}$

b Rearranged incorrectly, so incorrectly square rooted.

c
$$2\cos^2 x = 1$$
$$\cos^2 x = \frac{1}{2}$$
$$\cos x = \pm\frac{1}{\sqrt{2}}$$
$$x = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$$

24 a Not all solutions have been calculated. There will be four solutions in the given interval.

b $2\tan 2x = 5,\ 0 \leqslant x \leqslant 2\pi$

Let $X = 2x$

$2\tan X = 5,\ 0 \leqslant X \leqslant 4\pi$

$\tan X = 2.5$

$X = 1.19, 4.33, 7.47, 10.6$

$x = 0.595, 2.17, 3.74, 5.31$

25 a
$$5\sin x = 1 + \cos^2 x$$
$$5\sin x = 1 + 2(1 - \sin^2 x)$$
$$2\sin^2 x + 5\sin x - 3 = 0$$

b $2\sin^2 x + 5\sin x - 3 = 0,\ 0 \leqslant x \leqslant 2\pi$

$(2\sin x - 1)(\sin x + 3) = 0$

$\sin x = 0.5 \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

or $\sin x = -3$ (no solution)

26 a
$$4\sin^2 x + 9\cos x - 6 = 0$$
$$4(1 - \cos^2 x) + 9\cos x - 6 = 0$$
$$4 - 4\cos^2 x + 9\cos x - 6 = 0$$
$$4\cos^2 x - 9\cos x + 2 = 0$$

b $4\cos^2 x - 9\cos x + 2 = 0,\ 0 \leqslant \theta \leqslant 4\pi$

$(4\cos x - 1)(\cos x - 2) = 0$

$\cos x = 2$ (no solution)

or $\cos x = 0.25 \Rightarrow x = 1.3, 5.0, 7.6, 11.2$

27 a
$$\tan 2x = 5\sin 2x$$
$$\frac{\sin 2x}{\cos 2x} = 5\sin 2x$$
$$\sin 2x = 5\sin 2x\cos 2x$$
$$(1 - 5\cos 2x)\sin 2x = 0$$

b $\sin 2x(1 - 5\cos 2x) = 0,\ 0 \leqslant x \leqslant \pi$

Let $X = 2x$

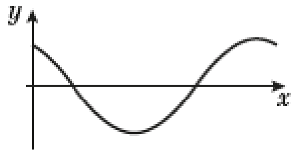
$\sin X(1 - 5\cos X) = 0,\ 0 \leqslant X \leqslant 2\pi$

$\sin X = 0 \Rightarrow X = 0, \pi, 2\pi$

or $\cos X = 0.2 \Rightarrow X = 1.37, 4.91$

$x = 0, 0.7, \frac{\pi}{2}, 2.5, \pi$

28 a



b $\left(0, \frac{\sqrt{3}}{2}\right), \left(\frac{\pi}{3}, 0\right), \left(\frac{4\pi}{3}, 0\right)$

c $\cos\left(x + \frac{\pi}{6}\right) = 0.65,\ 0 \leqslant x \leqslant 2\pi$

Let $X = x + \frac{\pi}{6}$

$\cos X = 0.65,\ \frac{\pi}{6} \leqslant x \leqslant \frac{13\pi}{6}$

$X = 0.863, 5.42$

$x = 0.34, 4.90$

29 $\sin\left(3x+\frac{\pi}{3}\right)=0.45, 0\le x\le\pi$

Let $X=3x+\frac{\pi}{3}, \frac{\pi}{3}\le X\le\frac{10\pi}{3}$

$\sin X=0.45$

$X=2.67, 6.75, 8.96$

$x=0.54, 1.90$ or 2.64 $(2\,\text{d.p.})$

Challenge

a $9\sin\theta\tan\theta+25\tan\theta=6$

When θ is small:

LHS $\approx 9\theta^2+25\theta$

so $9\theta^2+25\theta=6$

$9\theta^2+25\theta-6=0$

$(9\theta-2)(\theta+3)=0$

$\theta=\frac{2}{9}$ or $\theta=-3$

$\theta=\frac{2}{9}$ is ‘small’, so this value is valid.

$\theta=-3$ is not ‘small’, so this value is not valid. ‘Small’ in this context is ‘close to 0’.

b $2\tan\theta+3=5\cos4\theta$

When θ is small:

LHS $\approx 2\theta+3$ and RHS $\approx 5\left(1-\frac{(4\theta)^2}{2}\right)$

so $2\theta+3=5-40\theta^2$

$40\theta^2+2\theta-2=0$

$20\theta^2+\theta-1=0$

$(4\theta+1)(5\theta-1)=0$

$\theta=-\frac{1}{4}, \theta=\frac{1}{5}$

Both values of θ could be considered ‘small’ in this case so both solutions are valid.

c $\sin4\theta=37-2\cos2\theta$

When is small:

LHS $\approx 4\theta$

and RHS $\approx 37-2\left(1-\frac{(2\theta)^2}{2}\right)$

so $4\theta=37-2+4\theta^2$

$4\theta^2-4\theta+35=0$

$b^2-4ac<0$

So there are no solutions.

Trigonometric Functions 6A

1 a 300° is in the 4th quadrant

$$\sec 300° = \frac{1}{\cos 300°}$$

In 4th quadrant cos is +ve,

so $\sec 300°$ is +ve.

b 190° is in the 3rd quadrant

$$\text{cosec}\,190° = \frac{1}{\sin 190°}$$

In 3rd quadrant sin is $-$ve,

so $\text{cosec}\,190°$ is $-$ve.

c 110° is in the 2nd quadrant

$$\cot 110° = \frac{1}{\tan 110°}$$

In the 2nd quadrant tan is $-$ve,

so $\cot 110°$ is $-$ve.

d 200° is in the 3rd quadrant

tan is +ve in the 3rd quadrant,

so cot 200° is+ve.

e 95° is in the 2nd quadrant

cos is $-$ve in the 2nd quadrant,

so sec 95° is $-$ve.

2 a $\sec 100° = \dfrac{1}{\cos 100°} = -5.76$ (3 s.f.)

b $\text{cosec}\,260° = \dfrac{1}{\sin 260°} = -1.02$ (3 s.f.)

c $\text{cosec}\,280° = \dfrac{1}{\sin 280°} = -1.02$ (3 s.f.)

d $\cot 550° = \dfrac{1}{\tan 550°} = 5.67$ (3 s.f.)

e $\cot \dfrac{4\pi}{3} = \dfrac{1}{\tan \frac{4\pi}{3}} = 0.577$ (3 s.f.)

f $\sec 2.4\,\text{rad} = \dfrac{1}{\cos 2.4\,\text{rad}} = -1.36$ (3 s.f.)

g $\text{cosec}\dfrac{11\pi}{10} = \dfrac{1}{\sin \frac{11\pi}{10}} = -3.24$ (3 s.f.)

h $\sec 6\,\text{rad} = \dfrac{1}{\cos 6\,\text{rad}} = 1.04$ (3 s.f.)

3 a $\text{cosec}\,90° = \dfrac{1}{\sin 90°} = \dfrac{1}{1} = 1$

(refer to graph of $y = \sin\theta$)

b $\cot 135° = \dfrac{1}{\tan 135°} = \dfrac{1}{-\tan 45°} = \dfrac{1}{-1} = -1$

c $\sec 180° = \dfrac{1}{\cos 180°} = \dfrac{1}{-1} = -1$

(refer to graph of $y = \cos\theta$)

d 240° is in the 3rd quadrant

$$\sec 240° = \frac{1}{\cos 240°} = \frac{1}{-\cos 60°} = \frac{1}{-\frac{1}{2}} = -2$$

e 300° is in the 4th quadrant

$$\text{cosec}\,300° = \frac{1}{\sin 300°} = \frac{1}{-\sin 60°}$$

$$= \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

f $-45°$ is in the 4th quadrant

$$\cot(-45°) = \frac{1}{\tan(-45°)} = \frac{1}{-\tan 45°}$$

$$= \frac{1}{-1} = -1$$

g $\sec 60° = \dfrac{1}{\cos 60°} = \dfrac{1}{\frac{1}{2}} = 2$

h $-210°$ is in the 2nd quadrant

$$\text{cosec}(-210°) = \frac{1}{\sin(-210°)}$$

$$= \frac{1}{\sin 30°} = \frac{1}{\frac{1}{2}} = 2$$

3 **i** 225° is in the 3rd quadrant

$$\sec 225° = \frac{1}{\cos 225°} = \frac{1}{-\cos 45°}$$
$$= \frac{1}{-\frac{1}{\sqrt{2}}} = -\sqrt{2}$$

j $\frac{4\pi}{3}$ is in the 3rd quadrant

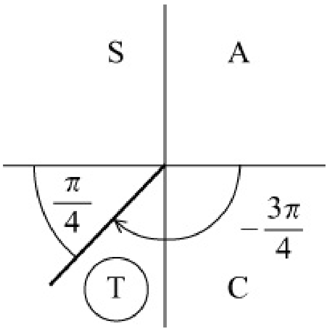
$$\cot \tfrac{4\pi}{3} = \frac{1}{\tan \frac{4\pi}{3}} = \frac{1}{\tan \frac{\pi}{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

k $\frac{11\pi}{6} = 2\pi - \frac{\pi}{6}$ (in the 4th quadrant)

$$\sec \frac{11\pi}{6} = \frac{1}{\cos \frac{11\pi}{6}} = \frac{1}{\cos \frac{\pi}{6}} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

l $-\frac{3\pi}{4}$ is in the 3rd quadrant

$$\operatorname{cosec}\left(-\frac{3\pi}{4}\right) = \frac{1}{\sin\left(-\frac{3\pi}{4}\right)} = \frac{1}{-\sin \frac{\pi}{4}}$$
$$= \frac{\mathbf{1}}{-\frac{1}{\sqrt{2}}} = -\sqrt{\mathbf{2}}$$



4 $$\operatorname{cosec}(\pi - x) \equiv \frac{1}{\sin(\pi - x)}$$
$$\equiv \frac{1}{\sin x}$$
$$\equiv \operatorname{cosec} x$$

5 $$\cot 30° \sec 30° = \frac{1}{\tan 30°} \times \frac{1}{\cos 30°}$$
$$= \frac{\sqrt{3}}{1} \times \frac{2}{\sqrt{3}}$$
$$= 2$$

6 $\frac{2\pi}{3} = \pi - \frac{\pi}{3}$ (in the 2nd quadrant)

$$\operatorname{cosec}\left(\frac{2\pi}{3}\right) + \sec\left(\frac{2\pi}{3}\right) = \frac{1}{\sin\left(\frac{2\pi}{3}\right)} + \frac{1}{\cos\left(\frac{2\pi}{3}\right)}$$
$$= \frac{\mathbf{1}}{\mathbf{sin}\left(\frac{\pi}{3}\right)} + \frac{\mathbf{1}}{-\mathbf{cos}\left(\frac{\pi}{3}\right)}$$
$$= \frac{1}{\frac{\sqrt{3}}{2}} + \frac{1}{-\frac{1}{2}}$$
$$= -2 + \frac{2}{\sqrt{3}}$$
$$= -2 + \frac{2}{3}\sqrt{3}$$

Challenge

a Triangles *OPB* and *OAP* are right-angled triangles as line *AB* is a tangent to the unit circle at *P*.

Using triangle OBP, $OB\cos\theta = 1$

$$\Rightarrow OB = \frac{1}{\cos\theta} = \sec\theta$$

b $\angle POA = 90° - \theta \Rightarrow \angle OAP = \theta$

Using triangle OAP, $OA\sin\theta = 1$

$$\Rightarrow OA = \frac{1}{\sin\theta} = \operatorname{cosec}\theta$$

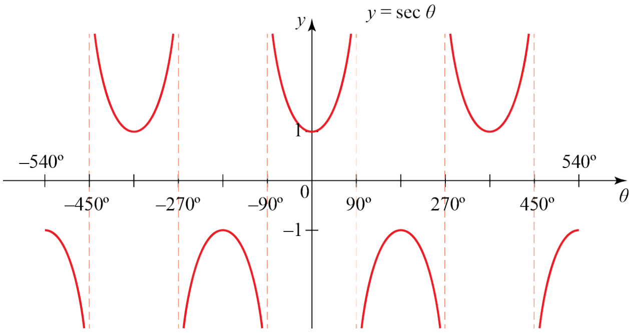
c Using Pythagoras' theorem,
$AP^2 = OA^2 - OP^2$

So, $AP^2 = \operatorname{cosec}^2\theta - 1$
$$= \frac{1}{\sin^2\theta} - 1$$
$$= \frac{1 - \sin^2\theta}{\sin^2\theta}$$
$$= \frac{\cos^2\theta}{\sin^2\theta}$$
$$= \cot^2\theta$$

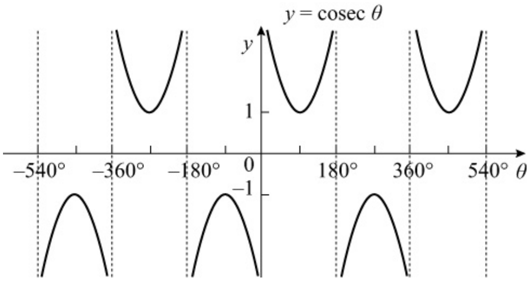
Therefore $AP = \cot\theta$

Trigonometric Functions 6B

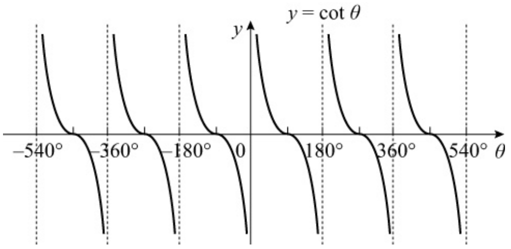
1 a



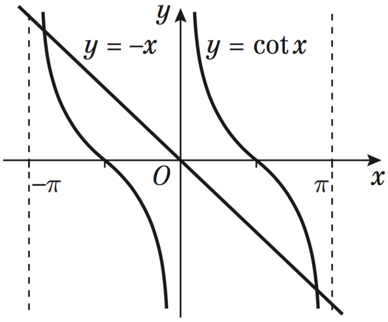
b



c

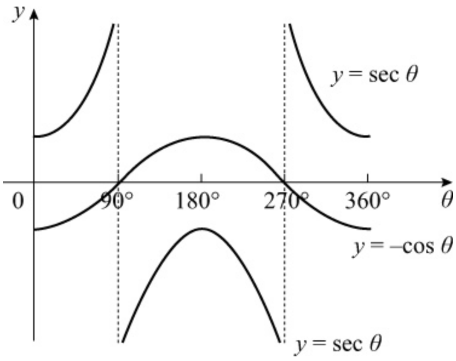


2 a



b 2 solutions

3 a



b You can see that the graphs of $y = \sec\theta$ and $y = -\cos\theta$ do not meet, so $\sec\theta = -\cos\theta$ has no solutions.

The same result can be found algebraically

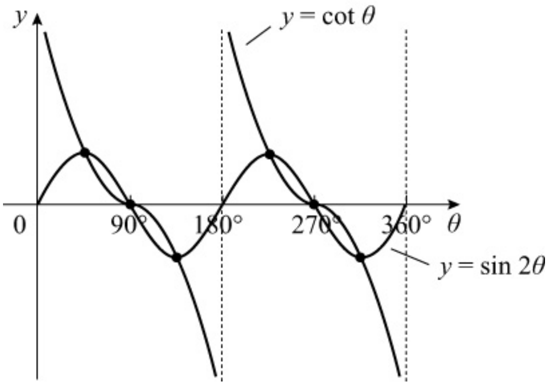
$\sec\theta = -\cos\theta$

$\Rightarrow \dfrac{1}{\cos\theta} = -\cos\theta$

$\Rightarrow \cos^2\theta = -1$

There are no solutions of this equation for real θ.

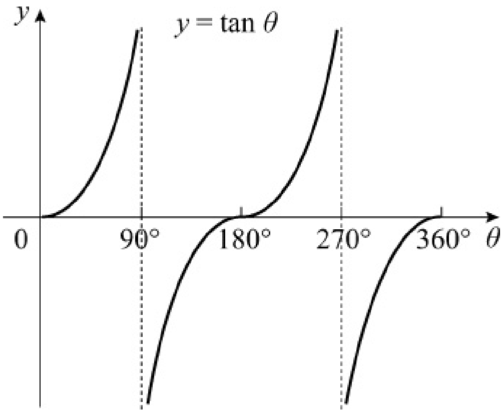
4 a

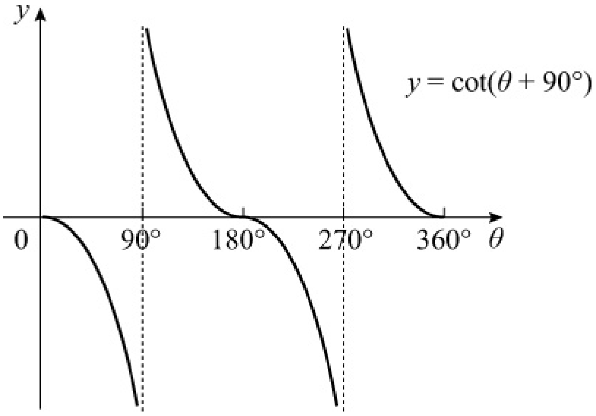


b The curves meet at the maxima and minima of $y = \sin 2\theta$, and on the θ-axis at odd integer multiples of 90°.

In the interval $0 \le \theta \le 360°$ there are 6 intersections. So there are 6 solutions of $\cot\theta = \sin 2\theta$ in the interval $0 \le \theta \le 360°$

5 a





b $y = \cot(\theta + 90°)$ is a reflection in the θ-axis of $y = \tan\theta$, so $\cot(\theta + 90°) = -\tan\theta$

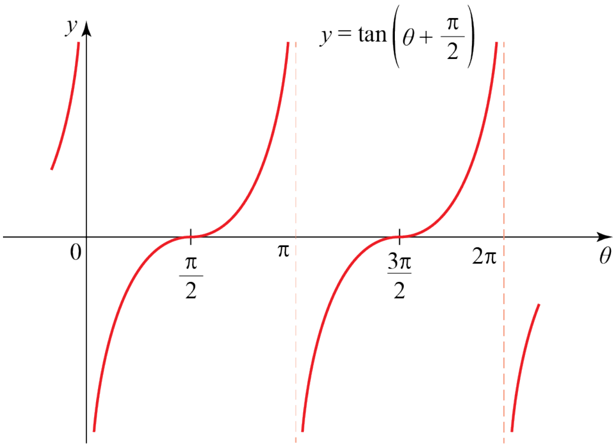
6 a i The graph of $y = \tan\left(\theta + \frac{\pi}{2}\right)$ is the same as that of $y = \tan\theta$ translated by the vector $\begin{pmatrix} -\frac{\pi}{2} \\ 0 \end{pmatrix}$, i.e by $\frac{\pi}{2}$ to the left.

ii The graph of $y = \cot(-\theta)$ is the same as that of $y = \cot\theta$ reflected in the y-axis.

iii The graph of $y = \operatorname{cosec}\left(\theta + \frac{\pi}{4}\right)$ is the same as that of $y = \operatorname{cosec}\theta$ translated by the vector $\begin{pmatrix} -\frac{\pi}{4} \\ 0 \end{pmatrix}$

iv The graph of $\sec\left(\theta - \frac{\pi}{4}\right)$ is the same as that of $y = \sec\theta$ translated by the vector $\begin{pmatrix} \frac{\pi}{4} \\ 0 \end{pmatrix}$

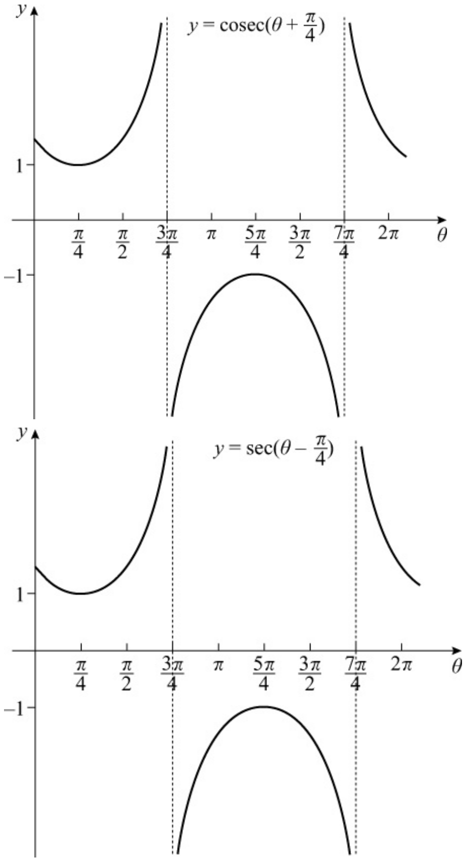
b



(reflection of $y = \cot\theta$ in the y-axis)

$\tan\left(\theta + \frac{\pi}{2}\right) = \cot(-\theta)$

6 b



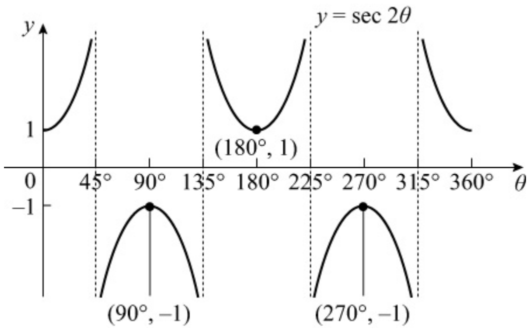
$$\text{cosec}\left(\theta+\frac{\pi}{4}\right)=\sec\left(\theta-\frac{\pi}{4}\right)$$

7 a A stretch of $y = \sec\theta$ in the θ direction with scale factor $\frac{1}{2}$

Minimum at (180°,1)

Maxima at (90°,−1) and (270°,−1)

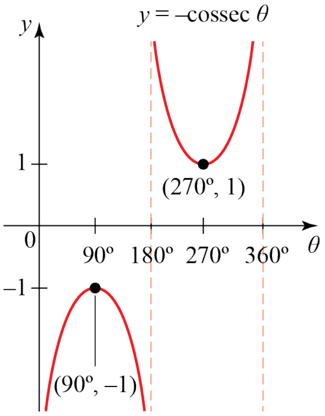
It meets the y-axis at (0, 1)



b Reflection in θ-axis of $y = \text{cosec}\,\theta$

Minimum at (270°, 1)

Maximum at (90°, −1)

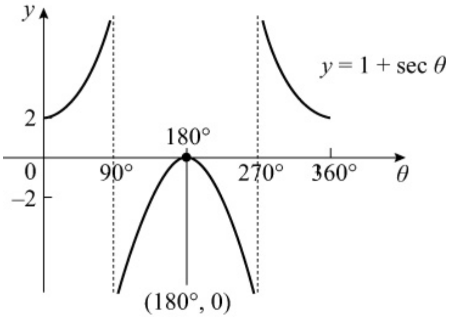


c Translation of $y = \sec\theta$ by the vector $\begin{pmatrix}0\\1\end{pmatrix}$, i.e. +1 in the y direction.

It meets x-axis at (180°, 0)

There is a maximum at (180°, 0)

It meets the y-axis at (0, 2)

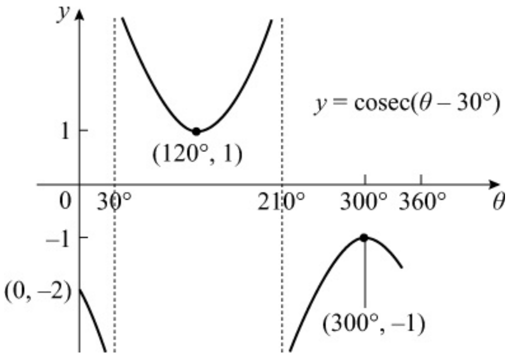


d Translation of $y = \operatorname{cosec}\theta$ by the vector $\begin{pmatrix}30\\0\end{pmatrix}$

Minimum at (120°, 1)

Maximum at $(300°, -1)$

It meets the y-axis at $(0, -2)$

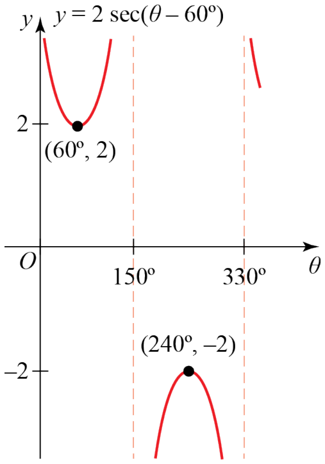


7 e $y = 2\sec(\theta - 60^\circ)$ is $y = \sec\theta$ translated by the vector $\begin{pmatrix}60\\0\end{pmatrix}$ and then stretched by a scale factor 2 in the y direction.

Minimum at (60°, 2)

Maximum at $(240°, -2)$

It meets the y-axis at (0, 4)

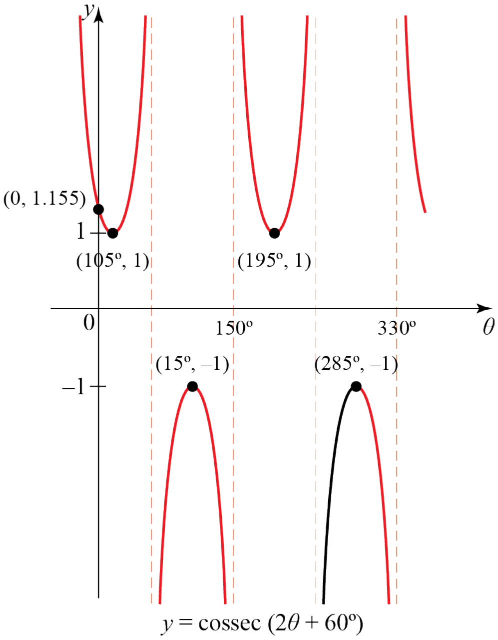


f $y = \operatorname{cosec}(2\theta + 60^\circ)$ is $\boldsymbol{y = \operatorname{cosec}\theta}$ translated by the vector $\begin{pmatrix}-60\\0\end{pmatrix}$ and then stretched by a scale factor $\frac{1}{2}$ in the θ direction.

Minima at (15°, 1), (195°, 1)

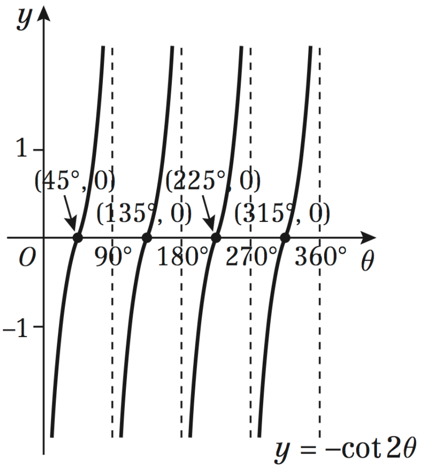
Maxima at $(105°, -1), (285°, -1)$

It meets the y-axis at $(0, 1.155)$



g $y=-\cot 2\theta$ is $y=\cot\theta$ stretched by a scale factor $\frac{1}{2}$ in the θ direction and then reflected in the x-axis.

It meets the θ-axis at (45°, 0), (135°, 0), (225°, 0) and (315°, 0)

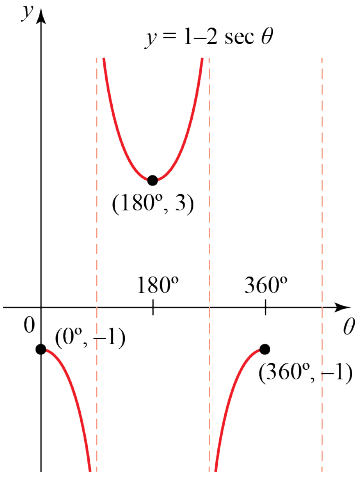


h $y=1-2\sec\theta=-2\sec\theta+1$ is $y=\sec\theta$ stretched by a scale factor 2 in the y direction, reflected in the x-axis and then translated by the vector $\begin{pmatrix}0\\1\end{pmatrix}$

Minima at (180°, 3)

Maxima at $(0,-1)$, $(360°,-1)$

It meets the y-axis at $(0,-1)$



8 **a** The period of $\sec\theta$ is 2π radians

$y=\sec 3\theta$ is a stretch of $y=\sec\theta$ with scale factor $\frac{1}{3}$ in the θ direction.

So the period of $\sec 3\theta$ is $\frac{2\pi}{3}$

b $\operatorname{cosec}\theta$ has a period of 2π

$\operatorname{cosec}\frac{1}{2}\theta$ is a stretch of $\operatorname{cosec}\theta$ in the θ direction with scale factor 2.

So the period of $\operatorname{cosec}\frac{1}{2}\theta$ is 4π

c $\cot\theta$ has a period of π

$2\cot\theta$ is a stretch in the y direction by scale factor 2. So the periodicity is not affected.

The period of $2\cot\theta$ is π

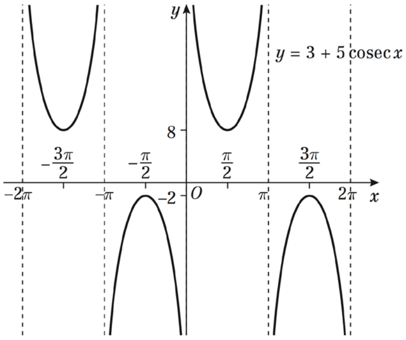
d $\sec\theta$ has a period of 2π

$\sec(-\theta)$ is a reflection in the y-axis.

So the periodicity is not affected.

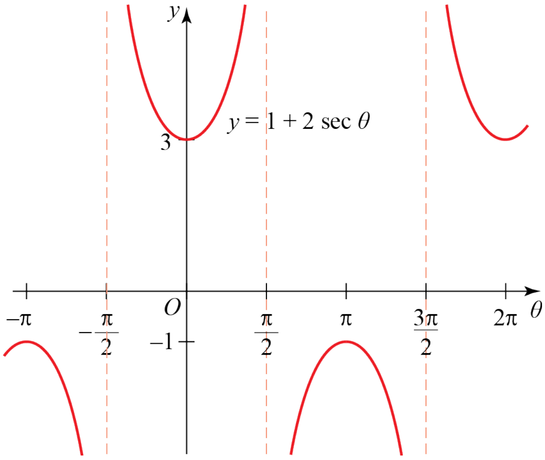
The period of $\sec(-\theta)$ is 2π

9 **a** $y=3+5\operatorname{cosec}\theta$ is $y=\operatorname{cosec}\theta$ stretched by a scale factor 5 in the y direction and then translated by the vector $\begin{pmatrix}0\\3\end{pmatrix}$



b $-2<k<8$

10 a



b The θ coordinates at points at which the gradient is zero are at the maxima and minima. These are $\theta = -\pi, 0, \pi, 2\pi$

c Minimum value of $\frac{1}{1+2\sec\theta}$

is where $1+2\sec\theta$ is a maximum.

So minimum value of $\frac{1}{1+2\sec\theta}$

is $\frac{1}{-1} = -1$

The first positive value of θ where this occurs is when $\theta = \pi$ (see diagram)

Maximum value of $\frac{1}{1+2\sec\theta}$

is where $1+2\sec\theta$ is a minimum.

So maximum value of $\frac{1}{1+2\sec\theta}$ is $\frac{1}{3}$

The first positive value of θ where this occurs is when $\theta = 2\pi$ (see diagram)

Trigonometric Functions 6C

1 **a** $\frac{1}{\sin^3\theta}=\left(\frac{1}{\sin\theta}\right)^3=\operatorname{cosec}^3\theta$

b $\frac{4}{\tan^6\theta}=4\times\left(\frac{1}{\tan\theta}\right)^6=4\cot^6\theta$

c $\frac{1}{2\cos^2\theta}=\frac{1}{2}\times\left(\frac{1}{\cos\theta}\right)^2=\frac{1}{2}\sec^2\theta$

d $\frac{1-\sin^2\theta}{\sin^2\theta}=\frac{\cos^2\theta}{\sin^2\theta}$

(using $\sin^2\theta+\cos^2\theta\equiv 1$)

So $\frac{1-\sin^2\theta}{\sin^2\theta}=\left(\frac{\cos\theta}{\sin\theta}\right)^2=\cot^2\theta$

e $\frac{\sec\theta}{\cos^4\theta}=\frac{1}{\cos\theta}\times\frac{1}{\cos^4\theta}=\frac{1}{\cos^5\theta}$

$=\left(\frac{1}{\cos\theta}\right)^5=\sec^5\theta$

f $\sqrt{\operatorname{cosec}^3\theta\cot\theta\sec\theta}$

$=\sqrt{\frac{1}{\sin^3\theta}\times\frac{\cos\theta}{\sin\theta}\times\frac{1}{\cos\theta}}=\sqrt{\frac{1}{\sin^4\theta}}$

$=\frac{1}{\sin^2\theta}=\left(\frac{1}{\sin\theta}\right)^2=\operatorname{cosec}^2\theta$

g $\frac{2}{\sqrt{\tan\theta}}=2\times\frac{1}{(\tan\theta)^{\frac{1}{2}}}=2\cot^{\frac{1}{2}}\theta$

h $\frac{\operatorname{cosec}^2\theta\tan^2\theta}{\cos\theta}=\frac{1}{\sin^2\theta}\times\frac{\sin^2\theta}{\cos^2\theta}\times\frac{1}{\cos\theta}$

$=\left(\frac{1}{\cos\theta}\right)^3=\sec^3\theta$

2 **a** $5\sin x=4\cos x$

$\Rightarrow\ 5=\frac{4\cos x}{\sin x}$ (divide by $\sin\ x$)

$\Rightarrow\ \frac{5}{4}=\cot x$ (divide by 4)

b $\tan x=-2$

$\Rightarrow\ \frac{1}{\tan x}=\frac{1}{-2}$

$\Rightarrow\ \cot x=-\frac{1}{2}$

c $3\frac{\sin x}{\cos x}=\frac{\cos x}{\sin x}$

$\Rightarrow\ 3\sin^2 x=\cos^2 x$

(multiply by $\sin x\cos x$)

$\Rightarrow\ 3=\frac{\cos^2 x}{\sin^2 x}$

(divide by $\sin^2 x$)

$\Rightarrow\ \left(\frac{\cos x}{\sin x}\right)^2=3$

$\Rightarrow\ \cot^2 x=3$

$\Rightarrow\ \cot x=\pm\sqrt{3}$

3 **a** $\sin\theta\cot\theta=\sin\theta\times\frac{\cos\theta}{\sin\theta}=\cos\theta$

b $\tan\theta\cot\theta=\tan\theta\times\frac{1}{\tan\theta}=1$

c $\tan 2\theta\operatorname{cosec}2\theta=\frac{\sin 2\theta}{\cos 2\theta}\times\frac{1}{\sin 2\theta}$

$=\frac{1}{\cos 2\theta}=\sec 2\theta$

d $\cos\theta\sin\theta(\cot\theta+\tan\theta)$

$=\cos\theta\sin\theta\left(\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}\right)$

$=\cos^2\theta+\sin^2\theta=1$

e $\sin^3 x\operatorname{cosec}x+\cos^3 x\sec x$

$=\sin^3 x\times\frac{1}{\sin x}+\cos^3 x\times\frac{1}{\cos x}$

$=\sin^2 x+\cos^2 x=1$

3 f $\sec A - \sec A \sin^2 A$

$= \sec A(1-\sin^2 A)$ (factorise)

$= \dfrac{1}{\cos A} \times \cos^2 A$

(using $\sin^2 A + \cos^2 A \equiv 1$)

$= \cos A$

g $\sec^2 x \cos^5 x + \cot x \operatorname{cosec} x \sin^4 x$

$= \dfrac{1}{\cos^2 x} \times \cos^5 x + \dfrac{\cos x}{\sin x} \times \dfrac{1}{\sin x} \times \sin^4 x$

$= \cos^3 x + \sin^2 x \cos x$

$= \cos x(\cos^2 x + \sin^2 x)$

$= \cos x$ (since $\cos^2 x + \sin^2 x \equiv 1$)

4 a LHS $\equiv \cos\theta + \sin\theta \tan\theta$

$\equiv \cos\theta + \sin\theta \dfrac{\sin\theta}{\cos\theta}$

$\equiv \dfrac{\cos^2\theta + \sin^2\theta}{\cos\theta}$

$\equiv \dfrac{1}{\cos\theta}$ (using $\sin^2\theta + \cos^2\theta \equiv 1$)

$\equiv \sec\theta \equiv$ RHS

b LHS $\equiv \cot\theta + \tan\theta$

$\equiv \dfrac{\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta}$

$\equiv \dfrac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$

$\equiv \dfrac{1}{\sin\theta\cos\theta}$

$\equiv \dfrac{1}{\sin\theta} \times \dfrac{1}{\cos\theta}$

$\equiv \operatorname{cosec}\theta \sec\theta \equiv$ RHS

c LHS $\equiv \operatorname{cosec}\theta - \sin\theta$

$\equiv \dfrac{1}{\sin\theta} - \sin\theta$

$\equiv \dfrac{1-\sin^2\theta}{\sin\theta}$

$\equiv \dfrac{\cos^2\theta}{\sin\theta}$

$\equiv \cos\theta \times \dfrac{\cos\theta}{\sin\theta}$

$\equiv \cos\theta \cot\theta \equiv$ RHS

d LHS $\equiv (1-\cos x)(1+\sec x)$

$\equiv 1 - \cos x + \sec x - \cos x \sec x$

(multiplying out)

$\equiv \sec x - \cos x$ (as $\cos x \sec x = 1$)

$\equiv \dfrac{1}{\cos x} - \cos x$

$\equiv \dfrac{1-\cos^2 x}{\cos x}$

$\equiv \dfrac{\sin^2 x}{\cos x}$

$\equiv \sin x \times \dfrac{\sin x}{\cos x}$

$\equiv \sin x \tan x \equiv$ RHS

e LHS $\equiv \dfrac{\cos x}{1-\sin x} + \dfrac{1-\sin x}{\cos x}$

$\equiv \dfrac{\cos^2 x + (1-\sin x)^2}{(1-\sin x)\cos x}$

$\equiv \dfrac{\cos^2 x + (1 - 2\sin x + \sin^2 x)}{(1-\sin x)\cos x}$

$\equiv \dfrac{2-2\sin x}{(1-\sin x)\cos x}$

(using $\sin^2 x + \cos^2 x \equiv 1$)

$\equiv \dfrac{2(1-\sin x)}{(1-\sin x)\cos x}$

(factorising)

$= \dfrac{2}{\cos x}$

$\equiv 2\sec x \equiv$ RHS

f LHS $\equiv \dfrac{\cos\theta}{1+\cot\theta}$

$\equiv \dfrac{\cos\theta}{1+\frac{1}{\tan\theta}}$

$\equiv \dfrac{\cos\theta}{\frac{\tan\theta+1}{\tan\theta}}$

$\equiv \dfrac{\cos\theta\tan\theta}{1+\tan\theta}$

$\equiv \dfrac{\cos\theta \times \frac{\sin\theta}{\cos\theta}}{1+\tan\theta}$

$\equiv \dfrac{\sin\theta}{1+\tan\theta} \equiv$ RHS

5 a $\sec\theta = \sqrt{2}$

$\Rightarrow \dfrac{1}{\cos\theta} = \sqrt{2}$

$\Rightarrow \cos\theta = \dfrac{1}{\sqrt{2}}$

Calculator value is $\theta = 45°$

$\cos\theta$ is positive

$\Rightarrow \theta$ is in 1st and 4th quadrants

Solutions are 45°, 315°

b $\operatorname{cosec}\theta = -3$

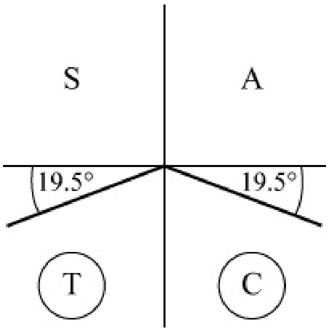
$\Rightarrow \dfrac{1}{\sin\theta} = -3$

$\Rightarrow \sin\theta = -\dfrac{1}{3}$

Calculator value is $\theta = -19.47°$ (2 d.p.)

$\sin\theta$ is negative

$\Rightarrow \theta$ is in 3rd and 4th quadrants



Solutions are 199°, 341° (3 s.f.)

c $5\cot\theta = -2$

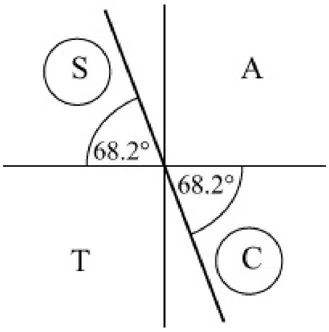
$\Rightarrow \cot\theta = -\dfrac{2}{5}$

$\Rightarrow \tan\theta = -\dfrac{5}{2}$

Calculator value is $\theta = -68.20°$ (2 d.p.)

$\tan\theta$ is negative

$\Rightarrow \theta$ is in 2nd and 4th quadrants



Solutions are 112°, 292° (3 s.f.)

d $\operatorname{cosec}\theta = 2$

$\Rightarrow \dfrac{1}{\sin\theta} = 2$

$\Rightarrow \sin\theta = \dfrac{1}{2}$

Calculator value is $\theta = 30°$

$\sin\theta$ is positive

$\Rightarrow \theta$ is in 1st and 2nd quadrants

Solutions are 30°, 150°

e $3\sec^2\theta = 4$

$\Rightarrow \sec^2\theta = \dfrac{4}{3}$

$\Rightarrow \cos^2\theta = \dfrac{3}{4}$

$\Rightarrow \cos\theta = \pm\dfrac{\sqrt{3}}{2}$

Calculator value for $\cos\theta = \dfrac{\sqrt{3}}{2}$ is $\theta = 30°$

As $\cos\theta$ is ±, θ is in all four quadrants

Solutions are 30°, 150°, 210°, 330°

f $5\cos\theta = 3\cot\theta$

$\Rightarrow 5\cos\theta = 3\dfrac{\cos\theta}{\sin\theta}$

Do not cancel $\cos\theta$ on each side.

Multiply through by $\sin\theta$.

$\Rightarrow 5\cos\theta\sin\theta = 3\cos\theta$

$\Rightarrow 5\cos\theta\sin\theta - 3\cos\theta = 0$

$\Rightarrow \cos\theta(5\sin\theta - 3) = 0$ (factorise)

So $\cos\theta = 0$ or $\sin\theta = \dfrac{3}{5}$

When $\cos\theta = 0$, $\theta = 90°, 270°$

When $\sin\theta = \dfrac{3}{5}$, $\theta = 36.9°, 143°$ (3 s.f.)

Solutions are 36.9°, 90°, 143°, 270°

5 **g** $\cot^2\theta - 8\tan\theta = 0$

$\Rightarrow \dfrac{1}{\tan^2\theta} - 8\tan\theta = 0$

$\Rightarrow 1 - 8\tan^3\theta = 0$

$\Rightarrow 8\tan^3\theta = 1$

$\Rightarrow \tan^3\theta = \dfrac{1}{8}$

$\Rightarrow \tan\theta = \dfrac{1}{2}$

Calculator value is $\theta = 26.57°$ (2 d.p.)

$\tan\theta$ is positive

$\Rightarrow \theta$ is in 1st and 3rd quadrants

Solutions are $26.57°$ and $(180° + 26.57°)$

So solutions are $26.6°,\ 207°$ (3 s.f.)

h $2\sin\theta = \operatorname{cosec}\theta$

$\Rightarrow 2\sin\theta = \dfrac{1}{\sin\theta}$

$\Rightarrow \sin^2\theta = \dfrac{1}{2}$

$\Rightarrow \sin\theta = \pm\dfrac{1}{\sqrt{2}}$

Calculator value for $\sin\theta = \dfrac{1}{\sqrt{2}}$ is $\theta = 45°$

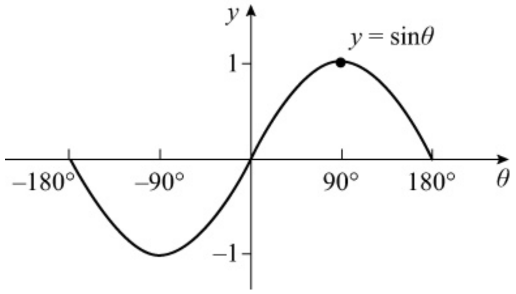
Solutions are in all four quadrants

Solutions are 45°, 135°, 225°, 315°

6 **a** $\operatorname{cosec}\theta = 1$

$\Rightarrow \sin\theta = 1$

$\Rightarrow \theta = 90°$



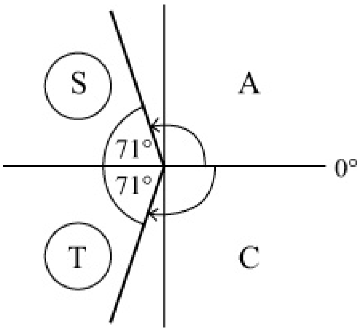
b $\sec\theta = -3$

$\Rightarrow \cos\theta = -\dfrac{1}{3}$

Calculator value is $\theta = 109°$ (3 s.f.)

$\cos\theta$ is negative

$\Rightarrow \theta$ is in 2nd and 3rd quadrants



Solutions are $109°,\ -109°$ (3 s.f.)

c $\cot\theta = 3.45$

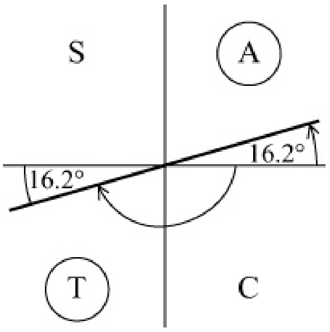
$\Rightarrow \dfrac{1}{\tan\theta} = 3.45$

$\Rightarrow \tan\theta = \dfrac{1}{3.45} = 0.2899$ (4 d.p.)

Calculator value is $\theta = 16.16°$ (2 d.p.)

$\tan\theta$ is positive

$\Rightarrow \theta$ is in 1st and 3rd quadrants



Solutions are $16.2°$ and $(-180° + 16.2°)$

So solutions are $16.2°,\ -164°$ (3 s.f.)

6 d $2\operatorname{cosec}^2\theta - 3\operatorname{cosec}\theta = 0$

$\Rightarrow \operatorname{cosec}\theta(2\operatorname{cosec}\theta - 3) = 0$ (factorise)

$\Rightarrow \operatorname{cosec}\theta = 0$ or $\operatorname{cosec}\theta = \dfrac{3}{2}$

$\Rightarrow \sin\theta = \dfrac{2}{3}$

$\operatorname{cosec}\theta = 0$ has no solutions

Calculator value for $\sin\theta = \frac{2}{3}$ is $\theta = 41.8°$

θ is in 1st and 2nd quadrants

Solutions are $41.8°$, $(180 - 41.8)°$

So solutions are $41.8°$, $138°$ (3 s.f.)

e $\sec\theta = 2\cos\theta$

$\Rightarrow \dfrac{1}{\cos\theta} = 2\cos\theta$

$\Rightarrow \cos^2\theta = \dfrac{1}{2}$

$\Rightarrow \cos\theta = \pm\dfrac{1}{\sqrt{2}}$

Calculator value for $\cos\theta = \frac{1}{\sqrt{2}}$ is $\theta = 45°$

θ is in all quadrants, but remember that solutions required for $-180° \le \theta \le 180°$

Solutions are $\pm 45°$, $\pm 135°$

f $3\cot\theta = 2\sin\theta$

$\Rightarrow 3\dfrac{\cos\theta}{\sin\theta} = 2\sin\theta$

$\Rightarrow 3\cos\theta = 2\sin^2\theta$

$\Rightarrow 3\cos\theta = 2(1 - \cos^2\theta)$

(use $\sin^2\theta + \cos^2\theta \equiv 1$)

$\Rightarrow 2\cos^2\theta + 3\cos\theta - 2 = 0$

$\Rightarrow (2\cos\theta - 1)(\cos\theta + 2) = 0$

$\Rightarrow \cos\theta = \dfrac{1}{2}$ or $\cos\theta = -2$

As $\cos\theta = -2$ has no solutions, $\cos\theta = \dfrac{1}{2}$

Solutions are $\pm 60°$

g $\operatorname{cosec} 2\theta = 4$

$\Rightarrow \sin 2\theta = \dfrac{1}{4}$

Remember that solutions are required in the interval $-180° \le \theta \le 180°$

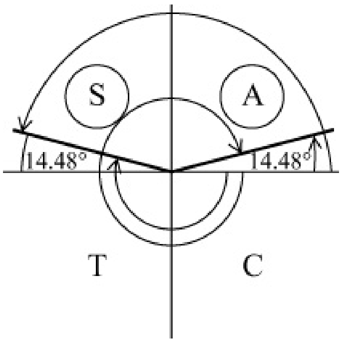
So $-360° \le 2\theta \le 360°$

Calculator value for $\sin 2\theta = \frac{1}{4}$ is

$2\theta = 14.48°$ (2 d.p.)

$\sin 2\theta$ is positive

$\Rightarrow 2\theta$ is in 1st and 2nd quadrants



$2\theta = -194.48°, \ -345.52°,$

$14.48°, \ 165.52°$

$\theta = -97.2°, \ -172.8°, \ 7.24°, \ 82.76°$

$= -173°, \ -97.2°, \ 7.24°, \ 82.8°$ (3 s.f.)

6 **h** $2\cot^2\theta-\cot\theta-5=0$

As this quadratic in $\cot\theta$ does not factorise, use the quadratic formula

$$\cot\theta=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

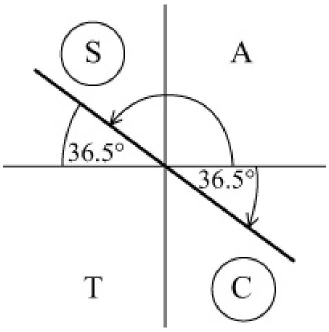
(You could change $\cot\theta$ to $\frac{1}{\tan\theta}$ and work with the quadratic $5\tan^2\theta+\tan\theta-2=0$)

So $\cot\theta=\dfrac{1\pm\sqrt{41}}{4}$

$=-1.3508,\ 1.8508$ (4 d.p.)

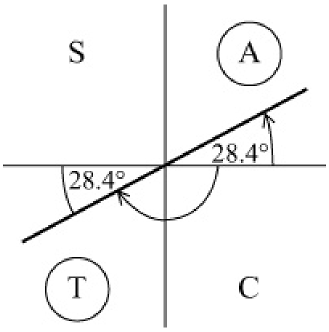
So $\tan\theta=-0.7403,\ 0.5403$ (4 d.p.)

The calculator value for $\tan\theta=-0.7403$ is $\theta=-36.51°$ (2 d.p.)



Solutions are $-36.5°,\ 143°$ (3 s.f.).

The calculator value for $\tan\theta=0.5403$ is $\theta=28.38°$ (2 d.p.)



Solutions are $28.4°,\ (-180+28.4)°$

Total set of solutions is

$-152°,\ -36.5°,\ 28.4°,\ 143°$ (3 s.f.)

7 **a** $\sec\theta=-1$

$\Rightarrow\ \cos\theta=-1$

$\Rightarrow\ \theta=\pi$

(refer to graph of $y=\cos\theta$)

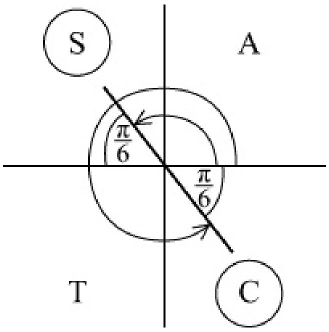
b $\cot\theta=-\sqrt{3}$

$\Rightarrow\ \tan\theta=-\dfrac{1}{\sqrt{3}}$

Calculator solution is $-\dfrac{\pi}{6}$

(you should know that $\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$)

$-\dfrac{\pi}{6}$ is not in the interval



Solutions are $\pi-\dfrac{\pi}{6},\ 2\pi-\dfrac{\pi}{6}=\dfrac{5\pi}{6},\ \dfrac{11\pi}{6}$

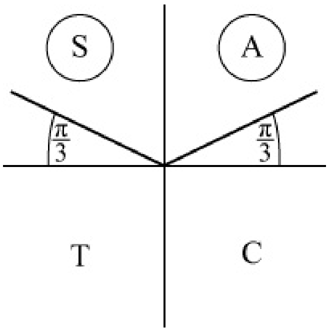
c $\operatorname{cosec}\dfrac{\theta}{2}=\dfrac{2\sqrt{3}}{3}$

$\Rightarrow\ \sin\dfrac{\theta}{2}=\dfrac{3}{2\sqrt{3}}=\dfrac{\sqrt{3}}{2}$

Remember that $0\le\theta\le2\pi$

so $0\le\dfrac{\theta}{2}\le\pi$

First solution for $\sin\dfrac{\theta}{2}=\dfrac{\sqrt{3}}{2}$ is $\dfrac{\theta}{2}=\dfrac{\pi}{3}$



So $\dfrac{\theta}{2}=\dfrac{\pi}{3},\ \dfrac{2\pi}{3}$

$\Rightarrow\ \theta=\dfrac{2\pi}{3},\ \dfrac{4\pi}{3}$

7 d $\sec\theta = \sqrt{2}\tan\theta$

$\Rightarrow \dfrac{1}{\cos\theta} = \sqrt{2}\dfrac{\sin\theta}{\cos\theta}$

$\Rightarrow 1 = \sqrt{2}\sin\theta \qquad (\cos\theta \neq 0)$

$\Rightarrow \sin\theta = \dfrac{1}{\sqrt{2}}$

Solutions are $\dfrac{\pi}{4}, \dfrac{3\pi}{4}$

8 a In the right-angled triangle ABD

$\dfrac{AB}{AD} = \cos\theta$

$\Rightarrow AD = \dfrac{6}{\cos\theta} = 6\sec\theta$

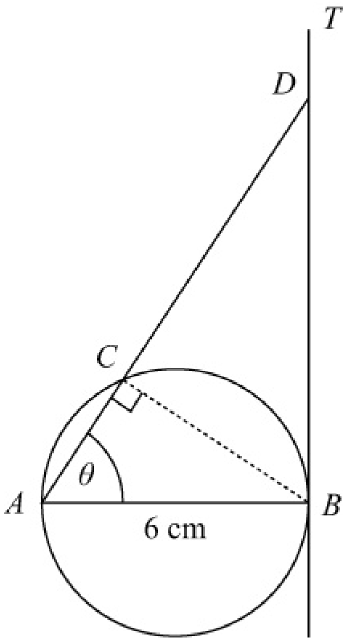
In the right-angled triangle ACB

$\dfrac{AC}{AB} = \cos\theta$

$\Rightarrow AC = 6\cos\theta$

$CD = AD - AC$

$= 6\sec\theta - 6\cos\theta = 6(\sec\theta - \cos\theta)$



b As $16 = 6\sec\theta - 6\cos\theta$

$\Rightarrow 8 = \dfrac{3}{\cos\theta} - 3\cos\theta$

$\Rightarrow 8\cos\theta = 3 - 3\cos^2\theta$

$\Rightarrow 3\cos^2\theta + 8\cos\theta - 3 = 0$

$\Rightarrow (3\cos\theta - 1)(\cos\theta + 3) = 0$

$\Rightarrow \cos\theta = \dfrac{1}{3} \quad \text{as } \cos\theta \neq -3$

From (a) $AC = 6\cos\theta = 6\times\dfrac{1}{3} = 2$ cm

9 a $\dfrac{\operatorname{cosec}x - \cot x}{1-\cos x} \equiv \dfrac{\dfrac{1}{\sin x} - \dfrac{\cos x}{\sin x}}{1-\cos x}$

$\equiv \dfrac{1}{\sin x}\times\dfrac{1-\cos x}{1-\cos x}$

$\equiv \operatorname{cosec}x$

b By part a equation becomes

$\cos\text{ec}\, x = 2$

$\Rightarrow \dfrac{1}{\sin x} = 2$

$\Rightarrow \sin x = \dfrac{1}{2}$

$\sin x$ is positive, so x is in

1st and 2nd quadrants

$x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

10 a $\dfrac{\sin x\tan x}{1-\cos x} - 1 \equiv \dfrac{\sin^2 x}{\cos x(1-\cos x)} - 1$

$\equiv \dfrac{\sin^2 x - \cos x + \cos^2 x}{\cos x(1-\cos x)}$

$\equiv \dfrac{1-\cos x}{\cos x(1-\cos x)}$

$\equiv \dfrac{1}{\cos x}$

$\equiv \sec x$

b Need to solve $\sec x = -\dfrac{1}{2}$

$\Rightarrow \cos x = -2$

which has no solutions.

11 $\dfrac{1+\cot x}{1+\tan x}=5$

$$\Rightarrow \frac{1+\dfrac{\cos x}{\sin x}}{1+\dfrac{\sin x}{\cos x}}=5$$

$$\Rightarrow \frac{\dfrac{\sin x+\cos x}{\sin x}}{\dfrac{\cos x+\sin x}{\cos x}}=5$$

$$\Rightarrow \frac{\sin x+\cos x}{\sin x}\times\frac{\cos x}{\cos x+\sin x}=5$$

$$\Rightarrow \frac{\cos x}{\sin x}=5$$

$$\Rightarrow \cot x=5$$

$$\Rightarrow \tan x=\frac{1}{5}$$

Calculator solution is 11.3° (1 d.p.)

$\tan x$ is positive, so x is in

1st and 3rd quadrants

Solutions are $11.3°, 191.3°$ (1 d.p.)

Trigonometric Functions 6D

1 a Use $1+\tan^2\theta=\sec^2\theta$

with θ replaced with $\frac{1}{2}\theta$

$1+\tan^2\left(\frac{1}{2}\theta\right)=\sec^2\left(\frac{1}{2}\theta\right)$

b $(\sec\theta-1)(\sec\theta+1)$ (multiply out)

$$=\sec^2\theta-1$$
$$=(1+\tan^2\theta)-1$$
$$=\tan^2\theta$$

c $\tan^2\theta(\operatorname{cosec}^2\theta-1)$

$$=\tan^2\theta\left((1+\cot^2\theta)-1\right)$$
$$=\tan^2\theta\cot^2\theta$$
$$=\tan^2\theta\times\frac{1}{\tan^2\theta}$$
$$=1$$

d $(\sec^2\theta-1)\cot\theta$

$$=\tan^2\theta\cot\theta$$
$$=\tan^2\theta\times\frac{1}{\tan\theta}$$
$$=\tan\theta$$

e $(\operatorname{cosec}^2\theta-\cot^2\theta)^2$

$$=\left((1+\cot^2\theta)-\cot^2\theta\right)^2$$
$$=1^2=1$$

f $2-\tan^2\theta+\sec^2\theta$

$$=2-\tan^2\theta+(1+\tan^2\theta)$$
$$=2-\tan^2\theta+1+\tan^2\theta$$
$$=3$$

g $\frac{\tan\theta\sec\theta}{1+\tan^2\theta}$

$$=\frac{\tan\theta\sec\theta}{\sec^2\theta}$$
$$=\frac{\tan\theta}{\sec\theta}$$
$$=\tan\theta\cos\theta$$
$$=\frac{\sin\theta}{\cos\theta}\times\cos\theta$$
$$=\sin\theta$$

h $(1-\sin^2\theta)(1+\tan^2\theta)$

$$=\cos^2\theta\times\sec^2\theta$$
$$=\cos^2\theta\times\frac{1}{\cos^2\theta}$$
$$=1$$

i $\frac{\operatorname{cosec}\theta\cot\theta}{1+\cot^2\theta}$

$$=\frac{\operatorname{cosec}\theta\cot\theta}{\operatorname{cosec}^2\theta}$$
$$=\frac{1}{\operatorname{cosec}\theta}\times\cot\theta$$
$$=\frac{\sin\theta}{1}\times\frac{\cos\theta}{\sin\theta}$$
$$=\cos\theta$$

j $\sec^4\theta-2\sec^2\theta\tan^2\theta+\tan^4\theta$

$$=(\sec^2\theta-\tan^2\theta)^2 \quad \text{(factorise)}$$
$$=\left((1+\tan^2\theta)-\tan^2\theta\right)^2$$
$$=1^2=1$$

k $4\operatorname{cosec}^2 2\theta+4\operatorname{cosec}^2 2\theta\cot^2 2\theta$

$$=4\operatorname{cosec}^2 2\theta(1+\cot^2 2\theta)$$
$$=4\operatorname{cosec}^2 2\theta\operatorname{cosec}^2 2\theta$$
$$=4\operatorname{cosec}^4 2\theta$$

2 $\operatorname{cosec}x=\frac{k}{\operatorname{cosec}x}$

$$\Rightarrow\quad \operatorname{cosec}^2 x=k$$
$$\Rightarrow\quad 1+\cot^2 x=k$$
$$\Rightarrow\quad \cot^2 x=k-1$$
$$\Rightarrow\quad \cot x=\pm\sqrt{k-1}$$

3 **a** $\cot\theta = \sqrt{3} \qquad 90° < \theta < 180°$

$\Rightarrow \cot^2\theta = 3$

$\Rightarrow 1 + \cot^2\theta = 1 + 3 = 4$

$\Rightarrow \operatorname{cosec}^2\theta = 4$

$\Rightarrow \sin^2\theta = \frac{1}{4}$

$\Rightarrow \sin\theta = \frac{1}{2}$

(as θ is in 2nd quadrant, $\sin\theta$ is positive)

b Using $\sin^2\theta + \cos^2\theta = 1$

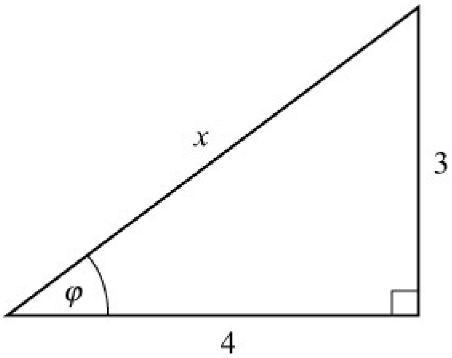
$\Rightarrow \cos^2\theta = 1 - \sin^2\theta = 1 - \frac{1}{4} = \frac{3}{4}$

$\Rightarrow \cos\theta = -\frac{\sqrt{3}}{4}$

(as θ is in 2nd quadrant, $\cos\theta$ is negative)

4 $\tan\theta = \frac{3}{4} \qquad 180° < \theta < 270°$

Draw a right-angled triangle where $\tan\varphi = \frac{3}{4}$



Using Pythagoras' theorem, $x = 5$

So $\cos\varphi = \frac{4}{5}$ and $\sin\varphi = \frac{3}{5}$

As θ is in the 3rd quadrant, both $\sin\theta$ and $\cos\theta$ are negative.

a $\sec\theta = \frac{1}{\cos\theta} = -\frac{1}{\cos\varphi} = -\frac{5}{4}$

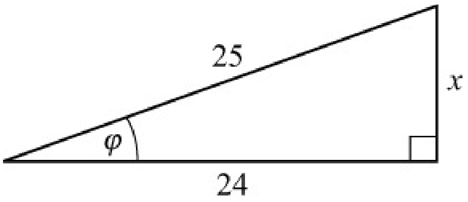
b $\cos\theta = -\cos\varphi = -\frac{4}{5}$

c $\sin\theta = -\sin\varphi = -\frac{3}{5}$

5 $\cos\theta = \frac{24}{25}$, θ reflex

As $\cos\theta$ is positive and θ reflex, θ is in the 4th quadrant.

Use right-angled triangle where $\cos\varphi = \frac{24}{25}$



Using Pythagoras' theorem,

$25^2 = x^2 + 24^2$

$\Rightarrow x^2 = 25^2 - 24^2 = 49$

$\Rightarrow x = 7$

So $\tan\varphi = \frac{7}{24}$ and $\sin\varphi = \frac{7}{25}$

As θ is in the 4th quadrant, both $\tan\theta$ and $\sin\theta$ are negative

a $\tan\theta = -\frac{7}{24}$

b $\operatorname{cosec}\theta = \frac{1}{\sin\theta} = -\frac{1}{\frac{7}{25}} = -\frac{25}{7}$

6 **a** LHS $\equiv \sec^4\theta - \tan^4\theta$

$\equiv (\sec^2\theta - \tan^2\theta)(\sec^2\theta + \tan^2\theta)$

(difference of two squares)

$\equiv (1)(\sec^2\theta + \tan^2\theta)$

(as $1 + \tan^2\theta \equiv \sec^2\theta$

$\Rightarrow \sec^2\theta - \tan^2\theta \equiv 1$)

$\equiv \sec^2\theta + \tan^2\theta \equiv$ RHS

b LHS $\equiv \operatorname{cosec}^2 x - \sin^2 x$

$\equiv (1 + \cot^2 x) - (1 - \cos^2 x)$

$\equiv 1 + \cot^2 x - 1 + \cos^2 x$

$\equiv \cot^2 x + \cos^2 x \equiv$ RHS

6 c LHS $\equiv \sec^2 A(\cot^2 A - \cos^2 A)$

$$\equiv \frac{1}{\cos^2 A}\left(\frac{\cos^2 A}{\sin^2 A} - \cos^2 A\right)$$

$$\equiv \frac{1}{\sin^2 A} - 1 \equiv \operatorname{cosec}^2 A - 1$$

(use $1 + \cot^2\theta = \operatorname{cosec}^2\theta$)

$$\equiv 1 + \cot^2 A - 1$$

$$\equiv \cot^2 A \equiv \text{ RHS}$$

d RHS $\equiv (\sec^2\theta - 1)(1 - \sin^2\theta)$

$$\equiv \tan^2\theta \times \cos^2\theta$$

(use $1 + \tan^2\theta \equiv \sec^2\theta$ and $\cos^2\theta + \sin^2\theta \equiv 1$)

$$\equiv \frac{\sin^2\theta}{\cos^2\theta} \times \cos^2\theta \equiv \sin^2\theta$$

$$\equiv 1 - \cos^2\theta \equiv \text{ LHS}$$

e LHS $\equiv \dfrac{1 - \tan^2 A}{1 + \tan^2 A} \equiv \dfrac{1 - \tan^2 A}{\sec^2 A}$

$$\equiv \frac{1}{\sec^2 A}(1 - \tan^2 A)$$

$$\equiv \cos^2 A\left(1 - \frac{\sin^2 A}{\cos^2 A}\right)$$

$$\equiv \cos^2 A - \sin^2 A$$

$$\equiv (1 - \sin^2 A) - \sin^2 A$$

$$\equiv 1 - 2\sin^2 A \equiv \text{ RHS}$$

f RHS $\equiv \sec^2\theta \operatorname{cosec}^2\theta$

$$\equiv \sec^2\theta(1 + \cot^2\theta)$$

$$\equiv \sec^2\theta + \frac{1}{\cos^2\theta} \times \frac{\cos^2\theta}{\sin^2\theta}$$

$$\equiv \sec^2\theta + \frac{1}{\sin^2\theta}$$

$$\equiv \sec^2\theta + \operatorname{cosec}^2\theta \equiv \text{ LHS}$$

Alternatively

LHS $\equiv \sec^2\theta + \operatorname{cosec}^2\theta \equiv \dfrac{1}{\cos^2\theta} + \dfrac{1}{\sin^2\theta}$

$$\equiv \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta\sin^2\theta} \equiv \frac{1}{\cos^2\theta\sin^2\theta}$$

$$\equiv \sec^2\theta \operatorname{cosec}^2\theta \equiv \text{ RHS}$$

g LHS $\equiv \operatorname{cosec} A \sec^2 A$

$$\equiv \operatorname{cosec} A(1 + \tan^2 A)$$

$$\equiv \operatorname{cosec} A + \frac{1}{\sin A} \times \frac{\sin^2 A}{\cos^2 A}$$

$$\equiv \operatorname{cosec} A + \frac{\sin A}{\cos^2 A}$$

$$\equiv \operatorname{cosec} A + \frac{\sin A}{\cos A} \times \frac{1}{\cos A}$$

$$\equiv \operatorname{cosec} A + \tan A \sec A \equiv \text{ RHS}$$

h LHS $\equiv (\sec\theta - \sin\theta)(\sec\theta + \sin\theta)$

$$\equiv \sec^2\theta - \sin^2\theta$$

$$\equiv (1 + \tan^2\theta) - (1 - \cos^2\theta)$$

$$\equiv 1 + \tan^2\theta - 1 + \cos^2\theta$$

$$\equiv \tan^2\theta + \cos^2\theta \equiv \text{ RHS}$$

7 $3\tan^2\theta + 4\sec^2\theta = 5$

$$\Rightarrow\ 3\tan^2\theta + 4(1 + \tan^2\theta) = 5$$

$$\Rightarrow\ 3\tan^2\theta + 4 + 4\tan^2\theta = 5$$

$$\Rightarrow\ 7\tan^2\theta = 1$$

$$\Rightarrow\ \tan^2\theta = \frac{1}{7}$$

$$\Rightarrow\ \cot^2\theta = 7$$

$$\Rightarrow\ \operatorname{cosec}^2\theta - 1 = 7$$

$$\Rightarrow\ \operatorname{cosec}^2\theta = 8$$

$$\Rightarrow\ \sin^2\theta = \frac{1}{8}$$

As θ is obtuse (in the 2nd quadrant), $\sin\theta$ is positive.

So $\sin\theta = \sqrt{\dfrac{1}{8}} = \dfrac{1}{2\sqrt{2}} = \dfrac{\sqrt{2}}{4}$

8 **a** $\sec^2\theta = 3\tan\theta \qquad 0 \le \theta \le 360°$

$\Rightarrow\ 1+\tan^2\theta = 3\tan\theta$

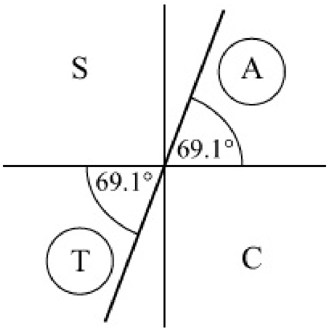
$\Rightarrow\ \tan^2\theta - 3\tan\theta + 1 = 0$

$\tan\theta = \dfrac{3\pm\sqrt{5}}{2}$

(equation does not factorise)

For $\tan\theta = \dfrac{3+\sqrt{5}}{2}$,

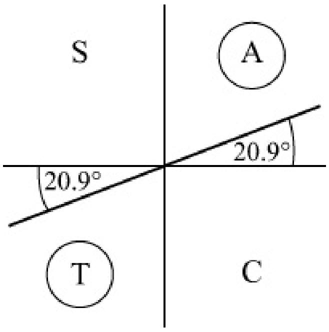
calculator value is 69.1° (3 s.f.)



Solutions are 69.1°, 249°

For $\tan\theta = \dfrac{3-\sqrt{5}}{2}$,

calculator value is 20.9° (3 s.f.)



Solutions are 20.9°, 201°

Set of solutions: 20.9°, 69.1°, 201°, 249° (3 s.f.)

b $\tan^2\theta - 2\sec\theta + 1 = 0 \qquad -\pi \le \theta \le \pi$

$\Rightarrow\ (\sec^2\theta - 1) - 2\sec\theta + 1 = 0$

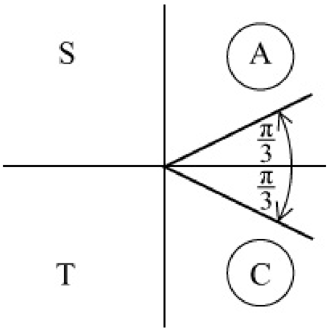
$\Rightarrow\ \sec^2\theta - 2\sec\theta = 0$

$\Rightarrow\ \sec\theta(\sec\theta - 2) = 0$

$\Rightarrow\ \sec\theta = 2 \quad$ (as $\sec\theta$ cannot be 0)

$\Rightarrow\ \cos\theta = \dfrac{1}{2}$

$\Rightarrow\ \theta = -\dfrac{\pi}{3},\ \dfrac{\pi}{3}$



c $\operatorname{cosec}^2\theta + 1 = 3\cot\theta \qquad -180° \le \theta \le 180°$

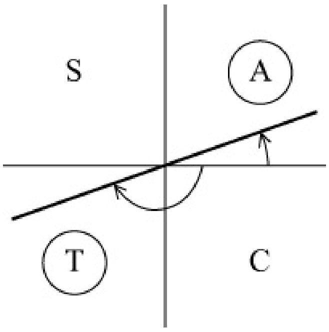
$\Rightarrow\ (1+\cot^2\theta) + 1 = 3\cot\theta$

$\Rightarrow\ \cot^2\theta - 3\cot\theta + 2 = 0$

$\Rightarrow\ (\cot\theta - 1)(\cot\theta - 2) = 0$

$\Rightarrow\ \cot\theta = 1$ or $\cot\theta = 2$

$\Rightarrow\ \tan\theta = 1$ or $\tan\theta = \dfrac{1}{2}$



$\tan\theta = 1 \ \Rightarrow\ \theta = -135°,\ 45°$

$\tan\theta = \dfrac{1}{2} \Rightarrow\ \theta = -153°,\ 26.6°$ (3 s.f.)

8 **d** $\cot\theta = 1 - \operatorname{cosec}^2\theta \qquad 0 \le \theta \le 2\pi$

$\Rightarrow \cot\theta = 1 - (1 + \cot^2\theta)$

$\Rightarrow \cot\theta = -\cot^2\theta$

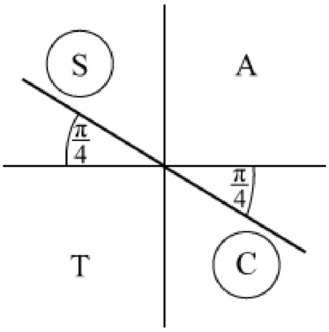
$\Rightarrow \cot^2\theta + \cot\theta = 0$

$\Rightarrow \cot\theta(\cot\theta + 1) = 0$

$\Rightarrow \cot\theta = 0 \text{ or } \cot\theta = -1$

For $\cot\theta = 0$ refer to graph: $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

For $\cot\theta = -1, \ \tan\theta = -1$



So $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

Set of solutions: $\frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$

e $3\sec\frac{1}{2}\theta = 2\tan^2\frac{1}{2}\theta \qquad 0 \le \theta \le 360°$

$\Rightarrow 3\sec\frac{1}{2}\theta = 2\left(\sec^2\frac{1}{2}\theta - 1\right)$

(use $1 + \tan^2 A \equiv \sec^2 A$ with $A = \frac{1}{2}\theta$)

$\Rightarrow 2\sec^2\frac{1}{2}\theta - 3\sec\frac{1}{2}\theta - 2 = 0$

$\Rightarrow \left(2\sec\frac{1}{2}\theta + 1\right)\left(\sec\frac{1}{2}\theta - 2\right) = 0$

$\Rightarrow \sec\frac{1}{2}\theta = -\frac{1}{2} \text{ or } \sec\frac{1}{2}\theta = 2$

Only $\sec\frac{1}{2}\theta = 2$ applies as

$\sec A \le -1$ or $\sec A \ge 1$

$\Rightarrow \cos\frac{1}{2}\theta = \frac{1}{2}$

As $0 \le \theta \le 360°$ so $0 \le \frac{1}{2}\theta \le 180°$

Calculator value is 60°

This is the only value in the interval.

So $\frac{1}{2}\theta = 60°$

$\Rightarrow \theta = 120°$

f $(\sec\theta - \cos\theta)^2 = \tan\theta - \sin^2\theta \quad 0 \le \theta \le \pi$

$\Rightarrow \sec^2\theta - 2\sec\theta\cos\theta + \cos^2\theta$

$= \tan\theta - \sin^2\theta$

$\Rightarrow \sec^2\theta - 2 + \cos^2\theta = \tan\theta - \sin^2\theta$

$\left(\sec\theta\cos\theta = \frac{1}{\cos\theta} \times \cos\theta = 1\right)$

$\Rightarrow (1 + \tan^2\theta) - 2 + (\cos^2\theta + \sin^2\theta)$

$= \tan\theta$

$\Rightarrow 1 + \tan^2\theta - 2 + 1 = \tan\theta$

$\Rightarrow \tan^2\theta - \tan\theta = 0$

$\Rightarrow \tan\theta(\tan\theta - 1) = 0$

$\Rightarrow \tan\theta = 0 \text{ or } \tan\theta = 1$

$\tan\theta = 0 \Rightarrow \theta = 0, \pi$

$\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

Set of solutions: $0, \frac{\pi}{4}, \pi$

g $\tan^2 2\theta = \sec 2\theta - 1 \qquad 0 \le \theta \le 180°$

$\Rightarrow \sec^2 2\theta - 1 = \sec 2\theta - 1$

$\Rightarrow \sec^2 2\theta - \sec 2\theta = 0$

$\Rightarrow \sec 2\theta(\sec 2\theta - 1) = 0$

$\Rightarrow \sec 2\theta = 0$ (not possible)

or $\sec 2\theta = 1$

$\Rightarrow \cos 2\theta = 1 \qquad 0 \le 2\theta \le 360°$

Refer to graph of $y = \cos\theta$

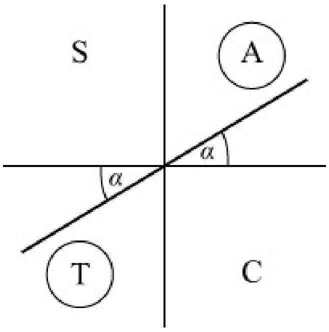
$\Rightarrow 2\theta = 0°, 360°$

$\Rightarrow \theta = 0°, 180°$

8 **h** $\sec^2\theta-(1+\sqrt{3})\tan\theta+\sqrt{3}=1$

for $0\le\theta\le 2\pi$

$$\Rightarrow\ (1+\tan^2\theta)-(1+\sqrt{3})\tan\theta+\sqrt{3}=1$$
$$\Rightarrow\ \tan^2\theta-(1+\sqrt{3})\tan\theta+\sqrt{3}=0$$
$$\Rightarrow\ (\tan\theta-\sqrt{3})(\tan\theta-1)=0$$
$$\Rightarrow\ \tan\theta=\sqrt{3}\ \text{or}\ \tan\theta=1$$



First answer for $\tan\theta=\sqrt{3}$ is $\frac{\pi}{3}$

Second solution is $\pi+\frac{\pi}{3}=\frac{4\pi}{3}$

First answer for $\tan\theta=1$ is $\frac{\pi}{4}$

Second solution is $\pi+\frac{\pi}{4}=\frac{5\pi}{4}$

Set of solutions: $\frac{\pi}{4},\ \frac{\pi}{3},\ \frac{5\pi}{4},\ \frac{4\pi}{3}$

9 **a** $\tan^2 k=2\sec k$

$$\Rightarrow\ (\sec^2 k-1)=2\sec k$$
$$\Rightarrow\ \sec^2 k-2\sec k-1=0$$
$$\Rightarrow\ \sec k=\frac{2\pm\sqrt{8}}{2}=\frac{2\pm2\sqrt{2}}{2}=1\pm\sqrt{2}$$

As $\sec k$ has no values between -1 and 1

$\sec k=1+\sqrt{2}$

b $\cos k=\frac{1}{1+\sqrt{2}}=\frac{\sqrt{2}-1}{(1+\sqrt{2})(\sqrt{2}-1)}$

$$=\frac{\sqrt{2}-1}{2-1}=\sqrt{2}-1$$

c Solutions of

$\tan^2 k=2\sec k,\ \ 0\le k\le 360°$

are solutions of $\cos k=\sqrt{2}-1$

Calculator solution is 65.5° (1 d.p.)

$$\Rightarrow\ k=65.5°,\ 360°-65.5°$$
$$=65.5°,\ 294.5°\ \text{(1 d.p.)}$$

10 **a** As $a=4\sec x$

$$\Rightarrow\ \sec x=\frac{a}{4}$$
$$\Rightarrow\ \cos x=\frac{4}{a}$$

As $\cos x=b$

$$\Rightarrow\ b=\frac{4}{a}$$

b $c=\cot x$

$$\Rightarrow\ c^2=\cot^2 x$$
$$\Rightarrow\ \frac{1}{c^2}=\tan^2 x$$
$$\Rightarrow\ \frac{1}{c^2}=\sec^2 x-1$$

(use $1+\tan^2 x\equiv\sec^2 x$)

$$\Rightarrow\ \frac{1}{c^2}=\frac{a^2}{16}-1\qquad\left(\sec x=\frac{a}{4}\right)$$
$$\Rightarrow\ 16=a^2c^2-16c^2\ \ \text{(multiply by }16c^2\text{)}$$
$$\Rightarrow\ c^2(a^2-16)=16$$
$$\Rightarrow\ c^2=\frac{16}{a^2-16}$$

11 **a** $x=\sec\theta+\tan\theta$

$$\frac{1}{x}=\frac{1}{\sec\theta+\tan\theta}$$
$$=\frac{\sec\theta-\tan\theta}{(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)}$$
$$=\frac{\sec\theta-\tan\theta}{\sec^2\theta-\tan^2\theta}$$
$$=\sec\theta-\tan\theta$$

(as $1+\tan^2\theta\equiv\sec^2\theta$

$\Rightarrow\ \sec^2\theta-\tan^2\theta\equiv1$)

11 b $x+\frac{1}{x}=\sec\theta+\tan\theta+\sec\theta-\tan\theta$

$$=2\sec\theta$$

$$\Rightarrow\left(x+\frac{1}{x}\right)^2=4\sec^2\theta$$

$$\Rightarrow x^2+2x\times\frac{1}{x}+\frac{1}{x^2}=4\sec^2\theta$$

$$\Rightarrow x^2+\frac{1}{x^2}+2=4\sec^2\theta$$

12 $2\sec^2\theta-\tan^2\theta=p$

$$\Rightarrow 2(1+\tan^2\theta)-\tan^2\theta=p$$

$$\Rightarrow 2+2\tan^2\theta-\tan^2\theta=p$$

$$\Rightarrow \tan^2\theta=p-2$$

$$\Rightarrow \cot^2\theta=\frac{1}{p-2}\quad(p\neq 2)$$

$$\operatorname{cosec}^2\theta=1+\cot^2\theta=1+\frac{1}{p-2}$$

$$=\frac{(p-2)+1}{p-2}=\frac{p-1}{p-2},\ p\neq 2$$

Trigonometric Functions 6E

1 **a** arccos(0) is the angle α in $0 \le \alpha \le \pi$ for which $\cos\alpha = 0$

Refer to graph of $y = \cos\theta \Rightarrow \alpha = \frac{\pi}{2}$

So $\arccos(0) = \frac{\pi}{2}$

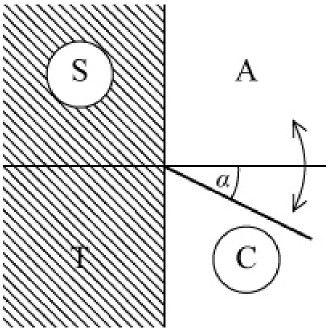
b arcsin(1) is the angle α in $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ for which $\sin\alpha = 1$

Refer to graph of $y = \sin\theta \Rightarrow \alpha = \frac{\pi}{2}$

So $\arcsin(1) = \frac{\pi}{2}$

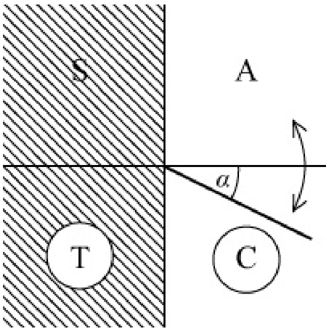
c arctan(−1) is the angle α in $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ for which $\tan\alpha = -1$

So $\arctan(-1) = -\frac{\pi}{4}$



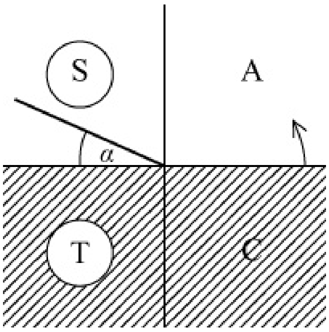
d $\arcsin\left(-\frac{1}{2}\right)$ is the angle α in $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ for which $\sin\alpha = -\frac{1}{2}$

So $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$



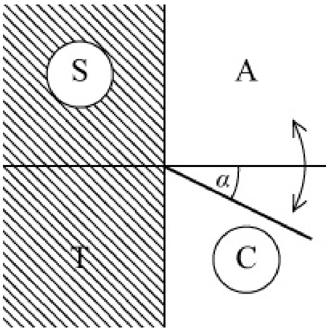
e $\arccos\left(-\frac{1}{\sqrt{2}}\right)$ is the angle α in $0 \le \alpha \le \pi$ for which $\cos\alpha = -\frac{1}{\sqrt{2}}$

So $\arccos\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$



f $\arctan\left(-\frac{1}{\sqrt{3}}\right)$ is the angle α in $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ for which $\tan\alpha = -\frac{1}{\sqrt{3}}$

So $\arctan\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$



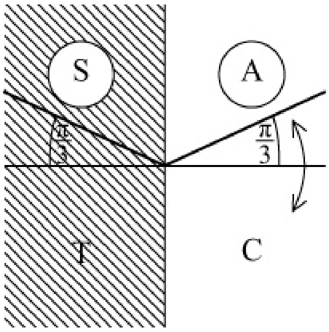
g $\arcsin\left(\sin\frac{\pi}{3}\right)$ is the angle α in $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ for which $\sin\alpha = \sin\frac{\pi}{3}$

So $\arcsin\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$

1 **h** $\arcsin\left(\sin\frac{2\pi}{3}\right)$ is the angle α in

$-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ for which $\sin\alpha = \sin\frac{2\pi}{3}$

So $\arcsin\left(\sin\frac{2\pi}{3}\right) = \frac{\pi}{3}$



2 **a** $\arcsin\left(\frac{1}{2}\right) + \arcsin\left(-\frac{1}{2}\right) = \frac{\pi}{6} + \left(-\frac{\pi}{6}\right) = 0$

b $\arccos\left(\frac{1}{2}\right) - \arccos\left(-\frac{1}{2}\right) = \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}$

c $\arctan(1) - \arctan(-1) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$

3 **a** $\sin\left(\arcsin\frac{1}{2}\right)$

$\arcsin\frac{1}{2} = \alpha$ where $\sin\alpha = \frac{1}{2}$,

and $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$

So $\arcsin\frac{1}{2} = \frac{\pi}{6}$

$\Rightarrow \sin\left(\arcsin\frac{1}{2}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$

b $\sin\left(\arcsin\left(-\frac{1}{2}\right)\right)$

$\arcsin\left(-\frac{1}{2}\right) = \alpha$

where $\sin\alpha = -\frac{1}{2}$, $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$

So $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

$\Rightarrow \sin\left(\arcsin\left(-\frac{1}{2}\right)\right) = \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

c $\tan\left(\arctan(-1)\right)$

$\arctan(-1) = \alpha$

where $\tan\alpha = -1$, $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$

So $\arctan(-1) = -\frac{\pi}{4}$

$\Rightarrow \tan\left(\arctan(-1)\right) = \tan\left(-\frac{\pi}{4}\right) = -1$

d $\cos(\arccos\ 0)$

$\arccos 0 = \alpha$ where $\cos\alpha = 0$, $0 \le \alpha \le \pi$

So $\arccos 0 = \frac{\pi}{2}$

$\Rightarrow \cos(\arccos 0) = \cos\frac{\pi}{2} = 0$

4 **a** $\sin\left(\arccos\frac{1}{2}\right)$

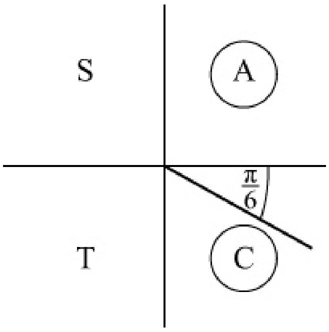
$\arccos\frac{1}{2} = \frac{\pi}{3}$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

4 **b** $\cos\left(\arcsin\left(-\frac{1}{2}\right)\right)$

$\arcsin\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$

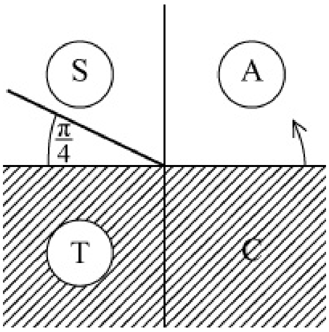
$\cos\left(-\frac{\pi}{6}\right)=+\frac{\sqrt{3}}{2}$



c $\tan\left(\arccos\left(-\frac{\sqrt{2}}{2}\right)\right)$

$\arccos\left(-\frac{\sqrt{2}}{2}\right)=\alpha$

where $\cos\alpha=-\frac{\sqrt{2}}{2},\ 0\leq\alpha\leq\pi$



So $\arccos\left(-\frac{\sqrt{2}}{2}\right)=\frac{3\pi}{4}$

$\tan\frac{3\pi}{4}=-1$

d $\sec(\arctan\sqrt{3})$

$\arctan\sqrt{3}=\frac{\pi}{3}$

(the angle whose $\tan$ is $\sqrt{3}$)

$\sec\frac{\pi}{3}=\frac{1}{\cos\frac{\pi}{3}}=\frac{1}{\frac{1}{2}}=2$

e $\operatorname{cosec}(\arcsin(-1))$

$\arcsin(-1)=\alpha$

where $\sin\alpha=-1,\ -\frac{\pi}{2}\leq\alpha\leq\frac{\pi}{2}$

So $\arcsin(-1)=-\frac{\pi}{2}$

$$\Rightarrow\ \operatorname{cosec}(\arcsin(-1))=\frac{1}{\sin\left(-\frac{\pi}{2}\right)}$$

$$=\frac{1}{-1}=-1$$

f $\sin\left(2\arcsin\left(\frac{\sqrt{2}}{2}\right)\right)$

$\arcsin\frac{\sqrt{2}}{2}=\frac{\pi}{4}$

So $\sin\left(2\arcsin\left(\frac{\sqrt{2}}{2}\right)\right)=\sin\frac{\pi}{2}=1$

5 As k is positive, the first two positive solutions of $\sin x=k$ are $\arcsin k$ and $\pi-\arcsin k$ i.e. α and $\pi-\alpha$

(Try a few examples, taking specific values for k).

6 **a** $\arcsin x$ is the angle α in $-\frac{\pi}{2}\leq\alpha\leq\frac{\pi}{2}$ such that $\sin\alpha=x$

In this case $x=\sin k$ where $0<k<\frac{\pi}{2}$

As $\sin$ is an increasing function

$\sin 0<x<\sin\frac{\pi}{2}$

$\Rightarrow\ 0<x<1$

b **i** $\cos k=\pm\sqrt{1-\sin^2 k}=\pm\sqrt{1-x^2}$

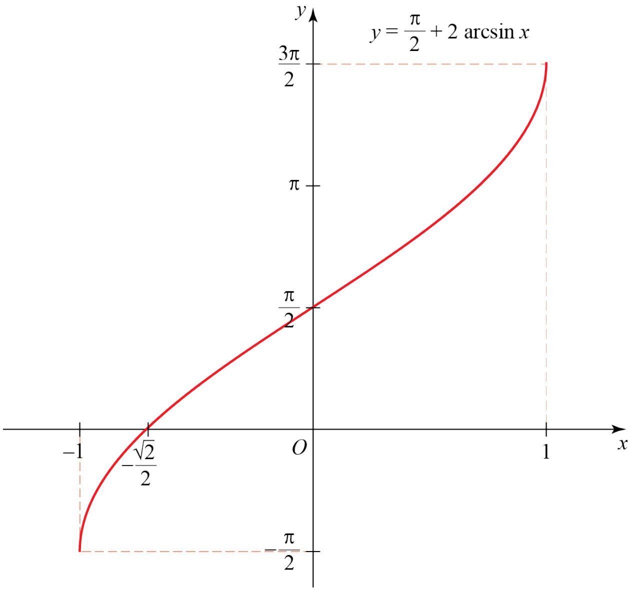
k is in the 1st quadrant $\Rightarrow$ $\cos k>0$

So $\cos k=\sqrt{1-x^2}$

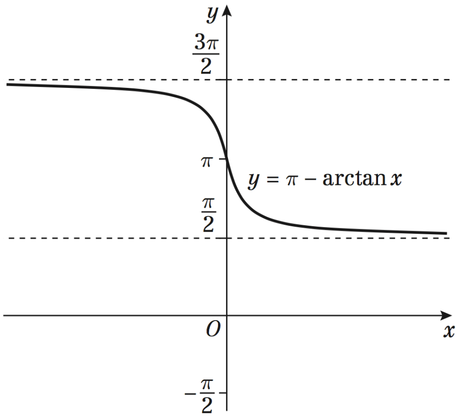
ii $\tan k=\frac{\sin k}{\cos k}=\frac{x}{\sqrt{1-x^2}}$

6 c k is now in the 4th quadrant, where $\cos k$ is positive. So the value of $\cos k$ remains the same and there is no change to $\tan k$.

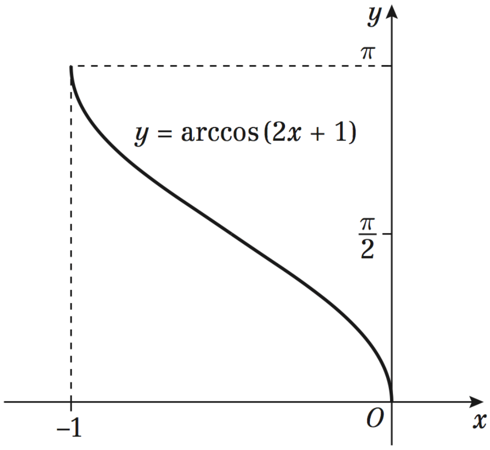
7 a The graph of $y = \frac{\pi}{2} + 2\arcsin x$ is $y = \arcsin x$ stretched by a scale factor 2 in the y direction and then translated by the vector $\begin{pmatrix} 0 \\ \frac{\pi}{2} \end{pmatrix}$



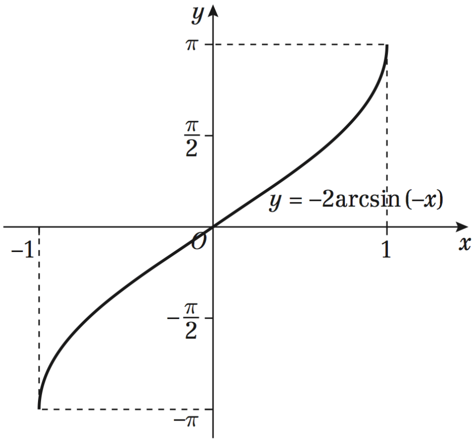
b The graph of $y = \pi - \arctan x$ is $y = \arctan x$ reflected in the x-axis and then translated by the vector $\begin{pmatrix} 0 \\ \pi \end{pmatrix}$



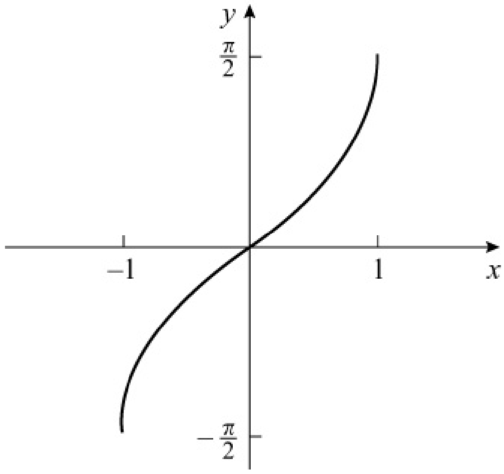
7 c The graph of $y = \arccos(2x+1)$ is $y = \arccos x$ translated by the vector $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and then stretched by scale factor $\frac{1}{2}$ in the x direction



d The graph of $y = -2\arcsin(-x)$ is $y = \arcsin x$ reflected in the y-axis, then reflected in the x-axis and then stretched by a scale factor 2 in the y direction



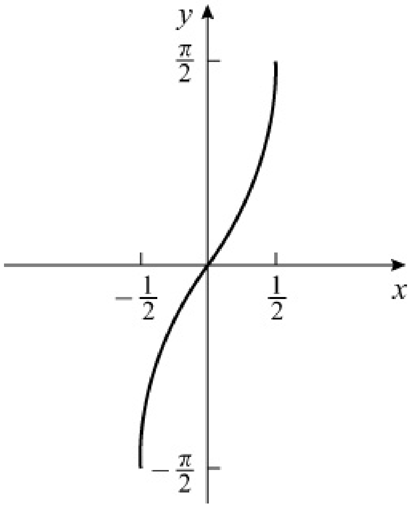
8 **a** $y = \arcsin x$



Range is $-\frac{\pi}{2} \le f(x) \le \frac{\pi}{2}$

b The graph of $y = f(2x)$ is the graph of $y = f(x)$ stretched in the x direction by scale factor $\frac{1}{2}$

$y = g(x)$



c $g : x \mapsto \arcsin 2x$

The domain is $-\frac{1}{2} \le x \le \frac{1}{2}$

d Let $y = \arcsin 2x$

$\Rightarrow \quad 2x = \sin y$

$\Rightarrow \quad x = \frac{1}{2}\sin y$

So $g^{-1} : x \mapsto \frac{1}{2}\sin x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

9 **a** Let $y = \arccos x$

As $0 \le x \le 1 \Rightarrow 0 \le y \le \frac{\pi}{2}$

$\cos y = x$, and using $\cos^2 y + \sin^2 y \equiv 1$

$\Rightarrow \sin^2 y = 1 - \cos^2 y$

$\Rightarrow \sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$

Note, $\sin y \ge 0$ since $0 \le y \le \frac{\pi}{2}$

so $\sin y \ne -\sqrt{1 - x^2}$

$\sin y = \sqrt{1 - x^2}$

$\Rightarrow y = \arcsin \sqrt{1 - x^2}$

Therefore, $\arccos x = \arcsin \sqrt{1 - x^2}$ for $0 \le x \le 1$

b For $-1 \le x \le 0$, $\frac{\pi}{2} \le \arccos x \le \pi$

But arcsin has a range of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

So $\arccos x \ne \arcsin \sqrt{1 - x^2}$, for $-1 \le x \le 0$

An alternative approach is to provide a counterexample.

Let $x = -\frac{1}{\sqrt{2}}$

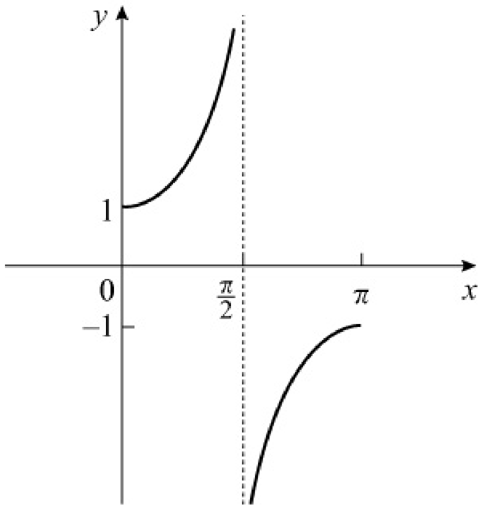
$\arccos x = \frac{3\pi}{4}$

$\arcsin \sqrt{1 - x^2} = \arcsin \frac{1}{\sqrt{2}} = \frac{\pi}{4}$

So $\arccos x \ne \arcsin \sqrt{1 - x^2}$

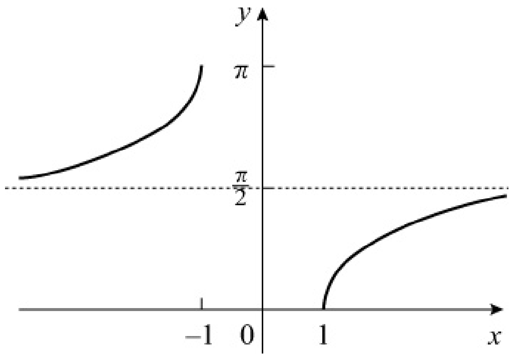
Challenge

a $y = \sec x$



b Reflect the graph drawn for part (a) in the line $y = x$

$y = \text{arc}\sec x,\ x \leqslant -1,\ x \geqslant 1$



Range is $0 \leqslant \text{arc}\sec x \leqslant \pi$, for $\text{arc}\sec x \neq \dfrac{\pi}{2}$

Trigonometric Functions Mixed Exercise

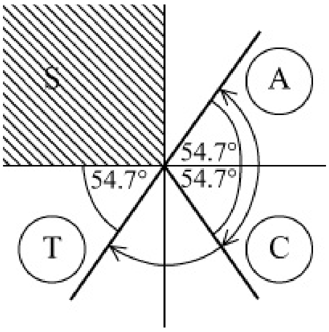
1 $\tan x = 2\cot x,\ \ -180° \le x \le 90°$

$$\Rightarrow\ \tan x = \frac{2}{\tan x}$$

$$\Rightarrow\ \tan^2 x = 2$$

$$\Rightarrow\ \tan x = \pm\sqrt{2}$$

Calculator value for $\tan x = +\sqrt{2}$ is 54.7° (1 d.p.)



Solutions are required in the 1st, 3rd and 4th quadrants.

Solution set is:

$-125.3°,\ -54.7°,\ 54.7°$ (1 d.p.)

2 $p = 2\sec\theta \ \Rightarrow\ \sec\theta = \dfrac{p}{2}$

$q = 4\cos\theta \ \Rightarrow\ \cos\theta = \dfrac{q}{4}$

$$\sec\theta = \frac{1}{\cos\theta} \ \Rightarrow\ \frac{p}{2} = \frac{1}{\frac{q}{4}} = \frac{4}{q} \ \Rightarrow\ p = \frac{8}{q}$$

3 $p = \sin\theta \ \Rightarrow\ \dfrac{1}{p} = \dfrac{1}{\sin\theta} = \operatorname{cosec}\theta$

$q = 4\cot\theta \ \Rightarrow\ \cot\theta = \dfrac{q}{4}$

Using $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$

$$\Rightarrow\ 1 + \frac{q^2}{16} = \frac{1}{p^2} \ \text{(multiply by } 16p^2\text{)}$$

$$\Rightarrow\ 16p^2 + p^2q^2 = 16$$

$$\Rightarrow\ p^2q^2 = 16 - 16p^2 = 16(1 - p^2)$$

4 a i $\operatorname{cosec}\theta = 2\cot\theta,\ \ 0 < \theta < 180°$

$$\Rightarrow\ \frac{1}{\sin\theta} = \frac{2\cos\theta}{\sin\theta}$$

$$\Rightarrow\ 2\cos\theta = 1$$

$$\Rightarrow\ \cos\theta = \frac{1}{2}$$

$$\Rightarrow\ \theta = 60°$$

ii $2\cot^2\theta = 7\operatorname{cosec}\theta - 8,\ \ 0 < \theta < 180°$

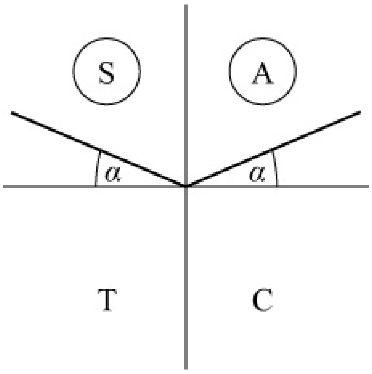
$$\Rightarrow\ 2(\operatorname{cosec}^2\theta - 1) = 7\operatorname{cosec}\theta - 8$$

$$\Rightarrow\ 2\operatorname{cosec}^2\theta - 7\operatorname{cosec}\theta + 6 = 0$$

$$\Rightarrow\ (2\operatorname{cosec}\theta - 3)(\operatorname{cosec}\theta - 2) = 0$$

$$\Rightarrow\ \operatorname{cosec}\theta = \frac{3}{2} \text{ or } \operatorname{cosec}\theta = 2$$

So $\sin\theta = \dfrac{2}{3}$ or $\sin\theta = \dfrac{1}{2}$



Solutions are $\alpha°$ and $(180 - \alpha)°$ where α is the calculator value.

$\sin\theta = \dfrac{2}{3}$

Calculator value is 41.8° (1 d.p.)

Solutions are 41.8°, 138.2°

$\sin\theta = \dfrac{1}{2}$

Calculator value is 30° (1 d.p.)

Solutions are 30°, 150°

Solution set is:

30°, 41.8°, 138.2°, 150°

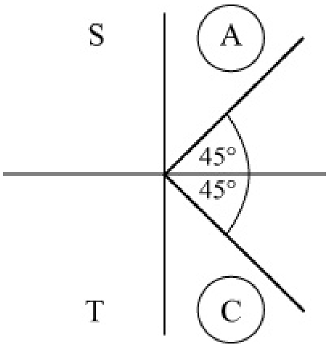
4 b i $\sec(2\theta - 15°) = \operatorname{cosec} 135°$,
$0 \le \theta \le 360°$

$$\Rightarrow \cos(2\theta - 15°) = \frac{1}{\operatorname{cosec} 135°}$$
$$= \sin 135° = \frac{\sqrt{2}}{2}$$

Solve $\cos(2\theta - 15°) = \frac{\sqrt{2}}{2}$, in the interval $-15° \le 2\theta - 15° \le 705°$

The calculator value is 45°

cos is positive, so $(2\theta - 15°)$ is in the 1st and 4th quadrants.



So $(2\theta - 15°) = 45°, 315°, 405°, 675°$

$\Rightarrow 2\theta = 60°, 330°, 420°, 690°$

$\Rightarrow \theta = 30°, 165°, 210°, 345°$

ii $\sec^2\theta + \tan\theta = 3, \ 0 \le \theta \le 360°$

$\Rightarrow 1 + \tan^2\theta + \tan\theta = 3$

$\Rightarrow \tan^2\theta + \tan\theta - 2 = 0$

$\Rightarrow (\tan\theta - 1)(\tan\theta + 2) = 0$

$\Rightarrow \tan\theta = 1 \ or \ \tan\theta = -2$

$\tan\theta = 1 \Rightarrow \theta = 45°, 180° + 45°$,
i.e. $45°, 225°$

$\tan\theta = -2$,
calculator value is $-63.4°$ (1 d.p.)

$\Rightarrow \theta = 180° + (-63.4°) = 116.6°$
$\theta = 360° + (-63.4°) = 296.6°$

Solution set is:
$45°, 116.6°, 225°, 296.6°$

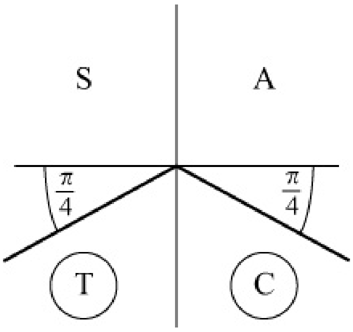
c i $\operatorname{cosec}\left(x + \frac{\pi}{15}\right) = -\sqrt{2}, \ 0 \le x \le 2\pi$

$$\Rightarrow \sin\left(x + \frac{\pi}{15}\right) = -\frac{1}{\sqrt{2}}$$

Calculator value is $-\frac{\pi}{4}$

$\sin\left(x + \frac{\pi}{15}\right)$ is negative,

so $x + \frac{\pi}{15}$ is in 3rd and 4th quadrants.



So $x + \frac{\pi}{15} = \frac{5\pi}{4}, \frac{7\pi}{4}$

$$\Rightarrow x = \frac{5\pi}{4} - \frac{\pi}{15}, \frac{7\pi}{4} - \frac{\pi}{15}$$
$$= \frac{75\pi - 4\pi}{60}, \frac{105\pi - 4\pi}{60}$$
$$= \frac{71\pi}{60}, \frac{101\pi}{60}$$

ii $\sec^2 x = \frac{4}{3}, \ 0 \le x \le 2\pi$

$$\Rightarrow \cos^2 x = \frac{3}{4}$$
$$\Rightarrow \cos x = \pm\frac{\sqrt{3}}{2}$$

Calculator value for $\cos x = +\frac{\sqrt{3}}{2}$ is $\frac{\pi}{6}$

As $\cos x$ is ±, x is in all four quadrants. Solution set is:

$$x = \frac{\pi}{6}, \pi - \frac{\pi}{6}, \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$$
$$= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

5 $5\sin x\cos y + 4\cos x\sin y = 0$

$$\Rightarrow \frac{5\sin x\cos y}{\sin x\sin y} + \frac{4\cos x\sin y}{\sin x\sin y} = 0$$

(divide by $\sin x\sin y$)

$$\Rightarrow \frac{5\cos y}{\sin y} + \frac{4\cos x}{\sin x} = 0$$

So $5\cot y + 4\cot x = 0$

As $\cot x = 2$

$5\cot y + 8 = 0$

$5\cot y = -8$

$\cot y = -\frac{8}{5}$

6 a LHS $\equiv (\tan\theta + \cot\theta)(\sin\theta + \cos\theta)$

$$\equiv \left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)(\sin\theta + \cos\theta)$$

$$\equiv \left(\frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}\right)(\sin\theta + \cos\theta)$$

$$\equiv \left(\frac{1}{\cos\theta\sin\theta}\right)(\sin\theta + \cos\theta)$$

$$\equiv \frac{\sin\theta}{\cos\theta\sin\theta} + \frac{\cos\theta}{\cos\theta\sin\theta}$$

$$\equiv \frac{1}{\cos\theta} + \frac{1}{\sin\theta}$$

$\equiv \sec\theta + \operatorname{cosec}\theta \equiv$ RHS

b LHS $\equiv \dfrac{\operatorname{cosec} x}{\operatorname{cosec} x - \sin x}$

$$\equiv \frac{\frac{1}{\sin x}}{\frac{1}{\sin x} - \sin x}$$

$$\equiv \frac{\frac{1}{\sin x}}{\frac{1-\sin^2 x}{\sin x}}$$

$$\equiv \frac{1}{\sin x} \times \frac{\sin x}{1-\sin^2 x}$$

$$\equiv \frac{1}{1-\sin^2 x}$$

$$\equiv \frac{1}{\cos^2 x}$$

(using $\sin^2 x + \cos^2 x \equiv 1$)

$\equiv \sec^2 x \equiv$ RHS

c LHS $\equiv (1-\sin x)(1+\operatorname{cosec} x)$

$\equiv 1 - \sin x + \operatorname{cosec} x - \sin x\operatorname{cosec} x$

$\equiv 1 - \sin x + \operatorname{cosec} x - 1$

$\left(\text{as } \operatorname{cosec} x = \dfrac{1}{\sin x}\right)$

$\equiv \operatorname{cosec} x - \sin x$

$$\equiv \frac{1}{\sin x} - \sin x$$

$$\equiv \frac{1-\sin^2 x}{\sin x}$$

$$\equiv \frac{\cos^2 x}{\sin x}$$

$$\equiv \frac{\cos x}{\sin x} \times \cos x$$

$\equiv \cos x\cot x \equiv$ RHS

d LHS $\equiv \dfrac{\cot x}{\operatorname{cosec} x - 1} - \dfrac{\cos x}{1+\sin x}$

$$\equiv \frac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x} - 1} - \frac{\cos x}{1+\sin x}$$

$$\equiv \frac{\frac{\cos x}{\sin x}}{\frac{1-\sin x}{\sin x}} - \frac{\cos x}{1+\sin x}$$

$$\equiv \frac{\cos x}{1-\sin x} - \frac{\cos x}{1+\sin x}$$

$$\equiv \frac{\cos x(1+\sin x) - \cos x(1-\sin x)}{(1-\sin x)(1+\sin x)}$$

$$\equiv \frac{2\cos x\sin x}{1-\sin^2 x}$$

$$\equiv \frac{2\cos x\sin x}{\cos^2 x}$$

$$\equiv 2\frac{\sin x}{\cos x}$$

$\equiv 2\tan x \equiv$ RHS

6 e LHS $\equiv \frac{1}{\operatorname{cosec}\theta-1}+\frac{1}{\operatorname{cosec}\theta+1}$

$$\equiv \frac{(\operatorname{cosec}\theta+1)+(\operatorname{cosec}\theta-1)}{(\operatorname{cosec}\theta-1)(\operatorname{cosec}\theta+1)}$$

$$\equiv \frac{2\operatorname{cosec}\theta}{\operatorname{cosec}^2\theta-1}$$

$$\equiv \frac{2\operatorname{cosec}\theta}{\cot^2\theta}$$

$(1+\cot^2\theta\equiv\operatorname{cosec}^2\theta)$

$$\equiv \frac{2}{\sin\theta}\times\frac{\sin^2\theta}{\cos^2\theta}$$

$$\equiv 2\times\frac{1}{\cos\theta}\times\frac{\sin\theta}{\cos\theta}$$

$\equiv 2\sec\theta\tan\theta\equiv$ RHS

f LHS $\equiv \frac{(\sec\theta-\tan\theta)(\sec\theta+\tan\theta)}{1+\tan^2\theta}$

$$\equiv \frac{\sec^2\theta-\tan^2\theta}{\sec^2\theta}$$

$$\equiv \frac{(1+\tan^2\theta)-\tan^2\theta}{\sec^2\theta}$$

$$\equiv \frac{1}{\sec^2\theta}$$

$\equiv \cos^2\theta\equiv$ RHS

7 a LHS $\equiv \frac{\sin x}{1+\cos x}+\frac{1+\cos x}{\sin x}$

$$\equiv \frac{\sin^2 x+(1+\cos x)^2}{(1+\cos x)\sin x}$$

$$\equiv \frac{\sin^2 x+1+2\cos x+\cos^2 x}{(1+\cos x)\sin x}$$

$$\equiv \frac{2+2\cos x}{(1+\cos x)\sin x}$$

$\left(\sin^2 x+\cos^2 x\equiv 1\right)$

$$\equiv \frac{2(1+\cos x)}{(1+\cos x)\sin x}$$

$$\equiv \frac{2}{\sin x}$$

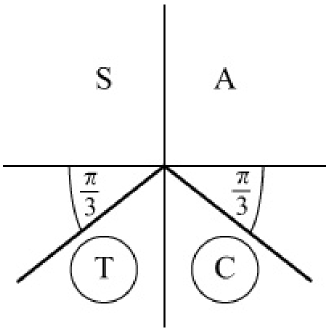
$\equiv 2\operatorname{cosec}x\equiv$ RHS

b Solve $2\operatorname{cosec}x=-\frac{4}{\sqrt{3}},\quad -2\pi\le x\le 2\pi$

$\Rightarrow\ \operatorname{cosec}x=-\frac{2}{\sqrt{3}}$

$\Rightarrow\ \sin x=-\frac{\sqrt{3}}{2}$

Calculator value is $-\frac{\pi}{3}$



Solutions in $-2\pi\le x\le 2\pi$ are

$-\frac{\pi}{3},\ -\pi+\frac{\pi}{3},\ \pi+\frac{\pi}{3},\ 2\pi-\frac{\pi}{3}$

i.e. $-\frac{\pi}{3}, -\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

8 RHS $\equiv(\operatorname{cosec}\theta+\cot\theta)^2$

$$\equiv\left(\frac{1}{\sin\theta}+\frac{\cos\theta}{\sin\theta}\right)^2$$

$$\equiv\frac{(1+\cos\theta)^2}{\sin^2\theta}$$

$$\equiv\frac{(1+\cos\theta)^2}{1-\cos^2\theta}$$

$$\equiv\frac{(1+\cos\theta)(1+\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}$$

$\equiv\frac{1+\cos\theta}{1-\cos\theta}\equiv$ LHS

9 **a** $\sec A = -3,\ \frac{\pi}{2} < A < \pi,$

i.e. A is in 2nd quadrant.

As $1 + \tan^2 A = \sec^2 A$

$1 + \tan^2 A = 9$

$\tan^2 A = 8$

$\tan A = \pm\sqrt{8} = \pm 2\sqrt{2}$

As A is in 2nd quadrant, $\tan A$ is negative.

So $\tan A = -2\sqrt{2}$

b $\sec A = -3$, so $\cos A = -\frac{1}{3}$

As $\tan A = \frac{\sin A}{\cos A}$

$\sin A = \cos A \times \tan A = -\frac{1}{3} \times -2\sqrt{2} = \frac{2\sqrt{2}}{3}$

So $\operatorname{cosec} A = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{2 \times 2} = \frac{3\sqrt{2}}{4}$

An alternative approach is to use the fact that $\operatorname{cosec}^2 \theta \equiv 1 + \cot^2 \theta$

$\operatorname{cosec}^2 A = 1 + \cot^2 A = 1 + \frac{1}{8} = \frac{9}{8}$

$\Rightarrow \operatorname{cosec} A = \pm\frac{3}{2\sqrt{2}} = \pm\frac{3\sqrt{2}}{4}$

As $\frac{\pi}{2} < A < \pi$, $\operatorname{cosec} A$ is positive, so

$\operatorname{cosec} A = \frac{3\sqrt{2}}{4}$

10 $\sec\theta = k,\ |k| \geq 1$

θ is in the 2nd quadrant

$\Rightarrow$ k is negative

a $\cos\theta = \frac{1}{\sec\theta} = \frac{1}{k}$

b Using $1 + \tan^2\theta = \sec^2\theta$

$\tan^2\theta = k^2 - 1$

c $\tan\theta = \pm\sqrt{k^2 - 1}$

In the 2nd quadrant, $\tan\theta$ is negative

So $\tan\theta = -\sqrt{k^2 - 1}$

$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{-\sqrt{k^2 - 1}} = -\frac{1}{\sqrt{k^2 - 1}}$

d Using $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$\operatorname{cosec}^2\theta = 1 + \frac{1}{k^2 - 1} = \frac{k^2 - 1 + 1}{k^2 - 1} = \frac{k^2}{k^2 - 1}$

So $\operatorname{cosec}\theta = \pm\frac{k}{\sqrt{k^2 - 1}}$

In the 2nd quadrant, $\operatorname{cosec}\theta$ is positive

As k is negative, $\operatorname{cosec}\theta = -\frac{k}{\sqrt{k^2 - 1}}$

11 $\sec\left(x + \frac{\pi}{4}\right) = 2,\ 0 \leq x \leq 2\pi$

$$\Rightarrow \cos\left(x + \frac{\pi}{4}\right) = \frac{1}{2},\ 0 \leq x \leq 2\pi$$

$$\Rightarrow x + \frac{\pi}{4} = \cos^{-1}\frac{1}{2},\ 2\pi - \cos^{-1}\frac{1}{2}$$

$$= \frac{\pi}{3},\ 2\pi - \frac{\pi}{3}$$

So $x = \frac{\pi}{3} - \frac{\pi}{4},\ \frac{5\pi}{3} - \frac{\pi}{4}$

$$= \frac{4\pi - 3\pi}{12},\ \frac{20\pi - 3\pi}{12}$$

$$= \frac{\pi}{12},\ \frac{17\pi}{12}$$

12 $\arcsin\left(\frac{1}{2}\right)$ is the angle α in the interval

$-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$ whose sine is $\frac{1}{2}$

So $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$

Similarly, $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

So $\arcsin\left(\frac{1}{2}\right) - \arcsin\left(-\frac{1}{2}\right) = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}$

13 $\sec^2 x - \frac{2\sqrt{3}}{3}\tan x - 2 = 0,\ 0 \le x \le 2\pi$

$$\Rightarrow\ (1+\tan^2 x) - \frac{2\sqrt{3}}{3}\tan x - 2 = 0$$

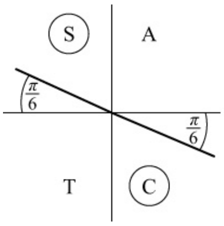
$$\Rightarrow\ \tan^2 x - \frac{2\sqrt{3}}{3}\tan x - 1 = 0$$

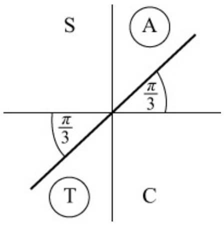
(This does factorise!)

$$\left(\tan x + \frac{\sqrt{3}}{3}\right)\left(\tan x - \sqrt{3}\right) = 0$$

$$\Rightarrow\ \tan x = -\frac{\sqrt{3}}{3}\ \text{or}\ \tan x = \sqrt{3}$$

Calculator values are $-\frac{\pi}{6}$ and $\frac{\pi}{3}$





Solution set: $\frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}$

14 a $\sec x \operatorname{cosec} x - 2\sec x - \operatorname{cosec} x + 2$

$$= \sec x(\operatorname{cosec} x - 2) - (\operatorname{cosec} x - 2)$$

$$= (\operatorname{cosec} x - 2)(\sec x - 1)$$

b So $\sec x \operatorname{cosec} x - 2\sec x - \operatorname{cosec} x + 2 = 0$

$$\Rightarrow\ (\operatorname{cosec} x - 2)(\sec x - 1) = 0$$

$$\Rightarrow\ \operatorname{cosec} x = 2\ \text{or}\ \sec x = 1$$

$$\Rightarrow\ \sin x = \frac{1}{2}\ \text{or}\ \cos x = 1$$

$\sin x = \frac{1}{2},\quad 0 \le x \le 360°$

$$\Rightarrow\ x = 30°, (180-30)°$$

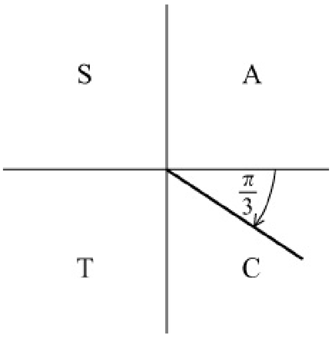
$\cos x = 1,\quad 0 \le x \le 360°,$

$$\Rightarrow\ x = 0°, 360°\quad \text{(from the graph)}$$

As $\operatorname{cosec} x$ is not defined for $x = 0°, 360°$, the equation is not defined for these values, so $x = 0°, 360°$ are not solutions

So the solution set is: 30°, 150°

15 $\arctan(x-2) = -\frac{\pi}{3}$

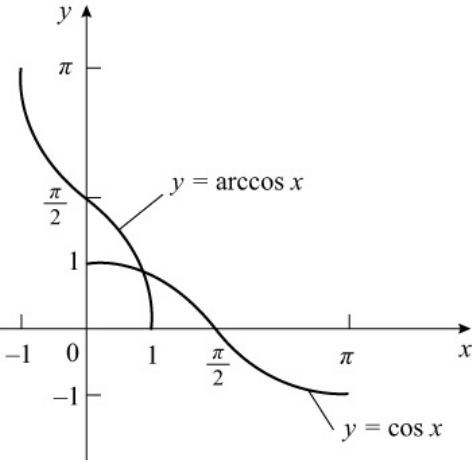


$$\Rightarrow\ x - 2 = \tan\left(-\frac{\pi}{3}\right)$$

$$\Rightarrow\ x - 2 = -\sqrt{3}$$

$$\Rightarrow\ x = 2 - \sqrt{3}$$

16



17 a As $1 + \tan^2 x \equiv \sec^2 x$

$\sec^2 x - \tan^2 x \equiv 1$

$$\Rightarrow\ (\sec x - \tan x)(\sec x + \tan x) \equiv 1$$

(difference of two squares)

As $\tan x + \sec x = -3$ is given,

so $-3(\sec x - \tan x) = 1$

$$\Rightarrow\ \sec x - \tan x = -\frac{1}{3}$$

17 b $\sec x + \tan x = -3$

and $\sec x - \tan x = -\frac{1}{3}$

i Add the equations

$\Rightarrow\ 2\sec x = -\frac{10}{3}\ \Rightarrow\ \sec x = -\frac{5}{3}$

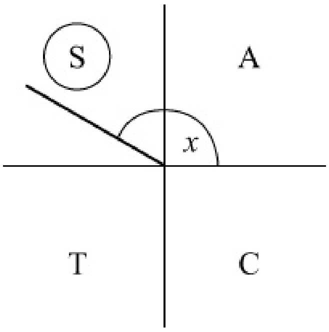
ii Subtract the equations

$\Rightarrow\ 2\tan x = -3 - \left(-\frac{1}{3}\right) = -\frac{8}{3}$

$\Rightarrow\ \tan x = -\frac{4}{3}$

c As $\sec x$ and $\tan x$ are both negative, $\cos x$ and $\tan x$ are both negative.

So x must be in the 2nd quadrant.



Solving $\tan x = -\frac{4}{3}$, where x is in the 2nd quadrant, gives

$x = 180° + (-53.1°) = 126.9°$ (1 d.p.)

18 $p = \sec\theta - \tan\theta,\ q = \sec\theta + \tan\theta$

Multiply together:

$pq = (\sec\theta - \tan\theta)(\sec\theta + \tan\theta)$

$\quad = \sec^2\theta - \tan^2\theta = 1$

(since $1 + \tan^2\theta = \sec^2\theta$)

$\Rightarrow\ p = \frac{1}{q}$

(There are several ways of solving this problem.)

19 a LHS $\equiv \sec^4\theta - \tan^4\theta$

$\equiv (\sec^2\theta - \tan^2\theta)(\sec^2\theta + \tan^2\theta)$

$\equiv 1 \times (\sec^2\theta + \tan^2\theta)$

(as $\sec^2\theta \equiv 1 + \tan^2\theta$

$\Rightarrow \sec^2\theta - \tan^2\theta \equiv 1$)

$\equiv \sec^2\theta + \tan^2\theta \equiv$ RHS

b $\sec^4\theta = \tan^4\theta + 3\tan\theta$

$\Rightarrow\ \sec^4\theta - \tan^4\theta = 3\tan\theta$

$\Rightarrow\ \sec^2\theta + \tan^2\theta = 3\tan\theta$

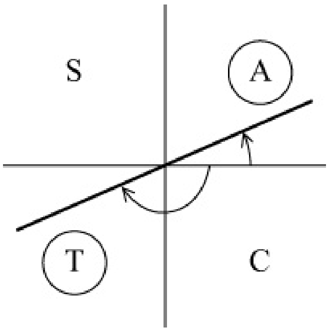
(using part (a))

$\Rightarrow\ (1 + \tan^2\theta) + \tan^2\theta = 3\tan\theta$

$\Rightarrow\ 2\tan^2\theta - 3\tan\theta + 1 = 0$

$\Rightarrow\ (2\tan\theta - 1)(\tan\theta - 1) = 0$

$\Rightarrow\ \tan\theta = \frac{1}{2}$ or $\tan\theta = 1$



In the interval $-180° \le \theta \le 180°$

$\tan\theta = \frac{1}{2} \Rightarrow\ \theta = \tan^{-1}\frac{1}{2}, -180° + \tan^{-1}\frac{1}{2}$

$= 26.6°, -153.4°$ (1 d.p.)

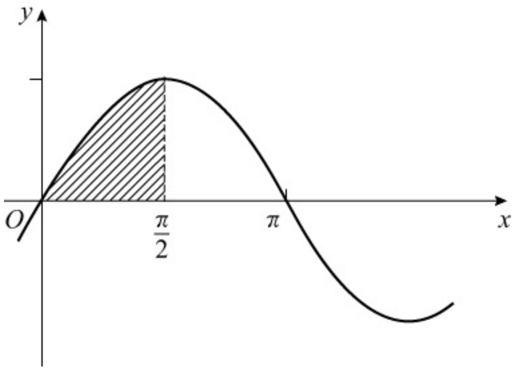
$\tan\theta = 1\ \Rightarrow \theta = \tan^{-1}1, -180° + \tan^{-1}1$

$= 45°,\ -135°$

Solution set is:

$-153.4°,\ -135°,\ 26.6°,\ 45°$

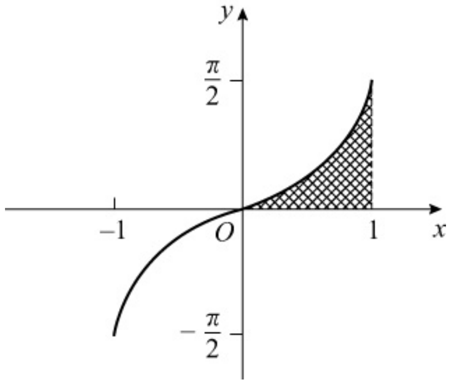
20 a $y = \sin x$



$\int_0^{\frac{\pi}{2}} \sin x\mathrm{d}x$ represents the area between

$y = \sin x,$ the x-axis and $x = \dfrac{\pi}{2}$

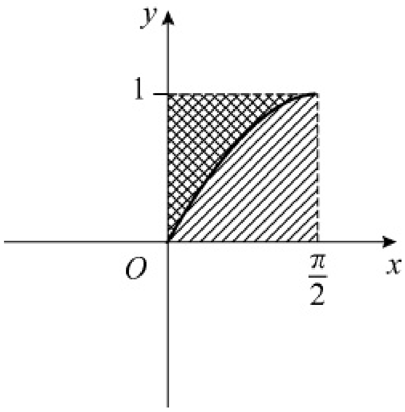
b $y = \arcsin x, \quad -1 \leq x \leq 1$



$\int_0^1 \arcsin x\mathrm{d}x$ represents the area between

$y = \arcsin x$, the x-axis and $x = 1$

c The curves are the same with the axes interchanged.
The shaded area in (b) could be added to the graph in (a) to form a rectangle with sides **1 and** $\frac{\pi}{2}$, as in the diagram.

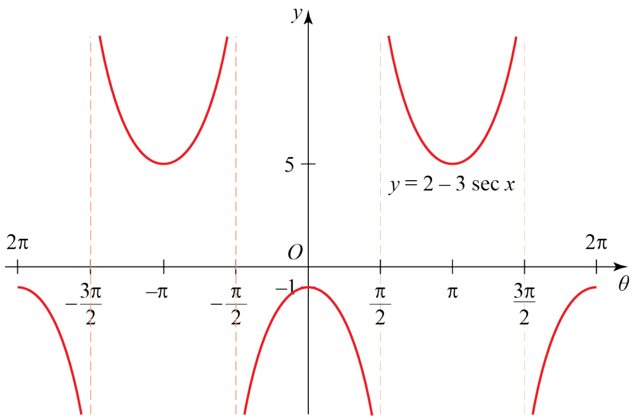


$$\text{Area of rectangle } = 1 \times \frac{\pi}{2} = \frac{\pi}{2}$$

$$\text{So } \int_0^{\frac{\pi}{2}} \sin x\mathrm{d}x + \int_0^1 \arcsin x\mathrm{d}x = \frac{\pi}{2}$$

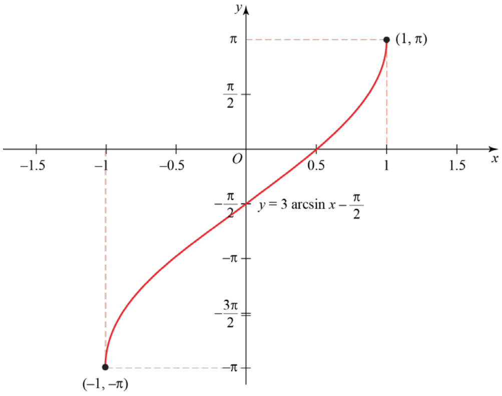
21
$$\cot 60° \sec 60° = \frac{1}{\tan 60°} \times \frac{1}{\cos 60°}$$
$$= \frac{1}{\sqrt{3}} \times \frac{1}{\frac{1}{2}} = \frac{2}{\sqrt{3}}$$
$$= \frac{2\sqrt{3}}{3}$$

22 a The graph of $y = 2 - 3\sec x$ is $y = \sec x$ stretched by a scale factor 3 in the y direction, then reflected in the x-axis and then translated by the vector $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.



b $-1 < k < 5$

23 a The graph of $y = 3\arcsin x - \frac{\pi}{2}$ is $y = \arcsin x$ stretched by a scale factor 3 in the y direction and then translated by the vector $\begin{pmatrix} 0 \\ -\frac{\pi}{2} \end{pmatrix}$



b Curve meets the x-axis when $y = 0$

$$\Rightarrow 3\arcsin x - \frac{\pi}{2} = 0$$

$$\Rightarrow \arcsin x = \frac{\pi}{6}$$

$$\Rightarrow \sin\frac{\pi}{6} = x$$

$$\Rightarrow x = \frac{1}{2}$$

Curve meets the x-axis at $\left(\frac{1}{2}, 0\right)$

24 a Let $y = \arccos x,\ 0 < x \le 1$

$$\Rightarrow \cos y = x$$

$$\Rightarrow \sin y = \sqrt{1-x^2}$$

Note that as $0 < x \le 1,\ 0 \le y < \frac{\pi}{2}$,

so $\sin y$ is positive

Thus $\tan y = \frac{\sqrt{1-x^2}}{x}$,

which is valid for $0 < x \le 1$

$$\Rightarrow y = \arctan\frac{\sqrt{1-x^2}}{x}$$

So $\arccos x = \arctan\frac{\sqrt{1-x^2}}{x}$ for $0 < x \le 1$

b Let $y = \arccos x,\ -1 \le x < 0$

$$\Rightarrow \cos y = x$$

$$\Rightarrow \sin y = \sqrt{1-x^2}$$

As $-1 \le x < 0,\ \frac{\pi}{2} < y \le \pi$,

so $\sin y$ is positive

$$\Rightarrow \tan y = \frac{\sin y}{\cos y} = \frac{\sqrt{1-x^2}}{x}$$

for $-1 \le x < 0,\ \frac{\pi}{2} < y \le \pi$

Note that as $y > \frac{\pi}{2}$, it is not in the range of $y = \arccos x$

However, from the tan curve, we know that $\tan(y - \pi) = \tan y$

So $\tan(y-\pi) = \frac{\sqrt{1-x^2}}{x}$

for $-1 \le x < 0,\ -\frac{\pi}{2} < y - \pi \le 0$

We can now use the inverse function

$$y - \pi = \arctan\frac{\sqrt{1-x^2}}{x}$$

So $y = \pi + \arctan\frac{\sqrt{1-x^2}}{x}$

for $-1 \le x < 0$

Thus $\arccos x = \pi + \arctan\frac{\sqrt{1-x^2}}{x}$

for $-1 \le x < 0$

Trigonometry and modelling 7A

1 a i $\angle FAB = \angle CAF + \angle BAC$

$= (\alpha - \beta) + \beta = \alpha$

So $\angle FAB = \alpha$

ii $\angle FAB$ and $\angle ABD$ are alternate angles

so $\angle FAB = \angle ABD$

so $\angle ABD = \alpha$

$\angle CBE = 90 - \alpha$, so $\angle ECB = 90 - (90 - \alpha) = \alpha$

iii $\cos\beta = \frac{AB}{1}$

So $AB = \cos\beta$

iv $\sin\beta = \frac{BC}{1}$

So $BC = \sin\beta$

b i $\angle ABD = \alpha$, so $\sin\alpha = \frac{AD}{AB}$

As $AB = \cos\beta$, this gives $\sin\alpha = \frac{AD}{\cos\beta}$

So $AD = \sin\alpha\cos\beta$

ii $\cos\alpha = \frac{BD}{AB} = \frac{BD}{\cos\beta}$

So $BD = \cos\alpha\cos\beta$

c i $\angle ECB = \alpha$, so $\cos\alpha = \frac{CE}{BC}$

As $BC = \sin\beta$, this gives $\cos\alpha = \frac{CE}{\sin\beta}$

So $CE = \cos\alpha\sin\beta$

ii $\sin\alpha = \frac{BE}{BC} = \frac{BE}{\sin\beta}$

So $BE = \sin\alpha\sin\beta$

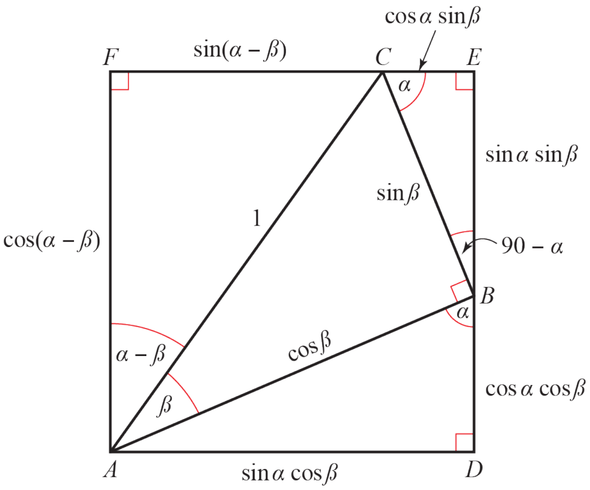
d i $\sin(\alpha - \beta) = \frac{FC}{1}$

So $FC = \sin(\alpha - \beta)$

1 d ii $\cos(\alpha-\beta)=\dfrac{FA}{1}$

So $FA=\cos(\alpha-\beta)$

e i The completed diagram should look like this:



$FC+CE=AD$, so $FC=AD-CE$

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

ii $AF=DB+BE$

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

2 $$\tan(A-B)=\frac{\sin(A-B)}{\cos(A-B)}$$
$$=\frac{\sin A\cos B-\cos A\sin B}{\cos A\cos B+\sin A\sin B}$$

Divide the numerator and denominator by cos A cos B

$$\tan(A-B)=\frac{\dfrac{\sin A\cos B}{\cos A\cos B}-\dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B}+\dfrac{\sin A\sin B}{\cos A\cos B}}=\frac{\dfrac{\sin A}{\cos A}-\dfrac{\sin B}{\cos B}}{1+\dfrac{\sin A\sin B}{\cos A\cos B}}$$

$$=\frac{\tan A-\tan B}{1+\tan A\tan B}$$ as required

3 $\sin(A+B)=\sin A\cos B+\cos A\sin B$

$\sin(P+(-Q))=\sin P\cos(-Q)+\cos P\sin(-Q)$

As $\cos(-P)=\cos P$ and $\sin(-P)=-\sin P$, this gives

$\sin(P-Q)\equiv\sin P\cos Q-\cos P\sin Q$

4 Example: $A = 60°,\ B = 30°$

$$\sin A = \frac{\sqrt{3}}{2},\ \sin B = \frac{1}{2}$$

$$\sin(A+B) = 1;\ \sin A + \sin B = \frac{\sqrt{3}}{2} + \frac{1}{2} \neq 1$$

This proves $\sin(A+B) = \sin A + \sin B$ is not true for all values.

There will be many values of A and B for which the statement is true, e.g. $A = -30°$ and $B = +30°$, and this is the danger of trying to prove a statement by taking particular examples. To prove a statement requires a sound argument; to disprove it only requires one counterexample.

5 $\cos(A-B) \equiv \cos A \cos B + \sin A \sin B$

Set $A = \theta,\ B = \theta$

$$\Rightarrow\ \cos(\theta - \theta) \equiv \cos\theta\cos\theta + \sin\theta\sin\theta$$

$$\Rightarrow\ \cos 0 \equiv \cos^2\theta + \sin^2\theta$$

So $\cos^2\theta + \sin^2\theta \equiv 1$ (since $\cos 0 = 1$)

6 **a** $\sin(A-B) \equiv \sin A\cos B - \cos A \sin B$

Set $A = \frac{\pi}{2},\ B = \theta$

$$\Rightarrow\ \sin\left(\frac{\pi}{2} - \theta\right) \equiv \sin\frac{\pi}{2}\cos\theta - \cos\frac{\pi}{2}\sin\theta$$

$$\Rightarrow\ \sin\left(\frac{\pi}{2} - \theta\right) \equiv \cos\theta$$

since $\sin\frac{\pi}{2} = 1, \cos\frac{\pi}{2} = 0$

b $\cos(A-B) \equiv \cos A \cos B + \sin A \sin B$

Set $A = \frac{\pi}{2},\ B = \theta$

$$\Rightarrow\ \cos\left(\frac{\pi}{2} - \theta\right) \equiv \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta$$

$$\Rightarrow\ \cos\left(\frac{\pi}{2} - \theta\right) \equiv \sin\theta$$

since $\cos\frac{\pi}{2} = 0, \sin\frac{\pi}{2} = 1$

7 $\sin\left(x + \frac{\pi}{6}\right) = \sin x \cos\frac{\pi}{6} + \cos x \sin\frac{\pi}{6}$

$$= \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x$$

8 $\cos\left(x+\frac{\pi}{3}\right)=\cos x\cos\frac{\pi}{3}-\sin x\sin\frac{\pi}{3}$

$=\frac{1}{2}\cos x-\frac{\sqrt{3}}{2}\sin x$

9 **a** Using $\sin(A+B)\equiv\sin A\cos B+\cos A\sin B$ gives

$\sin 15°\cos 20°+\cos 15°\sin 20°\equiv\sin(15°+20°)\equiv\sin 35°$

b Using $\sin(A-B)\equiv\sin A\cos B-\cos A\sin B$ gives

$\sin 58°\cos 23°-\cos 58°\sin 23°\equiv\sin(58°-23°)\equiv\sin 35°$

c Using $\cos(A+B)\equiv\cos A\cos B-\sin A\sin B$ gives

$\cos 130°\cos 80°-\sin 130°\sin 80°\equiv\cos(130°+80°)\equiv\cos 210°$

d Using $\tan(A-B)\equiv\frac{\tan A-\tan B}{1+\tan A\tan B}$ gives

$\frac{\tan 76°-\tan 45°}{1+\tan 76°\tan 45°}\equiv\tan(76°-45°)\equiv\tan 31°$

e Using $\cos(A-B)\equiv\cos A\cos B+\sin A\sin B$ gives

$\cos 2\theta\cos\theta+\sin 2\theta\sin\theta\equiv\cos(2\theta-\theta)\equiv\cos\theta$

f Using $\cos(A+B)\equiv\cos A\cos B-\sin A\sin B$ gives

$\cos 4\theta\cos 3\theta-\sin 4\theta\sin 3\theta\equiv\cos(4\theta+3\theta)\equiv\cos 7\theta$

g Using $\sin(A+B)\equiv\sin A\cos B+\cos A\sin B$ gives

$\sin\tfrac{1}{2}\theta\cos 2\tfrac{1}{2}\theta+\cos\tfrac{1}{2}\theta\sin 2\tfrac{1}{2}\theta\equiv\sin\left(\tfrac{1}{2}\theta+2\tfrac{1}{2}\theta\right)\equiv\sin 3\theta$

h Using $\tan(A+B)\equiv\frac{\tan A+\tan B}{1-\tan A\tan B}$ gives

$\frac{\tan 2\theta+\tan 3\theta}{1-\tan 2\theta\tan 3\theta}\equiv\tan(2\theta+3\theta)\equiv\tan 5\theta$

i Using $\sin(P-Q)\equiv\sin P\cos Q-\cos P\sin Q$ gives

$\sin(A+B)\cos B-\cos(A+B)\sin B\equiv\sin\left((A+B)-B\right)\equiv\sin A$

j Using $\cos(A+B)\equiv\cos A\cos B-\sin A\sin B$ gives

$\cos\left(\frac{3x+2y}{2}\right)\cos\left(\frac{3x-2y}{2}\right)-\sin\left(\frac{3x+2y}{2}\right)\sin\left(\frac{3x-2y}{2}\right)\equiv\cos\left(\left(\frac{3x+2y}{2}\right)+\left(\frac{3x-2y}{2}\right)\right)$

$\equiv\cos\left(\frac{6x}{2}\right)\equiv\cos 3x$

10 a Use the fact that $\frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} = \sin\frac{\pi}{4}$ to write

$$\frac{1}{\sqrt{2}}(\sin x + \cos x) = \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \sin x\cos\frac{\pi}{4} + \cos x\sin\frac{\pi}{4} = \sin\left(x + \frac{\pi}{4}\right)$$

or

$$\frac{1}{\sqrt{2}}(\sin x + \cos x) = \frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x = \cos x\cos\frac{\pi}{4} + \sin x\sin\frac{\pi}{4} = \cos\left(x - \frac{\pi}{4}\right)$$

b Use the fact that $\frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} = \sin\frac{\pi}{4}$ to write

$$\frac{1}{\sqrt{2}}(\cos x - \sin x) == \frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x = \cos x\cos\frac{\pi}{4} - \sin x\sin\frac{\pi}{4} = \cos\left(x + \frac{\pi}{4}\right)$$

c Use the fact that $\frac{1}{2} = \cos\frac{\pi}{3} = \sin\frac{\pi}{6}$ and $\frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} = \sin\frac{\pi}{3}$ to write

$$\frac{1}{2}(\sin x + \sqrt{3}\cos x) = \frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = \sin x\cos\frac{\pi}{3} + \cos x\sin\frac{\pi}{3} = \sin\left(x + \frac{\pi}{3}\right)$$

or

$$\frac{1}{2}(\sin x + \sqrt{3}\cos x) == \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = \cos x\cos\frac{\pi}{6} + \sin x\sin\frac{\pi}{6} = \cos\left(x - \frac{\pi}{6}\right)$$

d Use the fact that $\frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} = \sin\frac{\pi}{4}$ to write

$$\frac{1}{\sqrt{2}}(\sin x - \cos x) = \frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x = \sin x\cos\frac{\pi}{4} - \cos x\sin\frac{\pi}{4} = \sin\left(x - \frac{\pi}{4}\right)$$

11 $\cos y = \sin(x + y)$

$\Rightarrow \quad \cos y = \sin x\cos y + \cos x\sin y$

Divide throughout by $\cos x\cos y$

$$\frac{\cancel{\cos y}^{1}}{\cos x\cancel{\cos y}} = \frac{\sin x\cancel{\cos y}}{\cos x\cancel{\cos y}} + \frac{\cancel{\cos x}\sin y}{\cancel{\cos x}\cos y}$$

$\Rightarrow \quad \sec x = \tan x + \tan y$

$\Rightarrow \quad \tan y = \sec x - \tan x$

12 As $\tan(x - y) = 3$

so $\frac{\tan x - \tan y}{1 + \tan x\tan y} = 3$

$\Rightarrow \quad \tan x - \tan y = 3 + 3\tan x\tan y$

$\Rightarrow \quad 3\tan x\tan y + \tan y = \tan x - 3$

$\Rightarrow \quad \tan y(3\tan x + 1) = \tan x - 3$

$\Rightarrow \quad \tan y = \frac{\tan x - 3}{3\tan x + 1}$

13 $\sin x(\cos y + 2\sin y) = \cos x(2\cos y - \sin y)$

$$\Rightarrow \sin x\cos y + 2\sin x\sin y = 2\cos x\cos y - \cos x\sin y$$

$$\Rightarrow \sin x\cos y + \cos x\sin y = 2(\cos x\cos y - \sin x\sin y)$$

$$\Rightarrow \sin(x+y) = 2\cos(x+y)$$

$$\Rightarrow \frac{\sin(x+y)}{\cos(x+y)} = 2$$

$$\Rightarrow \tan(x+y) = 2$$

14 a $\tan(x - 45°) = \frac{1}{4}$

$$\Rightarrow \frac{\tan x - \tan 45°}{1 + \tan x\tan 45°} = \frac{1}{4}$$

$$\Rightarrow 4\tan x - 4 = 1 + \tan x \quad (\text{as } \tan 45° = 1)$$

$$\Rightarrow 3\tan x = 5$$

$$\Rightarrow \tan x = \frac{5}{3}$$

b $\sin(x - 60°) = 3\cos(x + 30°)$

$$\Rightarrow \sin x\cos 60° - \cos x\sin 60° = 3\cos x\cos 30° - 3\sin x\sin 30°$$

$$\Rightarrow \frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x = \frac{3\sqrt{3}}{2}\cos x - \frac{3}{2}\sin x$$

$$\Rightarrow 4\sin x = 4\sqrt{3}\cos x$$

$$\Rightarrow \frac{\sin x}{\cos x} = \frac{4\sqrt{3}}{4}$$

$$\Rightarrow \tan x = \sqrt{3}$$

c $\tan(x - 60°) = 2$

$$\Rightarrow \frac{\tan x - \tan 60°}{1 + \tan x\tan 60°} = 2$$

$$\Rightarrow \frac{\tan x - \sqrt{3}}{1 + \sqrt{3}\tan x} = 2 \quad (\text{as } \tan 60° = \sqrt{3})$$

$$\Rightarrow \tan x - \sqrt{3} = 2 + 2\sqrt{3}\tan x$$

$$\Rightarrow (2\sqrt{3} - 1)\tan x = -(2 + \sqrt{3})$$

$$\Rightarrow \tan x = -\frac{(2+\sqrt{3})}{2\sqrt{3}-1} = -\frac{(2+\sqrt{3})(2\sqrt{3}+1)}{(2\sqrt{3}-1)(2\sqrt{3}+1)}$$

$$= -\frac{8+5\sqrt{3}}{11}$$

15 $\tan\left(x+\frac{\pi}{3}\right)=\frac{1}{2}$

$$\Rightarrow \frac{\tan x+\tan\frac{\pi}{3}}{1-\tan x\tan\frac{\pi}{3}}=\frac{1}{2}$$

$$\Rightarrow \frac{\tan x+\sqrt{3}}{1-\sqrt{3}\tan x}=\frac{1}{2}\quad\left(\tan\frac{\pi}{3}=\sqrt{3}\right)$$

$$\Rightarrow 2\tan x+2\sqrt{3}=1-\sqrt{3}\tan x$$

$$\Rightarrow (2+\sqrt{3})\tan x=1-2\sqrt{3}$$

$$\Rightarrow \tan x=\frac{1-2\sqrt{3}}{2+\sqrt{3}}=\frac{(1-2\sqrt{3})(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})}$$

$$=\frac{2-4\sqrt{3}-\sqrt{3}+6}{1}=8-5\sqrt{3}$$

16 Write $\theta=\left(\theta+\frac{2\pi}{3}\right)-\frac{2\pi}{3}$ and $\theta+\frac{4\pi}{3}=\left(\theta+\frac{2\pi}{3}\right)+\frac{2\pi}{3}$

Now use the appropriate addition formulae for cos

$$\cos\left(\left(\theta+\frac{2\pi}{3}\right)-\frac{2\pi}{3}\right)=\cos\left(\theta+\frac{2\pi}{3}\right)\cos\frac{2\pi}{3}+\sin\left(\theta+\frac{2\pi}{3}\right)\sin\frac{2\pi}{3}$$

$$\cos\left(\left(\theta+\frac{2\pi}{3}\right)+\frac{2\pi}{3}\right)=\cos\left(\theta+\frac{2\pi}{3}\right)\cos\frac{2\pi}{3}-\sin\left(\theta+\frac{2\pi}{3}\right)\sin\frac{2\pi}{3}$$

Now add up all terms

$$\cos\theta+\cos\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{4\pi}{3}\right)$$

$$\equiv\cos\left(\left(\theta+\frac{2\pi}{3}\right)-\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\left(\theta+\frac{2\pi}{3}\right)+\frac{2\pi}{3}\right)$$

$$\equiv\cos\left(\theta+\frac{2\pi}{3}\right)\cos\frac{2\pi}{3}+\sin\left(\theta+\frac{2\pi}{3}\right)\sin\frac{2\pi}{3}+\cos\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{2\pi}{3}\right)\cos\frac{2\pi}{3}$$

$$-\sin\left(\theta+\frac{2\pi}{3}\right)\sin\frac{2\pi}{3}$$

$$\equiv 2\cos\left(\theta+\frac{2\pi}{3}\right)\cos\frac{2\pi}{3}+\cos\left(\theta+\frac{2\pi}{3}\right)$$

$$\equiv 0 \text{ as } \cos\frac{2\pi}{3}=-\frac{1}{2}$$

Challenge

a **i** Area $= \frac{1}{2}ab\sin\theta = \frac{1}{2}x(y\cos B)(\sin A)$

$$= \frac{1}{2}xy\sin A\cos B$$

ii Area $= \frac{1}{2}ab\sin\theta = \frac{1}{2}y(x\cos A)(\sin B)$

$$= \frac{1}{2}xy\cos A\sin B$$

iii Area $= \frac{1}{2}ab\sin\theta = \frac{1}{2}xy\sin(A+B)$

b Area $T_1 + T_2 =$ Area $T_1 +$ Area T_2

$$\Rightarrow \frac{1}{2}xy\sin(A+B) = \frac{1}{2}xy\sin A\cos B + \frac{1}{2}xy\cos A\sin B$$

$$\Rightarrow \sin(A+B) = \sin A\cos B + \cos A\sin B$$

Trigonometry and modelling 7B

1 **a** $\cos 15° = \cos(45° - 30°)$
$$= \cos 45° \cos 30° + \sin 45° \sin 30°$$
$$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2}$$
$$= \frac{\sqrt{2}\sqrt{3} + \sqrt{2}}{4} = \frac{\sqrt{2}\left(\sqrt{3}+1\right)}{4}$$

b $\sin 75° = \sin(45° + 30°)$
$$= \sin 45° \cos 30° + \cos 45° \sin 30°$$
$$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2}$$
$$= \frac{\sqrt{2}\sqrt{3} + \sqrt{2}}{4} = \frac{\sqrt{2}\left(\sqrt{3}+1\right)}{4}$$

Note $\sin 75° = \cos(90° - 75°) = \cos 15°$

c $\sin(120° + 45°)$
$$= \sin 120° \cos 45° + \cos 120° \sin 45°$$
$$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} + \left(-\frac{1}{2}\right) \times \frac{\sqrt{2}}{2}$$
$$= \frac{\sqrt{2}\sqrt{3} - \sqrt{2}}{4} = \frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4}$$

d $\tan 165° = \tan(120° + 45°)$
$$= \frac{\tan 120° + \tan 45°}{1 - \tan 120° \tan 45°}$$
$$\tan 120° = \frac{\sin 120°}{\cos 120°} = \frac{\sin 60°}{-\cos 60°}$$
$$= -\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

So $\tan 120° = \dfrac{-\sqrt{3}+1}{1+\sqrt{3}}$
$$= \frac{\left(1-\sqrt{3}+1\right)\left(\sqrt{3}-1\right)}{\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)}$$
$$= \frac{-4+2\sqrt{3}}{2}$$
$$= -2+\sqrt{3}$$

2 **a** Using $\sin(A+B)$ expansion
$$\sin 30° \cos 60° + \cos 30° \sin 60°$$
$$= \sin(30° + 60°) = \sin 90° = 1$$

b $\cos 110° \cos 20° + \sin 110° \sin 20°$
$$= \cos(110° - 20°) = \cos 90° = 0$$

c $\sin 33° \cos 27° + \cos 33° \sin 27°$
$$= \sin(33° + 27°) = \sin 60° = \frac{\sqrt{3}}{2}$$

d $\cos\frac{\pi}{8}\cos\frac{\pi}{8} - \sin\frac{\pi}{8}\sin\frac{\pi}{8}$
$$= \cos\left(\frac{\pi}{8} + \frac{\pi}{8}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

e $\sin 60° \cos 15° - \cos 60° \sin 15°$
$$= \sin(60° - 15°) = \sin 45° = \frac{\sqrt{2}}{2}$$

f $\cos 70° \cos 50° - \cos 70° \tan 70° \sin 50°$
$$= \cos 70° \cos 50° - \sin 70° \sin 50°$$

Simplifying as
$$\left(\cos\theta \times \tan\theta = \cancel{\cos\theta} \times \frac{\sin\theta}{\cancel{\cos\theta}} = \sin\theta\right)$$

So $\cos 70°(\cos 50° - \tan °\sin 50°)$
$$= \cos(70° + 50°)$$
$$= \cos 120° = -\cos 60° = -\frac{1}{2}$$

g $\dfrac{\tan 45° + \tan 15°}{1 - \tan 45° \tan 15°}$
$$= \tan(45+15)° = \tan 60° = \sqrt{3}$$

h Use the fact that $\tan 45° = 1$ to rewrite as
$$\frac{1 - \tan 15°}{1 + \tan 15°} = \frac{\tan 45° - \tan 15°}{1 + \tan 45° \tan 15°}$$
$$= \tan(45° - 15°) = \tan 30° = \frac{\sqrt{3}}{3}$$

i $\dfrac{\tan\frac{7\pi}{12} - \tan\frac{\pi}{3}}{1 + \tan\frac{7\pi}{12}\tan\frac{\pi}{3}} = \tan\left(\frac{7\pi}{12} - \frac{\pi}{3}\right)$
$$= \tan\frac{3\pi}{12} = \tan\frac{\pi}{4} = 1$$

2 **j** This is very similar to part (e) but to appreciate this you need to rewrite the equation as

$$\sqrt{3}\cos 15° - \sin 15°$$
$$\equiv 2\left(\frac{\sqrt{3}}{2}\cos 15° - \frac{1}{2}\sin 15°\right)$$
$$\equiv 2(\sin 60°\cos 15° - \cos 60°\sin 15°)$$
$$\equiv 2\sin(60-15)°$$
$$\equiv 2\sin 45°$$
$$= \sqrt{2}$$

3 **a** $\tan(45° + 30°) = \dfrac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$

b $\tan 75° = \dfrac{1 + \frac{\sqrt{3}}{3}}{1 - \frac{\sqrt{3}}{3}}$

$$= \frac{3+\sqrt{3}}{3-\sqrt{3}} = \frac{\left(3+\sqrt{3}\right)\left(3+\sqrt{3}\right)}{\left(3-\sqrt{3}\right)\left(3+\sqrt{3}\right)}$$
$$= \frac{12+6\sqrt{3}}{6} = 2+\sqrt{3}$$

4 $\cot(A+B) = 2$

$$\Rightarrow \tan(A+B) = \frac{1}{2}$$
$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{1}{2}$$

But as $\cot A = \dfrac{1}{4}$, then $\tan A = 4$.

So $\dfrac{4 + \tan B}{1 - 4\tan B} = \dfrac{1}{2}$

$$\Rightarrow 8 + 2\tan B = 1 - 4\tan B$$
$$\Rightarrow 6\tan B = -7$$
$$\Rightarrow \tan B = -\frac{7}{6}$$

So $\cot B = \dfrac{1}{\tan B} = -\dfrac{6}{7}$

5 **a** $\cos 105° = \cos(45° + 60°)$

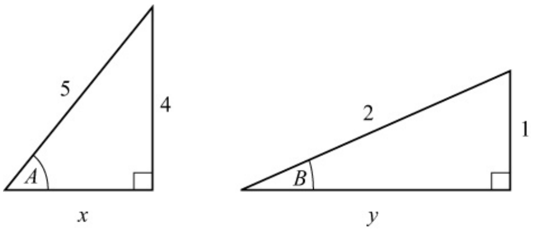
$$= \cos 45° \cos 60° - \sin 45° \sin 60°$$
$$= \frac{1}{\sqrt{2}} \times \frac{1}{2} - \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}$$
$$= \frac{1-\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}-\sqrt{6}}{4}$$

b $\sec 105° = \dfrac{1}{\cos 105°}$

$$= \frac{1}{\frac{\sqrt{2}-\sqrt{6}}{4}} = \frac{4}{\sqrt{2}-\sqrt{6}}$$
$$= \frac{4}{\sqrt{2}-\sqrt{6}} \times \frac{\sqrt{2}+\sqrt{6}}{\sqrt{2}+\sqrt{6}}$$
$$= \frac{4\left(\sqrt{2}+\sqrt{6}\right)}{-4} = -\sqrt{2}\left(1+\sqrt{3}\right)$$

So $a = 2$ and $b = 3$

6 Draw the right-angled triangles containing A and B



Using Pythagoras' theorem gives
$x = 3$ and $y = \sqrt{3}$

a $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$$= \frac{4}{5} \times \frac{\sqrt{3}}{2} + \frac{3}{5} \times \frac{1}{2} = \frac{4\sqrt{3}+3}{10}$$

b $\cos(A-B) = \cos A \cos B + \sin A \sin B$

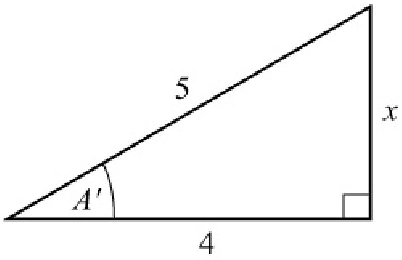
$$= \frac{3}{5} \times \frac{\sqrt{3}}{2} + \frac{4}{5} \times \frac{1}{2} = \frac{3\sqrt{3}+4}{10}$$

c $\sec(A-B) = \dfrac{1}{\cos(A-B)} = \dfrac{10}{3\sqrt{3}+4}$

$$= \frac{10(3\sqrt{3}-4)}{(3\sqrt{3}+4)(3\sqrt{3}-4)}$$
$$= \frac{10(3\sqrt{3}-4)}{27-16}$$
$$= \frac{10(3\sqrt{3}-4)}{11}$$

7 Let $A' = 180° - A$. As A is the second quadrant $\cos A' = -\cos A$

Draw a right-angled triangle where $\cos A' = \frac{4}{5}$



Using Pythagoras' theorem $x = 3$

So $\sin A' = \frac{3}{5}$, $\tan A' = \frac{3}{4}$

a As A is in the second quadrant,

$\sin A = \sin A'$, $\sin A = \frac{3}{5}$

b $\cos(\pi + A) = \cos\pi\cos A - \sin\pi\sin A$

$= -\cos A$

As $\cos\pi = -1$, $\sin\pi = 0$

So $\cos(\pi + A) = \frac{4}{5}$

c $$\sin\left(\frac{\pi}{3} + A\right) = \sin\frac{\pi}{3}\cos A + \cos\frac{\pi}{3}\sin A$$
$$= \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{4}{5}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{5}\right)$$
$$= \frac{3 - 4\sqrt{3}}{10}$$

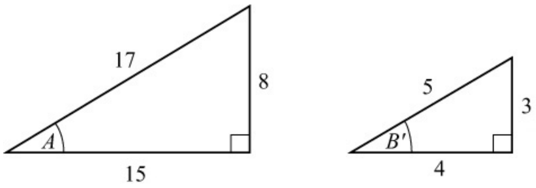
d As A is in the second quadrant,

$\tan A = -\tan A' = -\frac{3}{4}$

$$\tan\left(\frac{\pi}{4} + A\right) = \frac{\tan\frac{\pi}{4} + \tan A}{1 - \tan\frac{\pi}{4}\tan A}$$
$$= \frac{1 + \tan A}{1 - \tan A} = \frac{\frac{1}{4}}{\frac{7}{4}} = \frac{1}{7}$$

8 Let $B' = 180° - B$. As B is in the second quadrant $\cos B' = -\cos B$, $\sin B' = \sin B$ and $\tan B' = -\tan B$.

Drawing right-angled triangles for A and B', use Pythagoras' theorem to find the missing sides, which are 15 and 3.



So $\sin A = \frac{8}{17}$, $\cos A = \frac{15}{17}$, $\tan A = \frac{8}{15}$

and $\sin B = \frac{3}{5}$, $\cos B = -\frac{4}{5}$, $\tan B = -\frac{3}{4}$

a $$\sin(A - B) = \sin A\cos B - \cos A\sin B$$
$$= \left(\frac{8}{17}\right)\left(-\frac{4}{5}\right) - \left(\frac{15}{17}\right)\left(\frac{3}{5}\right)$$
$$= \frac{-32 - 45}{85} = -\frac{77}{85}$$

b $$\cos(A - B) = \cos A\cos B + \sin A\sin B$$
$$= \left(\frac{15}{17}\right)\left(-\frac{4}{5}\right) + \left(\frac{8}{17}\right)\left(\frac{3}{5}\right)$$
$$= \frac{-60 + 24}{85} = -\frac{36}{85}$$

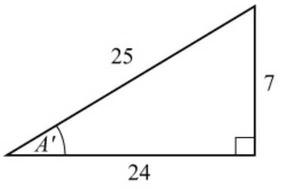
c $$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$
$$= \frac{\frac{8}{15} + \frac{3}{4}}{1 - \frac{24}{60}} = \frac{\frac{77}{60}}{\frac{36}{60}} = \frac{77}{36}$$

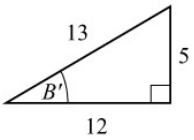
So $\cot(A - B) = \frac{1}{\tan(A - B)} = \frac{36}{77}$

9 Angle A is in the third quadrant as it is reflex and tan A is positive. Let $A' = A - 180°$, so $\sin A = -\sin A'$, $\cos A = -\cos A'$, $\tan A = \tan A'$

Let $B' = 180° - B$. As B is in the second quadrant $\cos B' = -\cos B$, $\sin B' = \sin B$ and $\tan B' = -\tan B$.

Drawing right-angled triangles for A' and B' use Pythagoras' theorem to find the missing sides, which are 25 and 12.





So $\sin A = -\dfrac{7}{25}$, $\cos A = -\dfrac{24}{25}$, $\tan A = \dfrac{7}{24}$

and $\sin B = \dfrac{5}{13}$, $\cos B = -\dfrac{12}{13}$, $\tan B = -\dfrac{5}{12}$

a $$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$= \left(-\frac{7}{25}\right)\left(-\frac{12}{13}\right) + \left(-\frac{24}{25}\right)\left(\frac{5}{13}\right)$$
$$= \frac{84-120}{325} = -\frac{36}{325}$$

b $$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}$$
$$= \frac{\frac{7}{24}+\frac{5}{12}}{1-\left(\frac{7}{24}\right)\left(\frac{5}{12}\right)} = \frac{\frac{17}{24}}{\frac{253}{288}} = \frac{204}{253}$$

c $$\operatorname{cosec}(A+B) = \frac{1}{\sin(A+B)} = -\frac{325}{36}$$

10 a $$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B}$$
$$= \frac{\frac{1}{5}+\frac{2}{3}}{1-\frac{1}{5}\times\frac{2}{3}} = \frac{\frac{13}{15}}{\frac{15-2}{15}} = \frac{\frac{13}{15}}{\frac{13}{15}} = 1$$

As tan ($A + B$) is positive, $A + B$ is in the first or third quadrants, but as A and B are both acute $A + B$ cannot be in the third quadrant, so $A + B = \tan^{-1}1 = 45°$

b A is reflex but $\tan A°$ is positive, so A is in the third quadrant, i.e. $180° < A < 270°$ and $0° < B < 90°$. As tan ($A + B$) is positive, $A + B$ is in the first or third quadrants.

As $180° < A + B < 360°$, it must be in the third quadrant, so $A + B = \tan^{-1}1 = 225°$

Trigonometry and modelling 7C

1 $\sin 2A = \sin(A+A) = \sin A\cos A + \cos A\sin A$
$= 2\sin A\cos A$

2 a $\cos 2A = \cos(A+A)$
$= \cos A\cos A - \sin A\sin A$
$= \cos^2 A - \sin^2 A$

b i $\cos 2A = \cos^2 A - \sin^2 A$
Use $\cos^2 A + \sin^2 A = 1$ to simplify, so
$\cos 2A = \cos^2 A - (1-\cos^2 A)$
$= 2\cos^2 A - 1$

ii $\cos 2A = \cos^2 A - \sin^2 A$
$= (1-\sin^2 A) - \sin^2 A$
$= 1 - 2\sin^2 A$

3 $\tan 2A = \tan(A+A)$

$$= \frac{\tan A + \tan A}{1 - \tan A\tan A} = \frac{2\tan A}{1-\tan^2 A}$$

4 a $2\sin 10°\cos 10° = \sin 20°$
(using $2\sin A\cos A \equiv \sin 2A$)

b $1 - 2\sin^2 25° = \cos 50°$
using $\cos 2A \equiv 1 - 2\sin^2 A$

c $\cos^2 40° - \sin^2 40° = \cos 80°$
using $\cos 2A \equiv \cos^2 A - \sin^2 A$

d $\dfrac{2\tan 5°}{1-\tan^2 5°} = \tan 10°$

using $\tan 2A \equiv \dfrac{2\tan A}{1-\tan^2 A}$

e $\dfrac{1}{2\sin 24.5°\cos 24.5°} = \dfrac{1}{\sin 49°}$
$= \text{cosec}\, 49°$

f $6\cos^2 30° - 3 = 3(2\cos^2 30° - 1)$
$= 3\cos 60°$

g $\dfrac{\sin 8°}{\sec 8°} = \sin 8°\cos 8°$

$$= \frac{1}{2}(2\sin 8°\cos 8°) = \frac{1}{2}\sin 16°$$

h $\cos^2\dfrac{\pi}{16} - \sin^2\dfrac{\pi}{16} = \cos\dfrac{2\pi}{16} = \cos\dfrac{\pi}{8}$

5 a $2\sin 22.5°\cos 22.5° = \sin 2\times 22.5°$

$$= \sin 45° = \frac{\sqrt{2}}{2}$$

b $2\cos^2 15° - 1 = \cos(2\times 15°)$

$$= \cos 30° = \frac{\sqrt{3}}{2}$$

c $(\sin 75° - \cos 75°)^2$
$= \sin^2 75° + \cos^2 75° - 2\sin 75°\cos 75°$
$= 1 - \sin(2\times 75°)$
as $\sin^2 75° + \cos^2 75° = 1$, and this gives
$(\sin 75° - \cos 75°)^2 = 1 - \sin 150°$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

d $\dfrac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}} = \tan\left(2\times\dfrac{\pi}{8}\right) = \tan\dfrac{\pi}{8} = 1$

6 a $(\sin A + \cos A)^2$
$= \sin^2 A + 2\sin A\cos A + \cos^2 A$
$= 1 + \sin 2A$

b $\left(\sin\dfrac{\pi}{8} + \cos\dfrac{\pi}{8}\right)^2$

$$= 1 + \sin\frac{\pi}{4} = 1 + \frac{\sqrt{2}}{2} = \frac{2+\sqrt{2}}{4}$$

7 a $\cos^2 3\theta - \sin^2 3\theta = \cos(2\times 3\theta) = \cos 6\theta$

b $6\sin 2\theta\cos 2\theta = 3(2\sin 2\theta\cos 2\theta)$
$= 3\sin(2\times 2\theta)$
$= 3\sin 4\theta$

c $\dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} = \tan\left(2\times\dfrac{\theta}{2}\right) = \tan\theta$

7 **d** $2-4\sin^2\frac{\theta}{2}=2\left(1-2\sin^2\left(\frac{\theta}{2}\right)\right)$

$=2\cos\left(2\times\frac{\theta}{2}\right)=2\cos\theta$

e $\sqrt{1+\cos 2\theta}=\sqrt{1+(2\cos^2\theta-1)}$

$=\sqrt{2\cos^2\theta}$

$=\sqrt{2}\cos\theta$

f $\sin^2\theta\cos^2\theta=\frac{1}{4}(4\sin^2\theta\cos^2\theta)$

$=\frac{1}{4}(2\sin\theta\cos\theta)^2$

$=\frac{1}{4}\sin^2 2\theta$

g

$4\sin\theta\cos\theta\cos 2\theta=2(2\sin\theta\cos\theta)\cos 2\theta$

$=2\sin 2\theta\cos 2\theta$

$=\sin 4\theta$

As $\sin 2A=2\sin A\cos A$ with $A=2\theta$

h $\frac{\tan\theta}{\sec^2\theta-2}=\frac{\tan\theta}{(1+\tan^2\theta)-2}$

$=\frac{\tan\theta}{\tan^2\theta-1}$

$=-\frac{\tan\theta}{1-\tan^2\theta}$

$=-\frac{1}{2}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$

$=-\frac{1}{2}\tan 2\theta$

i $\cos^4\theta-2\sin^2\theta\cos^2\theta+\sin^4\theta$

$=(\cos^2\theta-\sin^2\theta)^2$

$=\cos^2 2\theta$

8 $p=2\cos\theta \Rightarrow \cos\theta=\frac{p}{2}$

$\cos 2\theta=q$

Using $\cos 2\theta=2\cos^2\theta-1$

$\Rightarrow q=2\left(\frac{p}{2}\right)^2-1$

$\Rightarrow q=\frac{p^2}{2}-1$

9 **a** $\cos^2\theta=x,\ \cos 2\theta=1-y$

Using $\cos 2\theta=2\cos^2\theta-1$

$\Rightarrow 1-y=2x-1$

$\Rightarrow y=2-2x=2(1-x)$

Any form of this equation is correct

b $y=\cot 2\theta \Rightarrow \tan 2\theta=\frac{1}{y}$

$x=\tan\theta$

Using $\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$

$\Rightarrow \frac{1}{y}=\frac{2x}{1-x^2}$

$\Rightarrow 2xy=1-x^2$

Any form of this equation is correct

c $x=\sin\theta,\ y=2\sin\theta\cos\theta$

$\Rightarrow y=2x\cos\theta$

$\Rightarrow \cos\theta=\frac{y}{2x}$

Using $\sin^2\theta+\cos^2\theta\equiv 1$

$\Rightarrow x^2+\frac{y^2}{4x^2}=1$

$\Rightarrow 4x^4+y^2=4x^2$

or $y^2=4x^2(1-x^2)$

Any form of this equation is correct

d $x=3\cos 2\theta+1 \Rightarrow \cos 2\theta=\frac{x-1}{3}$

$y=2\sin\theta \Rightarrow \sin\theta=\frac{y}{2}$

Using $\cos 2\theta=1-2\sin^2\theta$

$\Rightarrow \frac{x-1}{3}=1-\frac{2y^2}{4}=1-\frac{y^2}{2}$

Multiplying both sides by 6 gives

$2(x-1)=6-3y^2$

$\Rightarrow 3y^2=6-2(x-1)=8-2x$

$\Rightarrow y^2=\frac{2(4-x)}{3}$

Any form of this equation is correct

10 $\cos 2x=2\cos^2 x-1$

$\cos 2x=2\left(\frac{1}{4}\right)^2-1=\frac{1}{8}-1=-\frac{7}{8}$

11 $\cos 2\theta = 1 - 2\sin^2\theta$

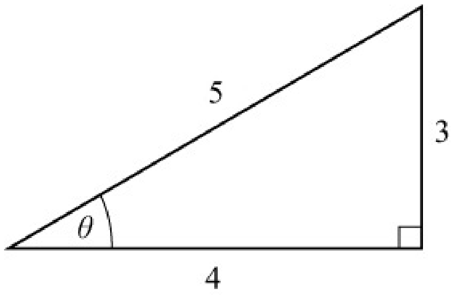
So $\dfrac{23}{25} = 1 - 2\sin^2\theta$

$$\Rightarrow\ 2\sin^2\theta = 1 - \frac{23}{25} = \frac{2}{25}$$

$$\Rightarrow\ \sin^2\theta = \frac{1}{25}$$

$$\Rightarrow\ \sin\theta = \pm\frac{1}{5}$$

12 Draw a right-angled triangle with θ as one of the angles. The hypotenuse is 5



So $\sin\theta = \dfrac{3}{5}$, $\cos\theta = \dfrac{4}{5}$, $\tan\theta = \dfrac{3}{4}$

a i $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{\frac{3}{2}}{1-\frac{9}{16}}$

$$= \frac{\frac{3}{2}}{\frac{7}{16}} = \frac{3}{2} \times \frac{16}{7} = \frac{24}{7}$$

ii $\sin 2\theta = 2\sin\theta\cos\theta = 2 \times \dfrac{3}{5} \times \dfrac{4}{5} = \dfrac{24}{25}$

iii $\cos 2\theta = \cos^2\theta - \sin^2\theta = \dfrac{16}{25} - \dfrac{9}{25} = \dfrac{7}{25}$

b

$$\sin 4\theta = 2\sin 2\theta\cos 2\theta$$

$$= 2 \times \frac{24}{25} \times \frac{7}{25} = \frac{336}{625}$$

13 a i $\cos 2A = 2\cos^2 A - 1$

$$= 2\left(-\frac{1}{3}\right)^2 - 1 = \frac{2}{9} - 1 = -\frac{7}{9}$$

ii $\cos 2A = 1 - 2\sin^2 A$

$$\Rightarrow -\frac{7}{9} = 1 - 2\sin^2 A$$

$$\Rightarrow 2\sin^2 A = 1 + \frac{7}{9} = \frac{16}{9}$$

$$\Rightarrow \sin^2 A = \frac{8}{9}$$

$$\Rightarrow \sin A = \pm\frac{\sqrt{8}}{3} = \pm\frac{2\sqrt{2}}{3}$$

But A is in the second quarter so $\sin A$ is positive, and the solution is

$$\sin A = \frac{2\sqrt{2}}{3}$$

iii $\operatorname{cosec} 2A = \dfrac{1}{\sin 2A} = \dfrac{1}{2\sin A\cos A}$

$$= \frac{1}{2 \times \frac{2\sqrt{2}}{3} \times \left(-\frac{1}{3}\right)}$$

$$= -\frac{9}{4\sqrt{2}} = -\frac{9\sqrt{2}}{8}$$

b $\tan 2A = \dfrac{\sin 2A}{\cos 2A} = \dfrac{-\frac{4\sqrt{2}}{9}}{-\frac{7}{9}}$

$$= -\frac{4\sqrt{2}}{9} \times -\frac{9}{7} = \frac{4\sqrt{2}}{7}$$

14 Using $\tan\theta = \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

$$\Rightarrow\ \frac{3}{4} = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$$

$$\Rightarrow\ 3 - 3\tan^2\frac{\theta}{2} = 8\tan\frac{\theta}{2}$$

$$\Rightarrow\ 3\tan^2\frac{\theta}{2} + 8\tan\frac{\theta}{2} - 3 = 0$$

$$\Rightarrow\ \left(3\tan\frac{\theta}{2} - 1\right)\left(\tan\frac{\theta}{2} + 3\right) = 0$$

so $\tan\dfrac{\theta}{2} = \frac{1}{3}$ or $\tan\dfrac{\theta}{2} = -3$

But $\pi < \theta < \dfrac{3\pi}{2}$ so $\dfrac{\pi}{2} < \dfrac{\theta}{2} < \dfrac{3\pi}{4}$

As $\dfrac{\theta}{2}$ is in the second quadrant, so $\tan\dfrac{\theta}{2}$ is negative, and the solution is $\tan\dfrac{\theta}{2} = -3$

15 $\cos x + \sin x = m$

$\cos x - \sin x = n$

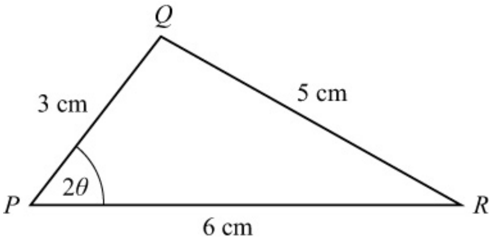
Multiply the equations

$$(\cos x + \sin x)(\cos x - \sin x) = mn$$

$$\Rightarrow \quad \cos^2 x - \sin^2 x = mn$$

$$\Rightarrow \quad \cos 2x = mn$$

16



a Using cosine rule with

$$\cos P = \frac{q^2 + r^2 - p^2}{2qr}$$

$$\cos 2\theta = \frac{36 + 9 - 25}{2 \times 6 \times 3} = \frac{20}{36} = \frac{5}{9}$$

b Using $\cos 2\theta = 1 - 2\sin^2\theta$

$$\frac{5}{9} = 1 - 2\sin^2\theta$$

$$\Rightarrow \quad 2\sin^2\theta = 1 - \frac{5}{9} = \frac{4}{9}$$

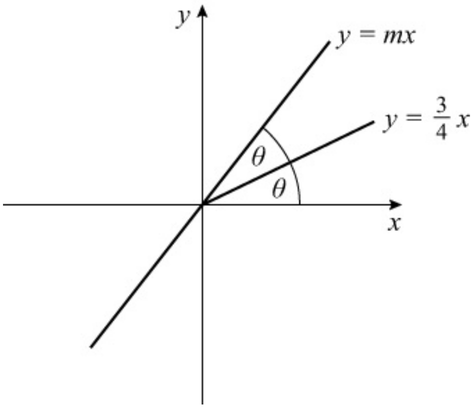
$$\Rightarrow \quad \sin^2\theta = \frac{2}{9}$$

$$\Rightarrow \quad \sin\theta = \pm\frac{\sqrt{2}}{3}$$

As 2θ is acute, θ must be in the first quadrant so $\sin\theta$ is positive, so

$$\sin\theta = \frac{\sqrt{2}}{3}$$

17 Sketch the problem,



a The gradient of line l is $\frac{3}{4}$, which is $\tan\theta$.

So $\tan\theta = \dfrac{3}{4}$

b The gradient of $y = mx$ is m and as $y = \frac{3}{4}x$ bisects the angle between $y = mx$ and the x-axis

$$m = \tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$= \frac{2 \times \frac{3}{4}}{1 - \left(\frac{3}{4}\right)^2} = \frac{\frac{3}{2}}{\frac{7}{16}} = \frac{3}{2} \times \frac{16}{7} = \frac{24}{7}$$

18 a $\cos 2A = \cos(A + A)$

$$= \cos A \cos A - \sin A \sin A$$

$$= \cos^2 A - \sin^2 A$$

$$= \cos^2 A - (1 - \cos^2 A)$$

$$= 2\cos^2 A - 1$$

b The lines intersect when

$$4\cos 2x = 6\cos^2 x - 3\sin 2x$$

This equation can be written as

$$\cos 2x + 3\cos 2x = 6\cos^2 x + 3\sin 2x$$

Use the fact that $3\cos 2x = 6\cos^2 x - 3$, so the equation becomes

$$\cos 2x + 6\cos^2 x - 3$$
$$= 6\cos x^2 - 3\sin 2x$$

$$\Rightarrow \cos 2x - 3 = 3\sin 2x$$

$$\Rightarrow \cos 2x + 3\sin 2x - 3 = 0$$

19 $\tan 2A \equiv \dfrac{\sin 2A}{\cos 2A} \equiv \dfrac{2\sin A\cos A}{\cos^2 A - \sin^2 A}$

$$\equiv \frac{\dfrac{2\sin A\cos A}{\cos^2 A}}{\dfrac{\cos^2 A - \sin^2 A}{\cos^2 A}}$$

$$\equiv \frac{\dfrac{2\sin A}{\cos A}}{1 - \dfrac{\sin^2 A}{\cos^2 A}}$$

$$\equiv \frac{2\tan A}{1 - \tan^2 A}$$

Trigonometry and modelling 7D

1 a $3\cos\theta = 2\sin(\theta + 60°)$

$$\Rightarrow\ 3\cos\theta = 2(\sin\theta\cos 60° + \cos\theta\sin 60°)$$

$$\Rightarrow\ 3\cos\theta = 2\left(\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta\right) = \sin\theta + \sqrt{3}\cos\theta$$

$$\Rightarrow\ \left(3-\sqrt{3}\right)\cos\theta = \sin\theta$$

$$\Rightarrow\ \tan\theta = 3-\sqrt{3} = 1.2679... \quad \left(\text{as } \tan\theta = \frac{\sin\theta}{\cos\theta}\right)$$

As $\tan\theta$ is positive, θ is in the first and third quadrants

$\theta = \tan^{-1}(1.2679),\ 180° + \tan^{-1}(1.2679)$

$\theta = 51.7°,\ 231.7°$

b $\sin(\theta + 30°) + 2\sin\theta = 0$

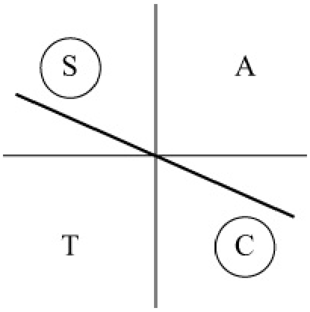
$$\Rightarrow\ \sin\theta\cos 30° + \cos\theta\sin 30° + 2\sin\theta = 0$$

$$\Rightarrow\ \frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta + 2\sin\theta = 0$$

$$\Rightarrow\ \left(4+\sqrt{3}\right)\sin\theta = -\cos\theta$$

$$\Rightarrow\ \tan\theta = -\frac{1}{4+\sqrt{3}}$$

As $\tan\theta$ is negative, θ is in the second and fourth quadrants



$\theta = \tan^{-1}\left(-\frac{1}{4+\sqrt{3}}\right) + 180°,\ \tan^{-1}\left(-\frac{1}{4+\sqrt{3}}\right) + 360°$

$\theta = 170.1°,\ 350.1°$

1 c $\cos(\theta+25°)+\sin(\theta+65°)=1$

$\Rightarrow \cos\theta\cos 25°-\sin\theta\sin 25°+\sin\theta\cos 65°+\cos\theta\sin 65°=1$

As $\sin(90-x)°=\cos x°$ and $\cos(90-x)°=\sin x°$

$\cos 25°=\sin 65°$ and $\sin 25°=\cos 65°$

So $\cos\theta\sin 65°-\sin\theta\cos 65°+\sin\theta\cos 65°+\cos\theta\sin 65°=1$

$\Rightarrow 2\cos\theta\sin 65°=1$

$$\Rightarrow \cos\theta=\frac{1}{2\sin 65°}=0.55168...$$

$\theta=\cos^{-1}(0.55168),\ 360°-\cos^{-1}(0.55168)$

$\theta=\ 56.5°,\ 303.5°$

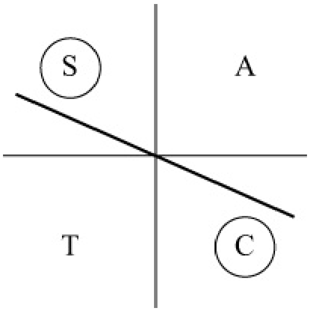
d $\cos\theta=\cos(\theta+60°)$

$$\Rightarrow \cos\theta=\cos\theta\cos 60°-\sin\theta\sin 60°=\frac{1}{2}\cos\theta-\frac{\sqrt{3}}{2}\sin\theta$$

$\Rightarrow \cos\theta=-\sqrt{3}\sin\theta$

$$\Rightarrow \tan\theta=-\frac{1}{\sqrt{3}} \qquad \left(\text{as } \tan\theta=\frac{\sin\theta}{\cos\theta}\right)$$

As $\tan\theta$ is negative, θ is in the second and fourth quadrants



$$\theta=\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+180°,\ \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+360°$$

$\theta=150.0°,\ 330.0°$

2 a
$$\sin\left(\theta+\frac{\pi}{4}\right)\equiv\sin\theta\cos\frac{\pi}{4}+\cos\theta\sin\frac{\pi}{4}$$
$$\equiv\frac{1}{\sqrt{2}}\sin\theta+\frac{1}{\sqrt{2}}\cos\theta\equiv\frac{1}{\sqrt{2}}(\sin\theta+\cos\theta)$$

2 b $\frac{1}{\sqrt{2}}(\sin\theta+\cos\theta)=\frac{1}{\sqrt{2}}$

$$\Rightarrow \sin\left(\theta+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

Find all answers for $\theta+\frac{\pi}{4}$. As $0\leq\theta\leq 2\pi$ so $\frac{\pi}{4}\leq\theta+\frac{\pi}{4}\leq 2\pi+\frac{\pi}{4}$

$\theta+\frac{\pi}{4}=\frac{\pi}{4},\frac{3\pi}{4},\frac{9\pi}{4}$, so $\theta=0,\frac{\pi}{2},2\pi$

c As $\sin\left(\theta+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}(\sin\theta+\cos\theta)$

When $\sin\theta+\cos\theta = 1$

$\sin\left(\theta+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$

So $\theta=0,\frac{\pi}{2},2\pi$

3 a $\cos\theta\cos 30°-\sin\theta\sin 30°=0.5$

$\Rightarrow \cos(\theta+30°)=0.5$

$\Rightarrow \theta+30°=60°,\ 300°$

$\Rightarrow \theta=30°,\ 270°$

b $\cos\theta\cos 30°-\sin\theta\sin 30°\equiv\cos\theta\times\frac{\sqrt{3}}{2}-\sin\theta\times\frac{1}{2}$

So $\cos\theta\cos 30°-\sin\theta\sin 30°=\frac{1}{2}$ is identical to $\sqrt{3}\cos\theta-\sin\theta=1$

Solutions are same as (a), i.e. 30°, 270°

4 a $3\sin(x-y)-\sin(x+y)=0$

$\Rightarrow 3\sin x\cos y-3\cos x\sin y-\sin x\cos y-\cos x\sin y=0$

$\Rightarrow 2\sin x\cos y=4\cos x\sin y$

$$\Rightarrow \frac{2\sin x\cos y}{\cos x\cos y}=\frac{4\cos x\sin y}{\cos x\cos y}$$

$$\Rightarrow \frac{2\sin x}{\cos x}=\frac{4\sin y}{\cos y}$$

$\Rightarrow 2\tan x=4\tan y$

b Put $y=45° \Rightarrow \tan x=2$

So $x=\tan^{-1}2,\ 180°+\tan^{-1}2$

$x=63.4°,\ 243.4°$ (1 d.p.)

5 **a** $\sin 2\theta = \sin\theta,\ 0 \le \theta \le 2\pi$

$\Rightarrow\ 2\sin\theta\cos\theta = \sin\theta$

$\Rightarrow\ 2\sin\theta\cos\theta - \sin\theta = 0$

$\Rightarrow\ \sin\theta(2\cos\theta - 1) = 0$

$\Rightarrow\ \sin\theta = 0 \text{ or } \cos\theta = \frac{1}{2}$

Solution set: $0,\ \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3},\ 2\pi$

b $\cos 2\theta = 1 - \cos\theta,\ \ -180° < \theta \le 180°$

$\Rightarrow\ 2\cos^2\theta - 1 = 1 - \cos\theta$

$\Rightarrow\ 2\cos^2\theta + \cos\theta - 2 = 0$

$\Rightarrow\ \cos\theta = \frac{-1 \pm \sqrt{17}}{4}$ (using the quadratic formula)

As $\frac{-1-\sqrt{17}}{4} \le -1$, this gives the only solution as $\cos\theta = \frac{-1+\sqrt{17}}{4} = 0.78077...$

As $\cos\theta$ is positive, θ is in the first and fourth quadrants

Using a calculator $\cos^{-1} 0.78077 = 38.7°$ (1 d.p.)

Solutions are $\pm 38.7°$

c $3\cos 2\theta = 2\cos^2\theta,\ \ \ 0 \le \theta < 360°$

$\Rightarrow\ 3(2\cos^2\theta - 1) = 2\cos^2\theta$

$\Rightarrow\ 6\cos^2\theta - 3 = 2\cos^2\theta$

$\Rightarrow\ 4\cos^2\theta = 3$

$\Rightarrow\ \cos^2\theta = \frac{3}{4}\ \Rightarrow\ \cos\theta = \pm\frac{\sqrt{3}}{2}$

θ will be in all four quadrants.

Solution set: 30°, 150°, 210°, 330°

d $\sin 4\theta = \cos 2\theta,\ \ \ \ 0 \le \theta \le \pi$

$\Rightarrow\ 2\sin 2\theta\cos 2\theta = \cos 2\theta$

$\Rightarrow\ \cos 2\theta(2\sin 2\theta - 1) = 0$

$\Rightarrow\ \cos 2\theta = 0 \text{ or } \sin 2\theta = \frac{1}{2}$

$\cos 2\theta = 0$ in $0 \le 2\theta \le 2\pi$

$\Rightarrow\ 2\theta = \frac{\pi}{2},\ \frac{3\pi}{2}\ \Rightarrow\ \theta = \frac{\pi}{4},\ \frac{3\pi}{4}$

$\sin 2\theta = \frac{1}{2}$ in $0 \le 2\theta \le 2\pi$

$\Rightarrow\ 2\theta = \frac{\pi}{6},\ \frac{5\pi}{6}\ \Rightarrow\ \theta = \frac{\pi}{12},\ \frac{5\pi}{12}$

Solution set: $\frac{\pi}{12},\ \frac{\pi}{4},\ \frac{5\pi}{12},\ \frac{3\pi}{4}$

5 **e** $3\cos\theta - \sin\frac{\theta}{2} - 1 = 0, \quad 0 \le \theta \le 720°$

$\Rightarrow\ 3\left(1 - 2\sin^2\frac{\theta}{2}\right) - \sin\frac{\theta}{2} - 1 = 0$

$\Rightarrow\ 6\sin^2\frac{\theta}{2} + \sin\frac{\theta}{2} - 2 = 0$

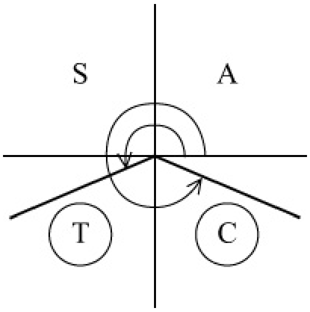
$\Rightarrow\ \left(3\sin\frac{\theta}{2} + 2\right)\left(2\sin\frac{\theta}{2} - 1\right) = 0$

$\Rightarrow\ \sin\frac{\theta}{2} = -\frac{2}{3}$ or $\sin\frac{\theta}{2} = \frac{1}{2}$

$\sin\frac{\theta}{2} = \frac{1}{2}$ in $0 \le \frac{\theta}{2} \le 360°$

$\Rightarrow\ \frac{\theta}{2} = 30°,\ 150° \ \Rightarrow\ \theta = 60°,\ 300°$

$\sin\frac{\theta}{2} = -\frac{2}{3}$ in $0 \le \frac{\theta}{2} \le 360°$



$\Rightarrow\ \frac{\theta}{2} = 180° - \sin^{-1}\left(-\frac{2}{3}\right),\ 360° + \sin^{-1}\left(-\frac{2}{3}\right) = \ 221.8°,\ 318.2°$ (1 d.p.)

$\Rightarrow\ \theta = \ 443.6°,\ 636.4°$

Solution set: $60°,\ 300°,\ 443.6°,\ 636.4°$

f $\cos^2\theta - \sin 2\theta = \sin^2\theta, \qquad 0 \le \theta \le \pi$

$\Rightarrow\ \cos^2\theta - \sin^2\theta = \sin 2\theta$

$\Rightarrow\ \cos 2\theta = \sin 2\theta$

$\Rightarrow\ \tan 2\theta = 1$ (divide both sides by $\cos 2\theta$)

$\tan 2\theta = 1$ in $0 \le 2\theta \le 2\pi$

$\Rightarrow\ 2\theta = \frac{\pi}{4},\ \frac{5\pi}{4}$

$\Rightarrow\ \theta = \frac{\pi}{8},\ \frac{5\pi}{8}$

5 **g** $2\sin\theta = \sec\theta, \quad 0 \le \theta \le 2\pi$

$$\Rightarrow \ 2\sin\theta = \frac{1}{\cos\theta}$$

$$\Rightarrow \ 2\sin\theta\cos\theta = 1$$

$$\Rightarrow \ \sin 2\theta = 1$$

$\sin 2\theta = 1$ in $0 \le 2\theta \le 4\pi$

$$\Rightarrow \ 2\theta = \frac{\pi}{2}, \ \frac{5\pi}{2}$$

$$\Rightarrow \ \theta = \frac{\pi}{4}, \ \frac{5\pi}{4}$$

h $2\sin 2\theta = 3\tan\theta, \quad 0 \le \theta < 360°$

$$\Rightarrow \ 4\sin\theta\cos\theta = \frac{3\sin\theta}{\cos\theta}$$

$$\Rightarrow \ 4\sin\theta\cos^2\theta = 3\sin\theta$$

$$\Rightarrow \ \sin\theta(4\cos^2\theta - 3) = 0$$

$$\Rightarrow \ \sin\theta = 0 \text{ or } \cos^2\theta = \frac{3}{4}$$

$\sin\theta = 0 \ \Rightarrow \ \theta = 0°, \ 180°$

$\cos^2\theta = \frac{3}{4} \ \Rightarrow \cos\theta = \pm\frac{\sqrt{3}}{2} \ \Rightarrow \ \theta = \ 30°, \ 150°, \ 210°, \ 330°$

Solution set: $0°, \ 30°, \ 150°, \ 180°, \ 210°, \ 330°$

5 i $2\tan\theta = \sqrt{3}(1-\tan\theta)(1+\tan\theta), \quad 0 \le \theta \le 2\pi$

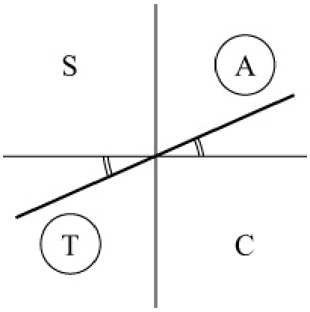
$\Rightarrow 2\tan\theta = \sqrt{3}(1-\tan^2\theta)$

$\Rightarrow \sqrt{3}\tan^2\theta + 2\tan\theta - \sqrt{3} = 0$

$\Rightarrow (\sqrt{3}\tan\theta - 1)(\tan\theta + \sqrt{3}) = 0$

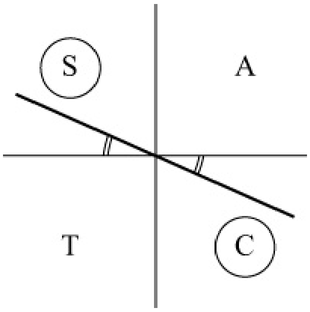
$\Rightarrow \tan\theta = \frac{1}{\sqrt{3}}$ or $\tan\theta = -\sqrt{3}$

$\tan\theta = \frac{1}{\sqrt{3}}, \ 0 \le \theta \le 2\pi$



$\Rightarrow \theta = \tan^{-1}\frac{1}{\sqrt{3}}, \ \pi + \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}, \ \frac{7\pi}{6}$

$\tan\theta = -\sqrt{3}, \quad 0 \le \theta \le 2\pi$



$\Rightarrow \theta = \pi + \tan^{-1}\left(-\sqrt{3}\right), \ 2\pi + \tan^{-1}\left(-\sqrt{3}\right) = \frac{2\pi}{3}, \ \frac{5\pi}{3}$

Solution set: $\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

5 **j** $\sin^2\theta = 2\sin 2\theta, \quad -180° < \theta \le 180°$

$\Rightarrow \sin^2\theta = 4\sin\theta\cos\theta$

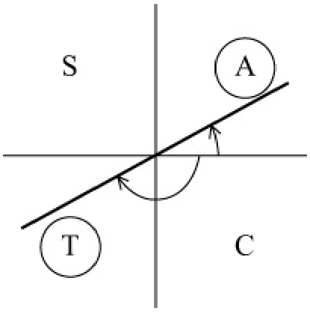
$\Rightarrow \sin\theta(\sin\theta - 4\cos\theta) = 0$

$\Rightarrow \sin\theta = 0 \text{ or } \sin\theta = 4\cos\theta$

$\Rightarrow \sin\theta = 0 \text{ or } \tan\theta = 4$

$\sin\theta = 0 \Rightarrow \theta = 0°, 180°$

$\tan\theta = 4 \Rightarrow \theta = \tan^{-1}4, -180° + \tan^{-1}4 = 76.0°, -104.0°$ (1 d.p.)



Solution set: $-104.0°, 0°, 76.0°$

k $4\tan\theta = \tan 2\theta, \quad 0 \le \theta < 360°$

$\Rightarrow 4\tan\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$\Rightarrow 2\tan\theta(1-\tan^2\theta) = \tan\theta$

$\Rightarrow \tan\theta(2 - 2\tan^2\theta - 1) = 0$

$\Rightarrow \tan\theta(1 - 2\tan^2\theta) = 0$

$\Rightarrow \tan\theta = 0 \ \textit{or} \ \tan\theta = \pm\sqrt{\frac{1}{2}}$

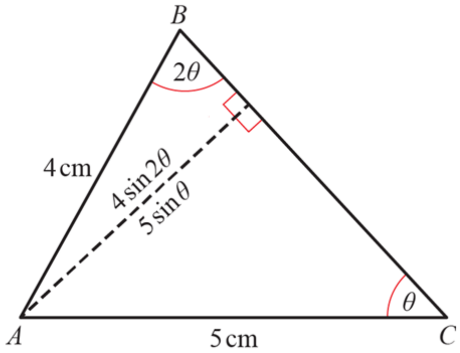
$\tan\theta = 0 \Rightarrow \theta = 0°, 180°$

$\tan\theta = \pm\sqrt{\frac{1}{2}}$

$\Rightarrow \theta = 35.3°, 144.7°, 215.3°, 324.7°$

Solution set: $0°, 35.3°, 144.7°, 180°, 215.3°, 324.7°$

6 Sketch $\Delta\, ABC$



$4\sin 2\theta = 5\sin\theta$

$\Rightarrow 8\sin\theta\cos\theta = 5\sin\theta$

$\Rightarrow 8\sin\theta\cos\theta - 5\sin\theta = 0$

$\Rightarrow \sin\theta(8\cos\theta - 5) = 0$

$\Rightarrow \sin\theta = 0$ or $\cos\theta = \frac{5}{8}$

As ABC is a triangle, $0 < \theta < 90°$, so $\theta = 0°$ or $180°$ are not possible solutions.

So $\theta = \cos^{-1}\left(\frac{5}{8}\right) = 51.3°$ (1 d.p.)

7 **a** As $5\sin 2\theta = 10\sin\theta\cos\theta$

$5\sin 2\theta + 4\sin\theta = 10\sin\theta\cos\theta + 4\sin\theta = 0$

$2\sin\theta(5\cos\theta + 2) = 0$

So $a = 2$, $b = 5$ and $c = 2$

b $2\sin\theta(5\cos\theta + 2) = 0, \quad 0 \le \theta < 360°$

$\Rightarrow \sin\theta = 0$ or $\cos\theta = -\frac{2}{5}$

$\sin\theta = 0 \;\Rightarrow \theta = 0°,\ 180°$

$\cos\theta = -\frac{2}{5} \;\Rightarrow \theta = \cos^{-1}\left(-\frac{2}{5}\right),\ 360° - \cos^{-1}\left(-\frac{2}{5}\right) = 113.6°,\ 246.4°$ (1 d.p.)

Solution set: $\theta = 0°,\ 113.6°,\ 180°,\ 246.4°$

8 **a** As $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = 1 - 2\sin^2\theta$

$\sin 2\theta + \cos 2\theta = 1$

$\Rightarrow 2\sin\theta\cos\theta + (1 - 2\sin^2\theta) = 1$

$\Rightarrow 2\sin\theta\cos\theta - 2\sin^2\theta = 0$

$\Rightarrow 2\sin\theta(\cos\theta - \sin\theta) = 0$

8 b $2\sin\theta(\cos\theta - \sin\theta) = 0, \quad 0 \le \theta < 360°$

$\Rightarrow \sin\theta = 0 \text{ or } \cos\theta = \sin\theta$

$\sin\theta = 0 \Rightarrow \theta = 0°, 180°$

$\cos\theta = \sin\theta \Rightarrow \tan\theta = 1 \Rightarrow \theta = 45°, 225°$

Solution set: $\theta = 0°, 45°, 180°, 225°$

9 a LHS $\equiv (\cos 2\theta - \sin 2\theta)^2$

$\equiv \cos^2 2\theta - 2\sin 2\theta\cos 2\theta + \sin^2 2\theta$

$\equiv (\cos^2 2\theta + \sin^2 2\theta) - (2\sin 2\theta\cos 2\theta)$

$\equiv 1 - \sin 4\theta \qquad (\sin^2 A + \cos^2 A \equiv 1, \ \sin 2A \equiv 2\sin A\cos A)$

$\equiv$ RHS

b You can use $(\cos 2\theta - \sin 2\theta)^2 = \frac{1}{2}$ but this also solves the equation

$$\cos 2\theta - \sin 2\theta = -\frac{1}{\sqrt{2}}$$

so you need to check your final answers.

As $(\cos 2\theta - \sin 2\theta)^2 \equiv 1 - \sin 4\theta$

$$\Rightarrow \frac{1}{2} = 1 - \sin 4\theta$$

$$\Rightarrow \sin 4\theta = \frac{1}{2}$$

$0 \le \theta < \pi$, so $0 \le 4\theta < 4\pi$

$$\Rightarrow 4\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

$$\Rightarrow \theta = \frac{\pi}{24}, \frac{5\pi}{24}, \frac{13\pi}{24}, \frac{17\pi}{24}$$

Checking these values in $\cos 2\theta - \sin 2\theta = \frac{1}{\sqrt{2}}$ eliminates $\frac{5\pi}{24}, \frac{13\pi}{24}$

which apply to $\cos 2\theta - \sin 2\theta = -\frac{1}{\sqrt{2}}$

Solutions are $\frac{\pi}{24}, \frac{17\pi}{24}$

10 a i RHS $\equiv \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

$$\equiv \frac{2\tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}}$$

$$\equiv \frac{2\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\times\cos^2\tfrac{\theta}{2}$$

$$\equiv 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}$$

$$\equiv \sin\theta \qquad (\sin 2A = 2\sin A\cos A)$$

$\equiv$ LHS

ii RHS $\equiv \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

$$\equiv \frac{1-\tan^2\frac{\theta}{2}}{\sec^2\frac{\theta}{2}}$$

$$\equiv \cos^2\tfrac{\theta}{2}\left(1-\tan^2\tfrac{\theta}{2}\right)$$

$$\equiv \cos^2\tfrac{\theta}{2}-\sin^2\tfrac{\theta}{2} \qquad \left(\tan^2\tfrac{\theta}{2}=\frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}\right)$$

$$\equiv \cos\theta \qquad (\cos 2A = \cos^2 A - \sin^2 A)$$

$\equiv$ LHS

b Let $\tan\dfrac{\theta}{2}=t$

i $\sin\theta+2\cos\theta=1$

$$\Rightarrow \frac{2t}{1+t^2}+\frac{2(1-t^2)}{1+t^2}=1$$

$$\Rightarrow 2t+2-2t^2=1+t^2$$

$$\Rightarrow 3t^2-2t-1=0$$

$$\Rightarrow (3t+1)(t-1)=0$$

$$\Rightarrow \tan\frac{\theta}{2}=-\frac{1}{3} \text{ or } \tan\frac{\theta}{2}=1, \quad 0\le\frac{\theta}{2}\le 180°$$

$$\tan\frac{\theta}{2}=1 \Rightarrow \frac{\theta}{2}=45° \Rightarrow \theta=90°$$

$$\tan\frac{\theta}{2}=-\frac{1}{3} \Rightarrow \frac{\theta}{2}=161.56° \Rightarrow \theta=323.1° \text{ (1 d.p.)}$$

Solution set: 90°, 323.1°

10 b ii $3\cos\theta - 4\sin\theta = 2$

$$\Rightarrow \frac{3(1-t^2)}{1+t^2} - \frac{4\times 2t}{1+t^2} = 2$$

$$\Rightarrow 3(1-t^2) - 8t = 2(1+t^2)$$

$$\Rightarrow 5t^2 + 8t - 1 = 0$$

$$\Rightarrow t = \frac{-8\pm\sqrt{84}}{10}$$

For $\tan\frac{\theta}{2} = \frac{-8+\sqrt{84}}{10} \qquad 0 \le \frac{\theta}{2} \le 180°$

$\frac{\theta}{2} = 6.65° \Rightarrow \theta = 13.3°$ (1 d.p.)

For $\tan\frac{\theta}{2} = \frac{-8-\sqrt{84}}{10} \qquad 0 \le \frac{\theta}{2} \le 180°$

$\frac{\theta}{2} = 120.2° \Rightarrow \theta = 240.4°$ (1 d.p.)

Solution set: 13.3°, 240.4°

11 a RHS $\equiv 1 + 2\cos 2x$

$$\equiv 1 + 2(\cos^2 x - \sin^2 x)$$

$$\equiv 1 + 2\cos^2 x - 2\sin^2 x$$

$$\equiv \cos^2 x + \sin^2 x + 2\cos^2 x - 2\sin^2 x \qquad (\text{using } \sin^2 x + \cos^2 x \equiv 1)$$

$$\equiv 3\cos^2 x - \sin^2 x$$

$\equiv$ LHS

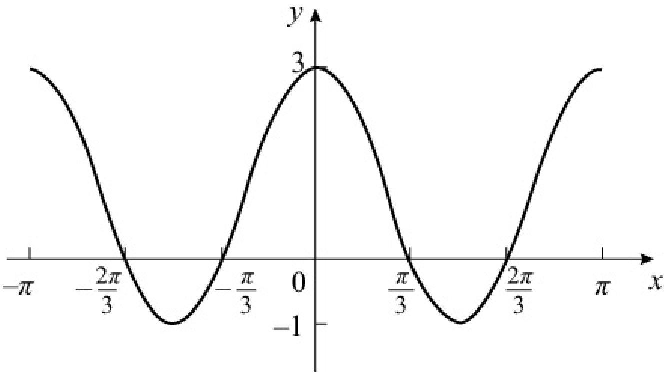
11 b $y = 3\cos^2 x - \sin^2 x$ is the same as $y = 1 + 2\cos 2x$

Using your work on transformations, this curve is the result of

(i) stretching $y = \cos x$ by scale factor $\frac{1}{2}$ in the x direction, then

(ii) stretching the result by scale factor 2 in the y direction, then

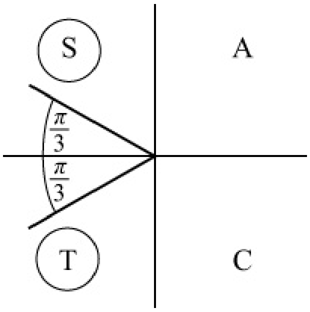
(iii) translating by 1 in the positive y direction.



The curve crosses y-axis at (0, 3). It crosses x-axis where $y = 0$

i.e. where $1 + 2\cos 2x = 0 \qquad -\pi \le x \le \pi$

$$\Rightarrow \cos 2x = -\frac{1}{2} \qquad -2\pi \le 2x \le 2\pi$$



So $2x = -\frac{4\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$

$$\Rightarrow x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$$

The curve meets the x-axis at $\left(-\frac{2\pi}{3}, 0\right), \left(-\frac{\pi}{3}, 0\right), \left(\frac{\pi}{3}, 0\right), \left(\frac{2\pi}{3}, 0\right)$

12 a $\cos^2 \frac{\theta}{2} = \frac{1 + \cos\theta}{2}, \ \sin^2 \frac{\theta}{2} = \frac{1 - \cos\theta}{2}$

So $2\cos^2 \frac{\theta}{2} - 4\sin^2 \frac{\theta}{2} = (1 + \cos\theta) - 2(1 - \cos\theta) = 3\cos\theta - 1$

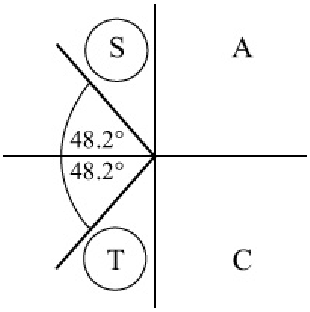
12 b $3\cos\theta - 1 = -3, \quad 0 \le \theta < 360°$

$\Rightarrow \; 3\cos\theta = -2$

$\Rightarrow \; \cos\theta = -\frac{2}{3}$

As $\cos\theta$ is negative, θ is in second and third quadrants.

Calculator value is $\cos^{-1}\left(-\frac{2}{3}\right) = 131.8°$ (1 d.p.)



Solutions are $131.8°,\; 360° - 131.8° = 131.8°,\; 228.2°$ (1 d.p.)

13 a As $\sin^2 A + \cos^2 A \equiv 1$ so $(\sin^2 A + \cos^2 A)^2 \equiv 1$

$\Rightarrow \; \sin^4 A + \cos^4 A + 2\sin^2 A\cos^2 A \equiv 1$

$\Rightarrow \; \sin^4 A + \cos^4 A \equiv 1 - 2\sin^2 A\cos^2 A$

$$\equiv 1 - \frac{1}{2}(4\sin^2 A\cos^2 A)$$

$$\equiv 1 - \frac{1}{2}\left((2\sin A\cos A)^2\right)$$

$$\equiv 1 - \frac{1}{2}\sin^2 2A$$

$$\equiv \frac{1}{2}(2 - \sin^2 2A)$$

b As $\cos 2A \equiv 1 - 2\sin^2 A$ so $\cos 4A \equiv 1 - 2\sin^2 2A$ so $\sin^2 2A \equiv \frac{1-\cos 4A}{2}$

$\Rightarrow$ from (a) $\sin^4 A + \cos^4 A \equiv \frac{1}{2}\left(2 - \frac{1-\cos 4A}{2}\right) \equiv \frac{1}{2}\left(\frac{4-1+\cos 4A}{2}\right) \equiv \frac{1}{4}\left(3+\cos 4A\right)$

13 c Using part (b)

$8\sin^4\theta + 8\cos^4\theta = 7$

$\Rightarrow\ 8\times\frac{1}{4}(3+\cos 4\theta) = 7$

$\Rightarrow\ 3+\cos 4\theta = \frac{7}{2}$

$\Rightarrow\ \cos 4\theta = \frac{1}{2}$

Solve $\cos 4\theta = \frac{1}{2}$ in $0 < 4\theta < 4\pi$

$\Rightarrow\ 4\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$

$\Rightarrow\ \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$

14 a $\cos 3\theta \equiv \cos(2\theta+\theta) \equiv \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$\equiv (\cos^2\theta - \sin^2\theta)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$

$\equiv \cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta$

$\equiv \cos^3\theta - 3\sin^2\theta\cos\theta$

$\equiv \cos^3\theta - 3(1-\cos^2\theta)\cos\theta$

$\equiv 4\cos^3\theta - 3\cos\theta$

b $6\cos\theta - 8\cos^3\theta + 1 = 0$, $0 < \theta < \pi$

$\Rightarrow 1 = 8\cos^3\theta - 6\cos\theta$

$\Rightarrow 4\cos^3\theta - 3\cos\theta = \frac{1}{2}$

$\Rightarrow \cos 3\theta = \frac{1}{2},\quad 0 < 3\theta < 3\pi$ using the result from part (a)

So $3\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$

$\Rightarrow \theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$

Trigonometry and modelling 7E

1 $5\sin\theta + 12\cos\theta \equiv R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

Comparing $\sin\theta$: $R\cos\alpha = 5$

Comparing $\cos\theta$: $R\sin\alpha = 12$

Divide the equations:

$$\frac{\sin\alpha}{\cos\alpha} = \frac{12}{5} \Rightarrow \tan\alpha = 2\frac{2}{5}$$

Square and add the equations:

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 5^2 + 12^2$$

$$R^2(\cos^2\alpha + \sin^2\alpha) = 13^2$$

$$R = 13$$

since $\cos^2\alpha + \sin^2\alpha \equiv 1$

2 $\sqrt{3}\sin\theta + \sqrt{6}\cos\theta$

$$\equiv 3\cos\theta\cos\alpha + 3\sin\theta\sin\alpha$$

Comparing $\sin\theta : \sqrt{3} = 3\sin\alpha$ (1)

Comparing $\cos\theta : \sqrt{6} = 3\cos\alpha$ (2)

Divide (1) by (2):

$$\tan\alpha = \frac{\sqrt{3}}{\sqrt{6}} = \frac{1}{\sqrt{2}}$$

So $\alpha = 35.3°$ (1 d.p.)

3 $2\sin\theta - \sqrt{5}\cos\theta$

$$\equiv -3\cos\theta\cos\alpha + 3\sin\theta\sin\alpha$$

Comparing $\sin\theta : 2 = 3\sin\alpha$ (1)

Comparing $\cos\theta : +\sqrt{5} = +3\cos\alpha$ (2)

Divide (1) by (2):

$$\tan\alpha = \frac{2}{\sqrt{5}}$$

So $\alpha = 41.8°$ (1 d.p.)

4 a Let $\cos\theta - \sqrt{3}\sin\theta \equiv R\cos(\theta + \alpha)$

$$\equiv R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$$

Compare $\cos\theta : R\cos\alpha = 1$ (1)

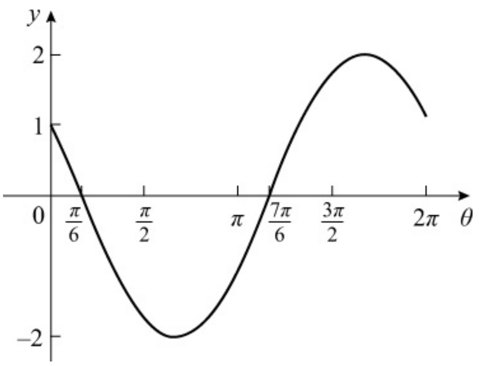
Compare $\sin\theta : R\sin\alpha = \sqrt{3}$ (2)

Divide (2) by (1) : $\tan\alpha = \sqrt{3} \Rightarrow \alpha = \dfrac{\pi}{3}$

Square and add : $R^2 = 1 + 3 = 4 \Rightarrow R = 2$

So $\cos\theta - \sqrt{3}\sin\theta \equiv 2\cos\left(\theta + \dfrac{\pi}{3}\right)$

b This is the graph of $y = \cos\theta$, translated by $\frac{\pi}{3}$ to the left and then stretched in the y direction by scale factor 2.



Meets y-axis at (0, 1)

Meets x-axis at $\left(\dfrac{\pi}{6}, 0\right)$, $\left(\dfrac{7\pi}{6}, 0\right)$

5 a Let $7\cos\theta - 24\sin\theta \equiv R\cos(\theta + \alpha)$

$$\equiv R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$$

Compare $\cos\theta$: $R\cos\alpha = 7$ (1)

Compare $\sin\theta$: $R\sin\alpha = 24$ (2)

Divide (2) by (1) : $\tan\alpha = \frac{24}{7}$

$\Rightarrow \alpha = 73.7°$ (1 d.p.)

Square and add: $R^2 = 24^2 + 7^2$

$\Rightarrow R = 25$

So $7\cos\theta - 24\sin\theta \equiv 25\cos(\theta + 73.7°)$

b Graph meets y-axis where $\theta = 0$, i.e. $y = 7\cos 0° - 24\sin 0° = 7$

so coordinates are (0, 7)

c Maximum value of $25\cos(\theta + 73.7°)$ is when $\cos(\theta + 73.7°) = 1$

So maximum is 25

Minimum value is $25(-1) = -25$

5 d i The line $y = 15$ will meet the graph twice in $0 < \theta < 360°$, so there are 2 solutions.

ii As the maximum value is 25 it can never be 26, so there are 0 solutions.

iii As -25 is a minimum, line $y = -25$ only meets curve once, so only 1 solution.

6 a Let $\sin\theta + 3\cos\theta \equiv R\sin(\theta+\alpha)$

$$\equiv R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$$

Comparing $\sin\theta$: $R\cos\alpha = 1$ (1)

Comparing $\cos\theta$: $R\sin\alpha = 3$ (2)

Divide (2) by (1)

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = 3$$

So $\alpha = 71.56°$ (2 d.p.)

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 1^2 + 3^2$$
$$R^2(\cos^2\alpha + \sin^2\alpha) = 10$$
$$R^2 = 10$$
$$R = \sqrt{10},\ \alpha = 71.6° \text{ (1 d.p.)}$$

b Use the value of α to 2 d.p. in calculating values of θ to avoid rounding errors

$$\sqrt{10}\sin(\theta + 71.56°) = 2$$
$$\sin(\theta + 71.56°) = \frac{2}{\sqrt{10}}$$
$$\sin^{-1}\left(\frac{2}{\sqrt{10}}\right) = 39.23° \text{ (2 d.p.)}$$

As $0 \le \theta < 360°$, the interval for $(\theta + 71.56°)$ is

$$71.56° \le \theta + 71.56° < 431.56°$$

So $\theta + 71.56° = 180° - 39.23°$,

and $\theta + 71.56° = 360° + 39.23°$

$$\theta + 71.56° = 140.77°, 399.23°$$
$$\theta = 69.2°, 327.7° \text{ (1 d.p.)}$$

7 a Set $\cos 2\theta - 2\sin 2\theta \equiv R\cos(2\theta + \alpha)$

$$\cos 2\theta - 2\sin 2\theta \equiv R\cos 2\theta\cos\alpha - R\sin 2\theta\sin\alpha$$

Comparing $\sin 2\theta$: $R\sin\alpha = 2$ (1)

Comparing $\cos 2\theta$: $R\cos\alpha = 1$ (2)

Divide (1) by (2)

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = 2$$

So $\alpha = 1.107$ (3 d.p.)

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 1^2 + 2^2$$
$$R^2(\cos^2\alpha + \sin^2\alpha) = 5$$
$$R = \sqrt{5}$$

So $\cos 2\theta - 2\sin 2\theta = \sqrt{5}\cos(2\theta + 1.107)$

b $\sqrt{5}\cos(2\theta + 1.107) = -1.5$

$$\cos(2\theta + 1.107) = -\frac{1.5}{\sqrt{5}}$$
$$\cos^{-1}\left(-\frac{1.5}{\sqrt{5}}\right) = 2.306 \text{ (3 d.p.)}$$

As $0 \le \theta < \pi$, the interval for $(2\theta + 1.107)$ is

$$1.107 \le 2\theta + 1.107 < 2\pi + 1.107$$

So $2\theta + 1.107 = 2.306, 2\pi - 2.306$

$$2\theta + 1.107 = 2.306, 3.977$$
$$\theta = 0.60, 1.44 \text{ (2 d.p.)}$$

8 a Write $6\sin x + 8\cos x$ in the form $R\sin(x+\alpha)$, where $R>0,\ 0<\alpha<90°$

So $6\sin x + 8\cos x$

$$\equiv R\sin x\cos\alpha + R\cos x\sin\alpha$$

Compare $\sin x$: $R\cos\alpha = 6$ (1)

Compare $\cos x$: $R\sin\alpha = 8$ (2)

Divide (2) by (1): $\tan\alpha = \frac{4}{3}$

$\Rightarrow\ \alpha = 53.13°$ (2 d.p.)

$R^2 = 6^2 + 8^2 \ \Rightarrow\ R = 10$

So $6\sin x + 8\cos x \equiv 10\sin(x + 53.13°)$

Solve $10\sin(x+53.13°) = 5\sqrt{3}$, in the interval $0 \le x \le 360°$

so $\sin(x+53.13°) = \frac{\sqrt{3}}{2}$

$\Rightarrow\ x + 53.13° = 60°,\ 120°$

$\Rightarrow\ x = 6.9°,\ 66.9°$ (1 d.p.)

b Let $2\cos 3\theta - 3\sin 3\theta \equiv R\cos(3\theta + \alpha)$

$$\equiv R\cos 3\theta\cos\alpha - R\sin 3\theta\sin\alpha$$

Compare $\cos 3\theta$: $R\cos\alpha = 2$ (1)

Compare $\sin 3\theta$: $R\sin\alpha = 3$ (2)

Divide (2) by (1) : $\tan\alpha = \frac{3}{2}$

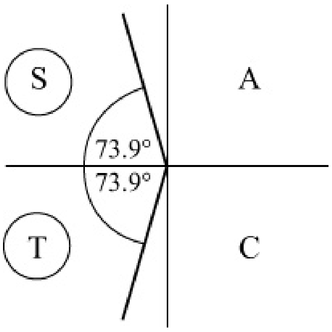
$\Rightarrow\ \alpha = 56.31°$ (2 d.p.)

$R^2 = 2^2 + 3^2 \ \Rightarrow\ R = \sqrt{13}$

Solve $\sqrt{13}\cos(3\theta + 56.31°) = -1$, in the interval $0 \le \theta \le 90°$

so $\cos(3\theta + 56.31°) = -\frac{1}{\sqrt{13}}$

for $56.31° \le 3\theta + 56.31° \le 326.31°$



$\Rightarrow\ 3\theta + 56.31° = 106.10°,\ 253.90°$

$\Rightarrow\ 3\theta = 49.8°,\ 197.6°$

$\Rightarrow\ \theta = 16.6°,\ 65.9°$ (1 d.p.)

c Let $8\cos\theta + 15\sin\theta \equiv R\cos(\theta - \alpha)$

$$\equiv R\sin x\cos\alpha + R\cos x\sin\alpha$$

Compare $\cos\theta$: $R\cos\alpha = 8$ (1)

Compare $\sin\theta$: $R\sin\alpha = 15$ (2)

Divide (2) by (1) : $\tan\alpha = \frac{15}{8}$

$\Rightarrow\ \alpha = 61.93°$ (2 d.p.)

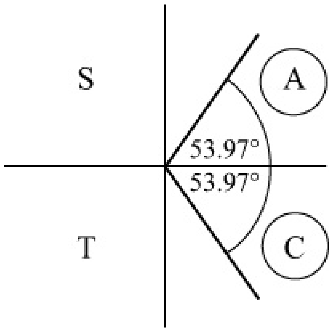
$R^2 = 8^2 + 15^2 \ \Rightarrow\ R = 17$

Solve $17\cos(\theta - 61.93°) = 10$, in the interval $0 \le \theta \le 360°$

So $\cos(\theta - 61.93°) = \frac{10}{17}$,

$-61.93° \le \theta - 61.93° \le 298.07°$

$\cos^{-1}\left(\frac{10}{17}\right) = 53.97°$ (2 d.p.)



So $\theta - 61.93° = -53.97°,\ +53.97°$

$\Rightarrow\ \theta = 8.0°,\ 115.9°$ (1 d.p.)

8 d Let $5\sin\frac{x}{2}-12\cos\frac{x}{2}$

$$\equiv R\sin\frac{x}{2}-\alpha$$

$$\equiv R\sin\frac{x}{2}\cos\alpha-R\cos\frac{x}{2}\sin\alpha$$

Compare $\sin\frac{x}{2}$: $R\cos\alpha=5$ (1)

Compare $\cos\frac{x}{2}$: $R\sin\alpha=12$ (2)

Divide (2) by (1) : $\tan\alpha=\frac{12}{5}$

$\Rightarrow\ \alpha=67.38°$ (2 d.p.)

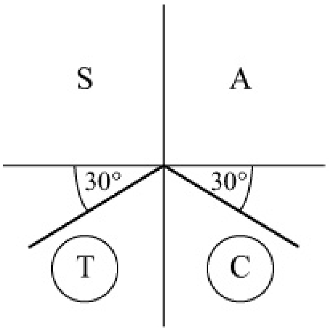
$R=13$

Solve $13\sin\left(\frac{x}{2}-67.38°\right)=-6.5$,

in the interval $-360°\leq x\leq 360°$

So $\sin\left(\frac{x}{2}-67.38°\right)=-\frac{1}{2}$,

$-247.4°\leq\frac{x}{2}-67.4°\leq 112.6°$



From quadrant diagram:

$\frac{x}{2}-67.38°=-150°,\ -30°$

$\Rightarrow\ \frac{x}{2}=-82.62°,\ 37.38°$

$\Rightarrow\ x=-165.2°,\ 74.8°$ (1 d.p.)

9 a Set $3\sin 3\theta-4\cos 3\theta\equiv R\sin(3\theta-\alpha)$

$3\sin 3\theta-4\cos 3\theta$

$$\equiv R\sin 3\theta\cos\alpha-R\cos 3\theta\sin\alpha$$

Compare $\sin 3\theta$: $R\cos\alpha=3$ (1)

Compare $\cos 3\theta$: $R\sin\alpha=4$ (2)

Divide (2) by (1) : $\tan\alpha=\frac{4}{3}$

$\Rightarrow\ \alpha=53.13°$ (2 d.p.)

$R^2=3^2+4^2=25\ \Rightarrow R=5$

So $3\sin 3\theta-4\cos 3\theta\equiv 5\sin(3\theta-53.13°)$

b The minimum value of $3\sin 3\theta-4\cos 3\theta$ is −5. This occurs when

$\sin(3\theta-53.13°)=-1$

$3\theta-53.13°=270°$

$\theta=107.7°$ (1 d.p.)

c $5\sin(3\theta-53.13°)=1$,

in the interval $0\leq\theta<180°$

So $\sin(3\theta-53.13°)=\frac{1}{5}$,

in the interval

$-53.13°\leq 3\theta-53.13°<506.87°$

$3\theta-53.13°=11.54°, 168.46°, 371.54°$

$\theta=21.6°, 73.9°, 141.6°$ (1 d.p.)

10 a As $\sin^2\theta=\frac{1-\cos 2\theta}{2}$ and

$\cos^2\theta=\frac{1+\cos 2\theta}{2}$

So $5\sin^2\theta-3\cos^2\theta+6\sin\theta\cos\theta$

$$\equiv 5\frac{1-\cos 2\theta}{2}-3\frac{1+\cos 2\theta}{2}$$

$$+3(2\sin\theta\cos\theta)$$

$$\equiv\frac{5}{2}-\frac{5}{2}\cos 2\theta-\frac{3}{2}-\frac{3}{2}\cos 2\theta+3\sin 2\theta$$

$$\equiv 1-4\cos 2\theta+3\sin 2\theta$$

10 b Write $3\sin 2\theta - 4\cos 2\theta$ in the form $R\sin(2\theta - \alpha)$

The maximum value of $R\sin(2\theta - \alpha)$ is R

The minimum value of $R\sin(2\theta - \alpha)$ is $-R$

You know that $R^2 = 3^2 + 4^2$ so $R = 5$

So maximum value of

$1 - 4\cos 2\theta + 3\sin 2\theta$ is $1 + 5 = 6$

and minimum value of

$1 - 4\cos 2\theta + 3\sin 2\theta$ is $1 - 5 = -4$

c $1 - 4\cos 2\theta - 3\sin 2\theta = -1$

$\Rightarrow 3\sin 2\theta - 4\cos 2\theta = -2$

Write $3\sin 2\theta - 4\cos 2\theta$ in the form $R\sin(2\theta - \alpha)$

So $R\sin(2\theta - \alpha) = -2$

$\Rightarrow 5\sin(2\theta - 53.13°) = -2$

(By solving in same way as Question 9, part a)

Look for solutions in the interval $-53.13° \leq 2\theta - 53.13° < 306.87°$

$2\theta - 53.13° = -23.58, 203.58$

$\theta = 14.8°, 128.4°$ (1 d.p.)

11 a Let $3\cos\theta + \sin\theta \equiv R\cos(\theta - \alpha)$

$\equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

Compare $\cos\theta$: $R\cos\alpha = 3$ (1)

Compare $\sin\theta$: $R\sin\alpha = 1$ (2)

Divide (2) by (1): $\tan\alpha = \dfrac{1}{3}$

$\Rightarrow \alpha = 18.43°$ (2 d.p.)

$R^2 = 3^2 + 1^2 = 10 \Rightarrow R = \sqrt{10} = 3.16$

Solve $\sqrt{10}\cos(\theta - 18.43°) = 2$,

in the interval $0 \leq \theta \leq 360°$

$\Rightarrow \cos(\theta - 18.43°) = \dfrac{2}{\sqrt{10}}$

$\Rightarrow \theta - 18.43° = 50.77°, 309.23°$

$\Rightarrow \theta = 69.2°, 327.7°$ (1 d.p.)

b Squaring $3\cos\theta = 2 - \sin\theta$

gives $9\cos^2\theta = 4 + \sin^2\theta - 4\sin\theta$

$\Rightarrow 9(1 - \sin^2\theta) = 4 + \sin^2\theta - 4\sin\theta$

$\Rightarrow 10\sin^2\theta - 4\sin\theta - 5 = 0$

c $10\sin^2\theta - 4\sin\theta - 5 = 0$

$\Rightarrow \sin\theta = \dfrac{4 \pm \sqrt{216}}{20}$

For $\sin\theta = \dfrac{4 + \sqrt{216}}{20}$, $\sin\theta$ is positive,

so θ is in the first and second quadrants.

$\Rightarrow \theta = 69.2°, 180° - 69.2°$

$= 69.2°, 110.8°$ (1 d.p.)

For $\sin\theta = \dfrac{4 - \sqrt{216}}{20}$, $\sin\theta$ is negative,

so θ is in the third and fourth quadrants.

$\Rightarrow \theta = 180° - (-32.3°), 360° + (-32.3°)$

$= 212.3°, 327.7°$ (1 d.p.)

So solutions of quadratic in (b) are

$69.2°, 110.8°, 212.3°, 327.7°$ (1 d.p.)

d In squaring the equation, you are also including the solutions to

$3\cos\theta = -(2 - \sin\theta)$,

which when squared produces the same quadratic. The extra two solutions satisfy this equation.

12 a $\cot\theta + 2 = \operatorname{cosec}\theta$

$$\Rightarrow \frac{\cos\theta}{\sin\theta} + 2 = \frac{1}{\sin\theta}$$

Multiplying both sides by $\sin\theta$ gives

$$\cos\theta + 2\sin\theta = 1$$

b $\cos\theta + 2\sin\theta = 1$

Set $2\sin\theta + \cos\theta \equiv R\sin(\theta+\alpha)$

$$\equiv R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$$

So $R\cos\alpha = 1$ and $R\sin\alpha = 1$

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{1}{2}$$

$$\alpha = \tan^{-1}\left(\frac{1}{2}\right) = 26.57° \text{ (2 d.p.)}$$

$$R^2 = 2^2 + 1^2 \Rightarrow R = \sqrt{5}$$

So $\sqrt{5}\sin(\theta + 26.57)° = 1$

$\sin(\theta + 26.57)° = \frac{1}{\sqrt{5}}$, in the interval

$$26.57° \leq \theta + 26.57° < 386.57°$$

$$\theta + 26.57 = 26.57, 153.43$$

$$\theta = 0°, 126.9° \text{ (1 d.p.)}$$

As both $\cot\theta$ and $\operatorname{cosec}\theta$ are undefined at 0, $\theta = 126.9°$ is the only solution.

13 a $\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right) + \left(\sqrt{3}-1\right)\sin\theta = 2$

$$\Rightarrow \sqrt{2}\cos\theta\cos\frac{\pi}{4} + \sqrt{2}\sin\theta\sin\frac{\pi}{4} + \sqrt{3}\sin\theta - \sin\theta = 2$$

$$\Rightarrow \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right) + \sqrt{3}\sin\theta - \sin\theta = 2$$

$$\Rightarrow \cos\theta + \sin\theta + \sqrt{3}\sin\theta - \sin\theta = 2$$

$$\Rightarrow \cos\theta + \sqrt{3}\sin\theta = 2$$

b $\cos\theta + \sqrt{3}\sin\theta = 2$

Set $\sqrt{3}\sin\theta + \cos\theta \equiv R\sin(\theta+\alpha)$

$$\equiv R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$$

So $R\cos\alpha = \sqrt{3}$ and $R\sin\alpha = 1$

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{1}{\sqrt{3}}$$

$$\alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$R^2 = \sqrt{3}^2 + 1^2 = 4 \Rightarrow R = 2$$

$$2\sin\left(\theta + \frac{\pi}{6}\right) = 2$$

$\sin\left(\theta + \frac{\pi}{6}\right) = 1$, in the interval

$$\frac{\pi}{6} \leq \theta + \frac{\pi}{6} \leq \frac{11\pi}{6}$$

$$\theta + \frac{\pi}{6} = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{3}$$

14 a Set $9\cos\theta + 40\sin\theta \equiv R\cos(\theta - \alpha)$

$$\equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$$

So $R\cos\alpha = 9$ and $R\sin\alpha = 40$

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{40}{9}$$

$$\alpha = \tan^{-1}\left(\frac{40}{9}\right)$$

So $\alpha = 77.320°$ (3 d.p.)

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 40^2 + 9^2$$

$$R^2(\cos^2\alpha + \sin^2\alpha) = 1681$$

$$R = 41$$

So $9\cos\theta + 40\sin\theta = 41\cos(\theta - 77.320°)$

b i $g(\theta) = \dfrac{18}{50 + 41\cos(\theta - 77.320°)}$

The minimum value of $g(\theta)$ is when $\cos(\theta - 77.320°) = 1$

So the minimum value is $\dfrac{18}{50+41} = \dfrac{18}{91}$

14 b ii The minimum occurs when

$$\cos(\theta - 77.320°) = 1$$

$$\theta - 77.320° = 0$$

$$\theta = 77.320°$$

15 a Set $12\cos 2\theta - 5\sin 2\theta \equiv R\cos(2\theta + \alpha)$

$$\equiv R\cos 2\theta \cos\alpha - R\sin 2\theta \sin\alpha$$

So $R\cos\alpha = 12$ and $R\sin\alpha = 5$

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{5}{12}$$

$$\alpha = \tan^{-1}\left(\frac{5}{12}\right)$$

So $\alpha = 22.62°$ (2 d.p.)

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 12^2 + 5^2$$

$$R^2(\cos^2\alpha + \sin^2\alpha) = 169$$

$$R = 13$$

b $13\cos(2\theta + 22.62°) = -6.5$

$\cos(2\theta + 22.62°) = -\frac{6.5}{13}$, in the interval

$$22.62° \le 2\theta + 22.62° < 382.62°$$

$$2\theta + 22.62° = 120°, 240°$$

$$\theta = 48.7°, 108.7° \text{ (1 d.p.)}$$

c $24\cos^2\theta - 10\sin\theta\cos\theta$

$$\equiv 24\left(\frac{\cos 2\theta + 1}{2}\right) - 5\sin 2\theta$$

$$\equiv 12\cos 2\theta - 5\sin 2\theta + 12$$

$a = 12$, $b = -5$ and $c = 12$

d $24\cos^2\theta - 10\sin\theta\cos\theta$

$$\equiv 12\cos 2\theta - 5\sin 2\theta + 12$$

From part (a)

$12\cos 2\theta - 5\sin 2\theta + 12$

$$= 13\cos(2\theta + 22.62°) + 12$$

The minimum value is therefore when $\cos(2\theta + 22.620°) = -1$

It is $13(-1) + 12 = -1$

Trigonometry and Modelling 7F

1 a $$\text{LHS} \equiv \frac{\cos 2A}{\cos A + \sin A}$$

$$\equiv \frac{\cos^2 A - \sin^2 A}{\cos A + \sin A}$$

$$\equiv \frac{(\cos A + \sin A)(\cos A - \sin A)}{\cos A + \sin A}$$

$$\equiv \cos A - \sin A \equiv \text{ RHS}$$

b $$\text{LHS} \equiv \frac{\sin B}{\sin A} - \frac{\cos B}{\cos A}$$

$$\equiv \frac{\sin B \cos A - \cos B \sin A}{\sin A \cos A}$$

$$\equiv \frac{\sin(B-A)}{\frac{1}{2}(2\sin A \cos A)}$$

$$\equiv \frac{2\sin(B-A)}{\sin 2A}$$

$$\equiv 2\operatorname{cosec} 2A \sin(B-A) \equiv \text{ RHS}$$

c $$\text{LHS} \equiv \frac{1-\cos 2\theta}{\sin 2\theta}$$

$$\equiv \frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta}$$

$$\equiv \frac{2\sin^2\theta}{2\sin\theta\cos\theta}$$

$$\equiv \frac{\sin\theta}{\cos\theta}$$

$$\equiv \tan\theta \equiv \text{ RHS}$$

d $$\text{LHS} \equiv \frac{\sec^2\theta}{1-\tan^2\theta}$$

$$\equiv \frac{1}{\cos^2\theta(1-\tan^2\theta)}$$

$$\equiv \frac{1}{\cos^2\theta - \sin^2\theta} \qquad \left(\text{as } \tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta}\right)$$

$$\equiv \frac{1}{\cos 2\theta}$$

$$\equiv \sec 2\theta \equiv \text{ RHS}$$

1 e
$$\begin{aligned}\text{LHS} &\equiv 2(\sin^3\theta\cos\theta+\cos^3\theta\sin\theta)\\ &\equiv 2\sin\theta\cos\theta(\sin^2\theta+\cos^2\theta)\\ &\equiv \sin 2\theta \qquad (\text{since } \sin^2\theta+\cos^2\theta\equiv 1)\\ &\equiv \text{ RHS}\end{aligned}$$

f
$$\begin{aligned}\text{LHS} &\equiv \frac{\sin 3\theta}{\sin\theta}-\frac{\cos 3\theta}{\cos\theta}\\ &\equiv \frac{\sin 3\theta\cos\theta-\cos 3\theta\sin\theta}{\sin\theta\cos\theta}\\ &\equiv \frac{\sin(3\theta-\theta)}{\frac{1}{2}\sin 2\theta}\\ &\equiv \frac{\sin 2\theta}{\frac{1}{2}\sin 2\theta}\\ &\equiv 2\equiv \text{ RHS}\end{aligned}$$

g
$$\begin{aligned}\text{LHS} &\equiv \operatorname{cosec}\theta-2\cot 2\theta\cos\theta\\ &\equiv \operatorname{cosec}\theta-2\frac{\cos 2\theta}{\sin 2\theta}\cos\theta\\ &\equiv \operatorname{cosec}\theta-\frac{2\cos 2\theta\cos\theta}{2\sin\theta\cos\theta}\\ &\equiv \frac{1}{\sin\theta}-\frac{\cos 2\theta}{\sin\theta}\\ &\equiv \frac{1-\cos 2\theta}{\sin\theta}\\ &\equiv \frac{1-(1-2\sin^2\theta)}{\sin\theta}\\ &\equiv \frac{2\sin^2\theta}{\sin\theta}\\ &\equiv 2\sin\theta\equiv \text{ RHS}\end{aligned}$$

h
$$\begin{aligned}\text{LHS} &\equiv \frac{\sec\theta-1}{\sec\theta+1}\\ &\equiv \frac{\frac{1}{\cos\theta}-1}{\frac{1}{\cos\theta}+1}\\ &\equiv \frac{1-\cos\theta}{1+\cos\theta}\\ &\equiv \frac{1-\left(1-2\sin^2\frac{\theta}{2}\right)}{1+(2\cos^2\frac{\theta}{2}-1)}\\ &\equiv \frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\\ &\equiv \tan^2\frac{\theta}{2}\equiv \text{ RHS}\end{aligned}$$

1 i LHS $\equiv \tan\left(\frac{\pi}{4}-x\right)$

$$\equiv \frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}$$

$$\equiv \frac{1-\tan x}{1+\tan x}$$

$$\equiv \frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}$$

$$\equiv \frac{\cos x-\sin x}{\cos x+\sin x}$$

$$\equiv \frac{\cos^2 x+\sin^2 x-2\sin x\cos x}{\cos^2 x-\sin^2 x} \quad \text{(multiply 'top and bottom' by } \cos x-\sin x)$$

$$\equiv \frac{1-\sin 2x}{\cos 2x} \equiv \text{RHS}$$

2 a LHS $\equiv \sin(A+60°)+\sin(A-60°)$

$$\equiv \sin A\cos 60°+\cos A\sin 60°+\sin A\cos 60°-\cos A\sin 60°$$

$$\equiv 2\sin A\cos 60°$$

$$\equiv \sin A \quad \left(\text{since } \cos 60°=\tfrac{1}{2}\right)$$

$$\equiv \text{RHS}$$

b LHS $\equiv \dfrac{\cos A}{\sin B}-\dfrac{\sin A}{\cos B}$

$$\equiv \frac{\cos A\cos B-\sin A\sin B}{\sin B\cos B}$$

$$\equiv \frac{\cos(A+B)}{\sin B\cos B}$$

$$\equiv \text{RHS}$$

c LHS $\equiv \dfrac{\sin(x+y)}{\cos x\cos y}$

$$\equiv \frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y}$$

$$\equiv \frac{\sin x\cos y}{\cos x\cos y}+\frac{\cos x\sin y}{\cos x\cos y}$$

$$\equiv \frac{\sin x}{\cos x}+\frac{\sin y}{\cos y}$$

$$\equiv \tan x+\tan y$$

$$\equiv \text{RHS}$$

2 **d** LHS $\equiv \dfrac{\cos(x+y)}{\sin x \sin y}+1$

$$\equiv \frac{\cos(x+y)+\sin x \sin y}{\sin x \sin y}$$

$$\equiv \frac{\cos x \cos y - \sin x \sin y + \sin x \sin y}{\sin x \sin y}$$

$$\equiv \frac{\cos x \cos y}{\sin x \sin y}$$

$$\equiv \cot x \cot y$$

$$\equiv \text{ RHS}$$

e LHS $\equiv \cos\left(\theta+\tfrac{\pi}{3}\right)+\sqrt{3}\sin\theta$

$$\equiv \cos\theta\cos\tfrac{\pi}{3}-\sin\theta\sin\tfrac{\pi}{3}+\sqrt{3}\sin\theta$$

$$\equiv \tfrac{1}{2}\cos\theta-\tfrac{\sqrt{3}}{2}\sin\theta+\sqrt{3}\sin\theta$$

$$\equiv \tfrac{\sqrt{3}}{2}\sin\theta+\tfrac{1}{2}\cos\theta$$

$$\equiv \sin\theta\cos\tfrac{\pi}{6}+\cos\theta\sin\tfrac{\pi}{6} \quad \left(\cos\tfrac{\pi}{6}=\tfrac{\sqrt{3}}{2}, \sin\tfrac{\pi}{6}=\tfrac{1}{2}\right)$$

$$\equiv \sin\left(\theta+\tfrac{\pi}{6}\right) \quad \left(\sin(A+B)\right)$$

$$\equiv \text{ RHS}$$

f LHS $\equiv \cot(A+B) \equiv \dfrac{\cos(A+B)}{\sin(A+B)}$

$$\equiv \frac{\cos A\cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B}$$

$$\equiv \frac{\dfrac{\cos A\cos B}{\sin A \sin B}-\dfrac{\sin A \sin B}{\sin A \sin B}}{\dfrac{\sin A\cos B}{\sin A \sin B}+\dfrac{\cos A \sin B}{\sin A \sin B}} \quad \text{(dividing top and bottom by } \sin A \sin B)$$

$$\equiv \frac{\cot A \cot B - 1}{\cot A + \cot B} \equiv \text{RHS}$$

2 **g** LHS $\equiv \sin^2(45+\theta)° + \sin^2(45-\theta)°$

$$\begin{aligned}
&\equiv (\sin 45° \cos\theta + \cos 45° \sin\theta)^2 + (\sin 45° \cos\theta - \cos 45° \sin\theta)^2 \\
&\equiv (\sin 45° \cos\theta + \sin 45° \sin\theta)^2 + (\sin 45° \cos\theta - \sin 45° \sin\theta)^2 \quad (\text{as } \sin 45° = \cos 45°) \\
&\equiv (\sin 45°)^2 \left((\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2\right) \\
&\equiv \tfrac{1}{2}(\cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta + \sin^2\theta) \\
&\equiv \tfrac{1}{2}\left(2\left(\sin^2\theta + \cos^2\theta\right)\right) \\
&\equiv \tfrac{1}{2}\times 2 \qquad (\sin^2\theta + \cos^2\theta \equiv 1) \\
&\equiv 1 \\
&\equiv \text{ RHS}
\end{aligned}$$

Alternatively as $\sin(90° - x) \equiv \cos x$, if $x = 45° + \theta°$ then $\sin(45° - \theta) \equiv \cos(45° + \theta)$ and original LHS becomes $\sin^2(45+\theta)° + \cos^2(45+\theta)°$, which $= 1$

h LHS $\equiv \cos(A+B)\cos(A-B)$

$$\begin{aligned}
&\equiv (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B) \\
&\equiv \cos^2 A\cos^2 B - \sin^2 A\sin^2 B \\
&\equiv \cos^2 A(1-\sin^2 B) - (1-\cos^2 A)\sin^2 B \\
&\equiv \cos^2 A - \cos^2 A\sin^2 B - \sin^2 B + \cos^2 A\sin^2 B \\
&\equiv \cos^2 A - \sin^2 B \\
&\equiv \text{ RHS}
\end{aligned}$$

3 **a** LHS $\equiv \tan\theta + \cot\theta$

$$\begin{aligned}
&\equiv \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \\
&\equiv \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} \\
&\equiv \frac{2}{2\sin\theta\cos\theta} \quad (\sin^2\theta + \cos^2\theta \equiv 1) \\
&\equiv \frac{2}{\sin 2\theta} \\
&\equiv 2\operatorname{cosec} 2\theta \equiv \text{ RHS}
\end{aligned}$$

b Use $\theta = 75°$

$$\Rightarrow \tan 75° + \cot 75° = 2\operatorname{cosec}150° = 2\times\frac{1}{\sin 150°} = 2\times\frac{1}{\frac{1}{2}} = 4$$

4 **a** $\sin 3\theta \equiv \sin(2\theta+\theta) \equiv \sin 2\theta\cos\theta + \cos 2\theta \sin\theta$

$$\equiv (2\sin\theta\cos\theta)\cos\theta + (\cos^2\theta - \sin^2\theta)\sin\theta$$

$$\equiv 2\sin\theta\cos^2\theta + \sin\theta\cos^2\theta - \sin^3\theta$$

$$\equiv 3\sin\theta\cos^2\theta - \sin^3\theta$$

b $\cos 3\theta \equiv \cos(2\theta+\theta) \equiv \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$$\equiv (\cos^2\theta - \sin^2\theta)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$$

$$\equiv \cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta$$

$$\equiv \cos^3\theta - 3\sin^2\theta\cos\theta$$

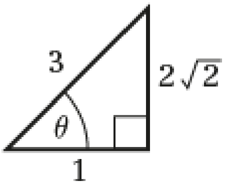
c $\tan 3\theta \equiv \dfrac{\sin 3\theta}{\cos 3\theta} \equiv \dfrac{3\sin\theta\cos^2\theta - \sin^3\theta}{\cos^3\theta - 3\sin^2\theta\cos\theta}$

$$\equiv \frac{\dfrac{3\sin\theta\cos^2\theta - \sin^3\theta}{\cos^3\theta}}{\dfrac{\cos^3\theta - 3\sin^2\theta\cos\theta}{\cos^3\theta}}$$

$$\equiv \frac{\dfrac{3\sin\theta}{\cos\theta} - \dfrac{\sin^3\theta}{\cos^3\theta}}{\dfrac{\cos^3\theta}{\cos^3\theta} - \dfrac{3\sin^2\theta}{\cos^2\theta}}$$

$$\equiv \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

d Sketch the right-angled triangle containing θ



This shows $\tan\theta = 2\sqrt{2}$

So $\tan 3\theta = \dfrac{3\left(2\sqrt{2}\right) - \left(2\sqrt{2}\right)^3}{1 - 3\left(2\sqrt{2}\right)^2} = \dfrac{6\sqrt{2} - 16\sqrt{2}}{1-24} = \dfrac{-10\sqrt{2}}{-23} = \dfrac{10\sqrt{2}}{23}$

5 **a** **i** Using $\cos 2A \equiv 2\cos^2 A - 1$ with $A = \dfrac{x}{2}$

$$\Rightarrow \quad \cos x \equiv 2\cos^2\frac{x}{2} - 1$$

$$\Rightarrow \quad 2\cos^2\frac{x}{2} \equiv 1 + \cos x$$

$$\Rightarrow \quad \cos^2\frac{x}{2} \equiv \frac{1+\cos x}{2}$$

5 a ii Using $\cos 2A \equiv 1 - 2\sin^2 A$

$$\Rightarrow \cos x \equiv 1 - 2\sin^2 \frac{x}{2}$$

$$\Rightarrow 2\sin^2 \frac{x}{2} \equiv 1 - \cos x$$

$$\Rightarrow \sin^2 \frac{x}{2} \equiv \frac{1-\cos x}{2}$$

b i Using (a) (i) $\cos^2 \frac{\theta}{2} = \frac{1.6}{2} = 0.8 = \frac{4}{5}$

$$\Rightarrow \cos\frac{\theta}{2} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5} \quad \left(\text{as } \frac{\theta}{2} \text{ acute}\right)$$

ii Using (a) (ii) $\sin^2 \frac{\theta}{2} = \frac{0.4}{2} = 0.2 = \frac{1}{5}$

$$\Rightarrow \sin\frac{\theta}{2} = \sqrt{\frac{1}{5}} = \frac{\sqrt{5}}{5}$$

iii $\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{\sqrt{5}}{5} \times \frac{5}{2\sqrt{5}} = \frac{1}{2}$

c Using (a) (i) and squaring

$$\cos^4 \frac{A}{2} \equiv \left(\frac{1+\cos A}{2}\right)^2 \equiv \frac{1+2\cos A+\cos^2 A}{4}$$

but using $\cos 2A \equiv 2\cos^2 A - 1$ gives

$$\cos^2 A \equiv \frac{1}{2}(1+\cos 2A)$$

So $\cos^4 \frac{A}{2} \equiv \frac{1+2\cos A+\frac{1}{2}(1+\cos 2A)}{4} \equiv \frac{2+4\cos A+1+\cos 2A}{8} \equiv \frac{3+4\cos A+\cos 2A}{8}$

6 LHS $\equiv \cos^4\theta \equiv (\cos^2\theta)^2 \equiv \left(\frac{1+\cos 2\theta}{2}\right)^2$

$$\equiv \frac{1}{4}(1+2\cos 2\theta+\cos^2 2\theta)$$

$$\equiv \frac{1}{4}+\frac{1}{2}\cos 2\theta+\frac{1}{4}\left(\frac{1+\cos 4\theta}{2}\right)$$

$$\equiv \frac{1}{4}+\frac{1}{2}\cos 2\theta+\frac{1}{8}+\frac{1}{8}\cos 4\theta$$

$$\equiv \frac{3}{8}+\frac{1}{2}\cos 2\theta+\frac{1}{8}\cos 4\theta \equiv \text{ RHS}$$

7 $\sin^2(x+y)-\sin^2(x-y)\equiv[\sin(x+y)+\sin(x-y)][\sin(x+y)-\sin(x-y)]$

$$\equiv[\sin x\cos y+\cos x\sin y+\sin x\cos y-\cos x\sin y][\sin x\cos y+\cos x\sin y-(\sin x\cos y-\cos x\sin y)]$$

$$\equiv[2\sin x\cos y][2\cos x\sin y]$$

$$\equiv[2\sin x\cos x][2\cos y\sin y]$$

$$\equiv\sin 2x\sin 2y$$

8 Let $\cos 2\theta-\sqrt{3}\sin 2\theta\equiv R\cos(2\theta+\alpha)\equiv R\cos 2\theta\cos\alpha-R\sin 2\theta\sin\alpha$

Compare $\cos 2\theta: R\cos\alpha=1$ (1)

Compare $\sin 2\theta: R\sin\alpha=\sqrt{3}$ (2)

Divide (2) by (1):

$\tan\alpha=\sqrt{3}\ \Rightarrow\ \alpha=\frac{\pi}{3}$

Square and add equations:

$R^2=1+3=4\ \Rightarrow\ R=2$

So $\cos 2\theta-\sqrt{3}\sin 2\theta\equiv 2\cos\left(2\theta+\frac{\pi}{3}\right)$

9 $4\cos\left(2\theta-\frac{\pi}{6}\right)\equiv 4\cos 2\theta\cos\frac{\pi}{6}+4\sin 2\theta\sin\frac{\pi}{6}$

$$\equiv 2\sqrt{3}\cos 2\theta+2\sin 2\theta$$

$$\equiv 2\sqrt{3}\left(1-2\sin^2\theta\right)+4\sin\theta\cos\theta$$

$$\equiv 2\sqrt{3}-4\sqrt{3}\sin^2\theta+4\sin\theta\cos\theta$$

10 a RHS $\equiv\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)$

$$\equiv\sqrt{2}\left(\sin\theta\cos\frac{\pi}{4}+\cos\theta\sin\frac{\pi}{4}\right)$$

$$\equiv\sqrt{2}\left(\sin\theta\times\frac{1}{\sqrt{2}}+\cos\theta\times\frac{1}{\sqrt{2}}\right)$$

$$\equiv\sin\theta+\cos\theta$$

$\equiv$ LHS

b RHS $\equiv 2\sin\left(2\theta-\frac{\pi}{6}\right)$

$$\equiv 2\left(\sin 2\theta\cos\frac{\pi}{6}-\cos 2\theta\sin\frac{\pi}{6}\right)$$

$$\equiv 2\left(\sin 2\theta\times\frac{\sqrt{3}}{2}-\cos 2\theta\times\frac{1}{2}\right)$$

$$\equiv\sqrt{3}\sin 2\theta-\cos 2\theta$$

$\equiv$ LHS

Challenge

1 a $\cos(A+B)-\cos(A-B)\equiv\cos A\cos B-\sin A\sin B-(\cos A\cos B+\sin A\sin B)$

$$\equiv -2\sin A\sin B$$

b Let $A+B=P$ and $A-B=Q$

Solving simultaneously gives

$$2A=P+Q$$

$$A=\frac{P+Q}{2}$$

and

$$2B=P-Q$$

$$B=\frac{P-Q}{2}$$

Substituting these into the identity from part a gives

$$\cos P-\cos Q\equiv -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$$

c Rearranging the identity from part **a** to give $\sin A\sin B\equiv -\frac{1}{2}\cos(A+B)+\frac{1}{2}\cos(A-B)$

$$3\sin x\sin 7x\equiv -\frac{3}{2}\cos(x+7x)+\frac{3}{2}\cos(x-7x)$$

$$\equiv -\frac{3}{2}\cos 8x+\frac{3}{2}\cos(-6x)$$

$$\equiv -\frac{3}{2}\cos 8x+\frac{3}{2}\cos(6x)\quad (\text{as } \cos(-x)\equiv\cos x)$$

$$\equiv -\frac{3}{2}(\cos 8x-\cos 6x)$$

2 a $\sin(A+B)+\sin(A-B)\equiv\sin A\cos B+\cos A\sin B+(\sin A\cos B-\cos A\sin B)$

$$\equiv 2\sin A\cos B$$

Let $A+B=P$ and $A-B=Q$

Solving simultaneously gives

$$2A=P+Q$$

$$A=\frac{P+Q}{2}$$

and

$$2B=P-Q$$

$$B=\frac{P-Q}{2}$$

Substituting these into the equation for $\sin(A+B)+\sin(A-B)$ gives

$$\sin P+\sin Q\equiv 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$$

2 b Let $\frac{11\pi}{24}=\frac{P+Q}{2}, \frac{5\pi}{24}=\frac{P-Q}{2}$

$\frac{22\pi}{24}=P+Q, \frac{10\pi}{24}=P-Q$

Solving simultaneously gives:

$2P=\frac{32\pi}{24},\ P=\frac{16\pi}{24}$

and

$2Q=\frac{12\pi}{24},\ Q=\frac{6\pi}{24}$

So $2\sin\frac{11\pi}{24}\cos\frac{5\pi}{24}=\sin\left(\frac{16\pi}{24}\right)+\sin\left(\frac{6\pi}{24}\right)=\sin\left(\frac{2\pi}{3}\right)+\sin\left(\frac{\pi}{4}\right)=\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}=\frac{\sqrt{3}+\sqrt{2}}{2}$

Trigonometry and modelling 7G

1 a The maximum height is at 0.25 m when $\sin(1800t)° = 1$

b $0.25\sin(1800t)° = 0.1$

$\sin(1800t)° = \frac{0.1}{0.25} = 0.4$

$1800t = 23.578$ (3 d.p.)

$t = 0.013099$ minutes

$= 0.8$ seconds (1 d.p.)

c The minimum height is at −0.25 m when $\sin(1800t)° = -1$

This occurs when $1800t = 270, 630$
$t = 0.15, 0.35$ minutes
Interval = 0.35 – 0.15 = 0.2 minutes
= 12 seconds

2 a The maximum displacement is at 0.03 radians when $\cos(25t) = 1$

b After 0.2 seconds

$\theta = 0.03\cos(25 \times 0.2) = 0.03 \times 0.28366$

$= 0.0085$ radians (2 s.f.)

c At $t = 0$, $\theta = 0.03\cos(25 \times 0) = 0.03$
To find when $\theta = 0.03$, solve

$0.03\cos(25t) = 0.03$

$\Rightarrow \cos(25t) = 1$

$\Rightarrow 25t = 0, 2\pi, 4\pi, ...$

$\Rightarrow t = 0, 0.251, 0.503, ...$ (3 d.p.)

The pendulum is first back to its starting position after 0.251 seconds.

d Solve $0.03\cos(25t) = 0.01$

$\cos(25t) = \frac{1}{3},\ 0 \le 25t \le 12.5$

$25t = 1.231, 2\pi - 1.231,\ 2\pi + 1.231,\ 4\pi - 1.231$

$25t = 1.231, 5.052, 7.514, 11.335$ (3 d.p.)

$t = 0.0492, 0.2021, 0.3006, 0.4534$ secs

3 a Beginning price when $t = 0$ is
$17.4 + 2\sin(0.7 \times 0 - 3) = £17.12$

End price when $t = 9$ is
$17.4 + 2\sin(0.7 \times 9 - 3) = £17.08$

b The maximum price of the stock is when $\sin(0.7t - 3) = 1$, so 17.4 + 2 = £19.40

This is when $\sin(0.7t - 3) = 1$

$0.7t - 3 = \frac{\pi}{2}$

$t = 6.5297$ (4 d.p.)

$t = 6$ hours 32 minutes

c Trader will show a £0.40 profit when
$17.4 + 2\sin(0.7t - 3) = £17.12 + £0.40$

$\Rightarrow \sin(0.7t - 3) = 0.06$

$\Rightarrow 0.7t - 3 = 0.060$ (3 d.p.)

$t = 4.371$

So trader should sell 4 hours 22 minutes after the market opens.

4 a The minimum temperature of the oven is when $\sin(2x - 3) = 1$
$T = 225 - 0.3 = 224.7$ °C

b Solve $225 - 0.3\sin(2x - 3) = 224.7$, for $0 \le x \le 10$

$\Rightarrow \sin(2x - 3) = 1,$

for $-1.5 \le 2x - 3 \le 17$

$2x - 3 = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$

$x = 2.285, 5.427, 8.569$ (3 d.p.)

$x = 2\text{m}\ 17\text{s}, 5\text{m}\ 26\text{s}, 8\text{m}\ 34\text{s}$

c $225 - 0.3\sin(2x - 3) = 225.2$, for $x \ge 0$

$\Rightarrow \sin(2x - 3) = -\frac{2}{3}$, for $2x - 3 \ge -3$

$2x - 3 = -2.412, -0.730...$ (3 d.p.)

So oven first reaches minimum temperature at $2x - 3 = -2.412$, so $x = 0.294$ minutes, which is 17.6 seconds (1 d.p.)

5 **a** Set $0.3\sin\theta - 0.4\cos\theta \equiv R\sin(\theta - \alpha)$

$$\equiv R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$$

So $R\cos\alpha = 0.3$ and $R\sin\alpha = 0.4$

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{0.4}{0.3}$$

$$\alpha = \tan^{-1}\left(\frac{4}{3}\right) = 53.13° \text{ (2 d.p.)}$$

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 0.3^2 + 0.4^2$$

$$R^2(\cos^2\alpha + \sin^2\alpha) = 0.25$$

$$R = 0.5$$

So $0.3\sin\theta - 0.4\cos\theta = 0.5\sin(\theta - 53.13°)$

b **i** The maximum value of $0.3\sin\theta - 0.4\cos\theta$ is when $\sin(\theta - 53.13°) = 1$, so the maximum value is 0.5

ii Solve $\sin(\theta - 53.13°) = 1$, for the interval $-53.13° < \theta - 53.13° < 126.87°$

$$\theta - 53.13° = 90°$$

$$\theta = 143.13°$$

c Using part (a)

$$23 + 0.3\sin(18x)° - 0.4\cos(18x)°$$

$$\equiv 23 + 0.5\sin(18x - 53.13°)$$

The minimum occurs when $\sin(18x - 53.13°) = -1$

So the minimum temperature is $23 - 0.5 = 22.5$°C

It occurs when $18x - 53.13 = 270$, $x = 17.95$ minutes (2 d.p.)

d At exactly 23 °C, $23 + 0.5\sin(18x - 53.13°) = 23$

Find solutions for $0.5\sin(18x - 53.13)° = 0$ for $0 \le x \le 60$, i.e. in the interval $-53.13 \le 18x - 53.13 \le 1026.87$

So $18x - 53.13 = 0, 180, 360, 540, 720, 900$

Solutions are 3, 13, 23, 33, 43, 53 minutes (nearest minute)

6 **a** Set $65\cos\theta - 20\sin\theta \equiv R\cos(\theta + \alpha)$

$$\equiv R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$$

So $R\cos\alpha = 65$ and $R\sin\alpha = 20$

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{20}{65}$$

$$\alpha = \tan^{-1}\left(\frac{20}{65}\right) = 0.2985 \text{ (4 d.p.)}$$

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 65^2 + 20^2$$

$$R^2(\cos^2\alpha + \sin^2\alpha) = 4625$$

$$R = 68.0074 \text{ (4 d.p.)}$$

So

$$65\cos\theta - 20\sin\theta$$

$$= 68.0074\cos(\theta + 0.2985)$$

b Using part (a)

$$70 - 65\cos 0.2t - 20\sin 0.2t$$

$$\equiv 70 - 68.0074\cos(0.2t + 0.2985)$$

The maximum height is when $\cos(0.2t + 0.2985) = -1$

So $H = 70 + 68.0074 = 138.0\,\text{m}$ (1 d.p.)

c Find consecutive times that the tourist is at the maximum height. This is when

$$\cos(0.2t + 0.2985) = -1$$

$$0.2t + 0.2985 = \pi, 3\pi$$

$$t = 14.216, 45.631 \text{ (3 d.p.)}$$

The time for one revolution is $45.631 - 14.216 = 31.4$ minutes (1 d.p.)

d Find the times the tourist is at 100 m

$$70 - 68.0074\cos(0.2t + 0.2985) = 100$$

$$\cos(0.2t + 0.2985) = -\frac{30}{68.0074} = -0.4411$$

$$0.2t + 0.2985 = 2.0277, 2\pi - 2.0277$$

$$0.2t + 0.2985 = 1.1139, 4.2555$$

$$t = 8.646, 19.785 \text{ (3 d.p.)}$$

Between these times the tourist is above 100 m because the highest point is reached at t = 14.216 minutes.

So time spent above 100 m in each revolution = 19.785 – 8.646 = 11.1 minutes (1 d.p.)

7 a Set $200\sin\theta - 150\cos\theta \equiv R\sin(\theta-\alpha)$

$$\equiv R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$$

So $R\cos\alpha = 200$ and $R\sin\alpha = 150$

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{150}{200}$$

$$\alpha = \tan^{-1}\left(\frac{3}{4}\right) = 0.6435 \text{ (4 d.p.)}$$

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 200^2 + 150^2$$

$$R^2(\cos^2\alpha + \sin^2\alpha) = 62500$$

$$R = 250$$

So $200\sin\theta - 150\cos\theta$

$$\equiv 250\sin(\theta - 0.6435)$$

b i $1700 + 200\sin\left(\frac{4\pi x}{25}\right) - 150\cos\left(\frac{4\pi x}{25}\right)$

$$\equiv 1700 + 250\sin\left(\frac{4\pi x}{25} - 0.6435\right)$$

The maximum value of E is when

$$\sin\left(\frac{4\pi x}{25} - 0.6435\right) = 1$$

So maximum value of E is $1700 + 250 = 1950$ V/m

ii This maximum occurs when

$$\sin\left(\frac{4\pi x}{25} - 0.6435\right) = 1$$

Look for solutions in the interval

$$-0.6435 \le \frac{4\pi x}{25} - 0.6435 < 4\pi - 0.6435$$

$$\frac{4\pi x}{25} - 0.6435 = \frac{\pi}{2}, \frac{5\pi}{2}$$

$$\frac{4\pi x}{25} = 2.2143, 8.4975$$

$$4\pi x = 55.3575, 212.4375$$

$$x = 4.41\text{cm}, 16.91\text{cm (2 d.p.)}$$

c Solve

$$1700 + 250\sin\left(\frac{4\pi x}{25} - 0.6435\right) = 1800$$

for the same interval as in part b ii

$$\sin\left(\frac{4\pi x}{25} - 0.6435\right) = \frac{100}{250} = 0.4$$

$$\frac{4\pi x}{25} - 0.6435 = 0.4115, \pi - 0.4115,$$

$$2\pi + 0.4115, 3\pi - 0.4115$$

$$= 0.4115, 2.7301, 6.6947, 9.0133$$

$$x = 2.10, 6.71, 14.60, 19.21 \text{ (2 d.p.)}$$

These results show where E = 1800 V/m. So, because of the shape of the sine curve, $E \le 1800$ V/m when $2.10 \le x \le 6.71$ and $14.60 \le x \le 19.21$

Challenge

a Energy $\propto E^2$ and Energy $\propto \frac{1}{t}$, so $E^2 = \frac{k}{t}$

When E = 1950, t = 20 seconds

$k = 1950^2 \times 20 = 76\,050\,000$

When t = 30, E = 1592.1683 V/m

Find where E = 1592.1683 V/m, using the formula for E from question 7:

$$1592.1683 = 1700 + 250\sin\left(\frac{4\pi x}{25} - 0.6435\right)$$

$$-0.4313 = \sin\left(\frac{4\pi x}{25} - 0.6435\right)$$

$$\frac{4\pi x}{25} - 0.6435 = -0.4460, 3.5876, 5.8372,$$

$$9.8707, 12.1204$$

This gives these results $0 \le x < 0.393$cm, $8.42\text{cm} < x < 12.9\text{cm}$ and $20.9\text{cm} < x < 25\text{cm}$

b Two limitations of the model are (i) assumes that the field strength is the same from the front to the back of the microwave and (ii) the microwave oven would not necessarily work exactly the same every time it is used.

Trigonometry and Modelling Mixed Exercise

1 a i $\sin 40° \cos 10° - \cos 40° \sin 10°$

$= \sin(40° - 10°) = \sin 30° = \frac{1}{2}$

ii $\frac{1}{\sqrt{2}} \cos 15° - \frac{1}{\sqrt{2}} \sin 15°$

$\cos 45° \cos 15° - \sin 45° \sin 15°$

$\cos(45° + 15°) = \cos 60° = \frac{1}{2}$

iii $\frac{1 - \tan 15°}{1 + \tan 15°} = \frac{\tan 45° - \tan 15°}{1 + \tan 45° \tan 15°}$

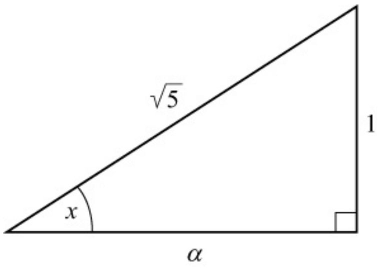
$= \tan(45° - 15°) = \tan 30° = \frac{\sqrt{3}}{3}$

2 As $\cos(x - y) = \sin y$

$\cos x \cos y + \sin x \sin y = \sin y \qquad (1)$

Draw a right-angled triangle,

where $\sin x = \frac{1}{\sqrt{5}}$



Using Pythagoras' theorem,

$a^2 = \left(\sqrt{5}\right)^2 - 1 = 4 \quad \Rightarrow \quad a = 2$

So $\cos x = \frac{2}{\sqrt{5}}$

Substitute into (1):

$\frac{2}{\sqrt{5}} \cos y + \frac{1}{\sqrt{5}} \sin y = \sin y$

$\Rightarrow 2\cos y + \sin y = \sqrt{5} \sin y$

$\Rightarrow 2\cos y = \sin y\left(\sqrt{5} - 1\right)$

$\Rightarrow \frac{2}{\left(\sqrt{5} - 1\right)} = \tan y \qquad \left(\tan y = \frac{\sin y}{\cos y}\right)$

$\Rightarrow \tan y = \frac{2\left(\sqrt{5} + 1\right)}{\left(\sqrt{5} - 1\right)\left(\sqrt{5} + 1\right)}$

$= \frac{2\left(\sqrt{5} + 1\right)}{4} = \frac{\sqrt{5} + 1}{2}$

3 a $\tan A = 2$, $\tan B = \frac{1}{3}$ since $y = \frac{1}{3}x - \frac{1}{3}$

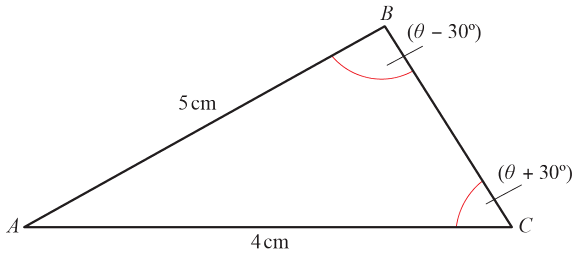
b The angle required is $(A – B)$

Using $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$= \frac{2 - \frac{1}{3}}{1 + 2 \times \frac{1}{3}} = \frac{\frac{5}{3}}{\frac{5}{3}} = 1$

$\Rightarrow A - B = 45°$

4



Using $\dfrac{\sin B}{b}=\dfrac{\sin C}{c}$

$$\Rightarrow \frac{\sin(\theta-30°)}{4}=\frac{\sin(\theta+30°)}{5}$$

$$\Rightarrow 5\sin(\theta-30°)=4\sin(\theta+30°)$$

$$\Rightarrow 5(\sin\theta\cos 30°-\cos\theta\sin 30°)$$

$$=4(\sin\theta\cos 30°+\cos\theta\sin 30°)$$

$$\Rightarrow \sin\theta\cos 30°=9\cos\theta\sin 30°$$

$$\Rightarrow \frac{\sin\theta}{\cos\theta}=9\frac{\sin 30°}{\cos 30°}=9\tan 30°$$

$$\Rightarrow \tan\theta=9\times\frac{\sqrt{3}}{3}=3\sqrt{3}$$

5 As the three values are consecutive terms of an arithmetic progression,

$$\sin(\theta-30°)-\sqrt{3}\cos\theta=\sin\theta-\sin(\theta-30°)$$

$$\Rightarrow 2\sin(\theta-30°)=\sin\theta+\sqrt{3}\cos\theta$$

$$\Rightarrow 2(\sin\theta\cos 30°-\cos\theta\sin 30°)$$

$$=\sin\theta+\sqrt{3}\cos\theta$$

$$\Rightarrow \sqrt{3}\sin\theta-\cos\theta=\sin\theta+\sqrt{3}\cos\theta$$

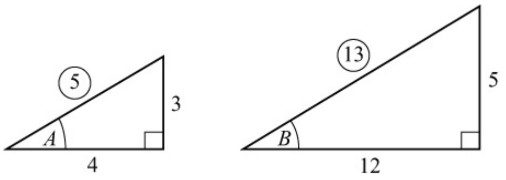
$$\Rightarrow \sin\theta\left(\sqrt{3}-1\right)=\cos\theta\left(\sqrt{3}+1\right)$$

$$\Rightarrow \tan\theta=\frac{\sqrt{3}+1}{\sqrt{3}-1}$$

Calculator value is $\theta=\tan^{-1}\dfrac{\sqrt{3}+1}{\sqrt{3}-1}=75°$

No other values as θ is acute.

6 **a**



$\sin A=\frac{3}{5},\cos A=\frac{4}{5}$ $\sin B=\frac{5}{13},\cos B=\frac{12}{13}$

i $\sin(A+B)=\sin A\cos B+\cos A\sin B$

$$=\frac{3}{5}\times\frac{12}{13}+\frac{4}{5}\times\frac{5}{13}=\frac{56}{65}$$

ii $\tan 2B=\dfrac{2\tan B}{1-\tan^2 B}$

$$=\frac{2\times\frac{5}{12}}{1-\left(\frac{5}{12}\right)^2}=\frac{\frac{5}{6}}{\frac{119}{144}}$$

$$=\frac{5}{6}\times\frac{144}{119}=\frac{120}{119}$$

b $\cos C=\cos\bigl(180°-(A+B)\bigr)$

$$=-\cos(A+B)$$

$$=-(\cos A\cos B-\sin A\sin B)$$

$$=-\left(\frac{4}{5}\times\frac{12}{13}-\frac{3}{5}\times\frac{5}{13}\right)$$

$$=-\frac{33}{65}$$

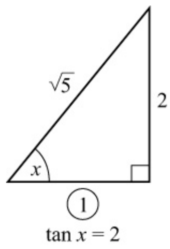
7 **a** $\cos 2x\equiv 1-2\sin^2 x$

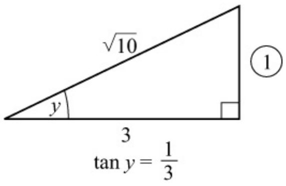
$$=1-2\left(\frac{2}{\sqrt{5}}\right)^2=1-\frac{8}{5}=-\frac{3}{5}$$

b $\cos 2y\equiv 2\cos^2 y-1$

$$=2\left(\frac{3}{\sqrt{10}}\right)^2-1=2\left(\frac{9}{10}\right)-1=\frac{4}{5}$$

7 c





i $\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$

$=\frac{2+\frac{1}{3}}{1-\frac{2}{3}}=\frac{\frac{7}{3}}{\frac{1}{3}}=7$

ii $\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}=\frac{\frac{5}{3}}{\frac{5}{3}}=1$

As x and y are acute, and $x>y$,
$x-y$ is acute

So $x-y=\frac{\pi}{4}$ $\left(\text{it cannot be }\frac{5\pi}{4}\right)$

8 a $\sin(x+y)=\sin x\cos y+\cos x\sin y$

$=\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$

$5\sin(x-y)\equiv5(\sin x\cos y-\cos x\sin y)$

$=5\left(\frac{1}{2}-\frac{1}{3}\right)=5\times\frac{1}{6}=\frac{5}{6}$

b $\frac{\sin x\cos y}{\cos x\sin y}=\frac{\frac{1}{2}}{\frac{1}{3}}=\frac{3}{2}$

$\Rightarrow\ \frac{\tan x}{\tan y}=\frac{3}{2}$

so $\tan x=\frac{3\tan y}{2}=\frac{3k}{2}$

c $\tan 2x=\frac{2\tan x}{1-\tan^2 x}=\frac{3k}{1-\frac{9}{4}k^2}$

$=\frac{12k}{4-9k^2}$

9 a $\sqrt{3}\sin2\theta+2\sin^2\theta=1$

$\sqrt{3}\sin2\theta=1-2\sin^2\theta=\cos2\theta$

$\frac{\sin2\theta}{\cos2\theta}=\frac{1}{\sqrt{3}}\ \Rightarrow\tan2\theta=\frac{1}{\sqrt{3}}$

b $\tan2\theta=\frac{1}{\sqrt{3}}$, for $0\le2\theta\le2\pi$

$2\theta=\frac{\pi}{6},\frac{7\pi}{6}\ \Rightarrow\theta=\frac{\pi}{12},\frac{7\pi}{12}$

10 a $\cos2\theta=5\sin\theta$

$\Rightarrow\cos2\theta-5\sin\theta=0$

$\Rightarrow1-2\sin^2\theta-5\sin\theta=0$

$\Rightarrow2\sin^2\theta+5\sin\theta-1=0$

$a=2$, $b=5$ and $c=-1$

b $2\sin^2\theta+5\sin\theta-1=0$

Using the quadratic formula

$\sin\theta=\frac{-5\pm\sqrt{5^2-4(2)(-1)}}{2(2)}$

$=\frac{-5\pm\sqrt{33}}{4}$

$\sin\theta=0.1861$, for $-\pi\le\theta\le\pi$

$\sin\theta$ is positive so solutions in the first and second quadrants

$\theta=\sin^{-1}0.1861,\ \pi-\sin^{-1}0.1861$

$\theta=0.187,\ 2.954$ (3 d.p.)

11 a $\cos(x-60°)=\cos x\cos60°+\sin x\sin60°$

$=\frac{1}{2}\cos x+\frac{\sqrt{3}}{2}\sin x$

So $2\sin x=\frac{1}{2}\cos x+\frac{\sqrt{3}}{2}\sin x$

$\Rightarrow\left(2-\frac{\sqrt{3}}{2}\right)\sin x=\frac{1}{2}\cos x$

$\Rightarrow\tan x=\frac{\frac{1}{2}}{2-\frac{\sqrt{3}}{2}}=\frac{1}{2}\times\frac{2}{4-\sqrt{3}}=\frac{1}{4-\sqrt{3}}$

b $\tan x=\frac{1}{4-\sqrt{3}}=0.44$ (2 d.p.), in the interval $0°\le\theta\le360°$

$\tan\theta$ is positive so solutions in the first and third quadrants

$x=23.8°,\ 203.8°$ (1 d.p.)

12 a $\cos(x+20°) = \sin(90° - 20° - x)$

$= \sin(70° - x)$

$= \sin 70° \cos x - \cos 70° \sin x \qquad (1)$

$4\sin(70° + x) = 4\sin 70° \cos x$
$+ 4\cos 70° \sin x \qquad (2)$

As (1) = (2)

$4\sin 70° \cos x + 4\cos 70° \sin x$
$= \sin 70° \cos x - \cos 70° \sin x$

$5\sin x \cos 70° = -3\sin 70° \cos x$

$\tan x = -\frac{3}{5}\tan 70°$

b $\tan x = -\frac{3}{5}\tan 70°$, for $0° \le \theta \le 180°$

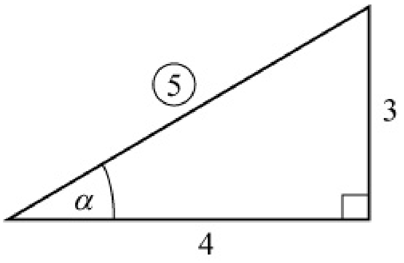
$\tan\theta$ is negative so the solution is in the second quadrant

$x = 180° + \tan^{-1}\left(-\frac{3}{5}\tan 70°\right)$

$x = 180° - \tan^{-1}(-1.648)$

$x = 180° - (-58.8°) = 121.2°$ (1 d.p.)

13 a Draw a right-angled triangle and find $\sin\alpha$ and $\cos\alpha$.



$\Rightarrow \quad \sin\alpha = \frac{3}{5}, \quad \cos\alpha = \frac{4}{5}$

$3\sin(\theta + \alpha) + 4\cos(\theta + \alpha)$

$\equiv 3(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$
$+ 4(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$

$\equiv 3\left(\frac{4}{5}\sin\theta + \frac{3}{5}\cos\theta\right)$
$+ 4\left(\frac{4}{5}\cos\theta - \frac{3}{5}\sin\theta\right)$

$\equiv \frac{12}{5}\sin\theta + \frac{9}{5}\cos\theta + \frac{16}{5}\cos\theta - \frac{12}{5}\sin\theta$

$\equiv \frac{25}{5}\cos\theta \equiv 5\cos\theta$

b $\cos(x + 270°)$

$\equiv \cos x° \cos 270° - \sin x° \sin 270°$

$= (-0.8)(0) - (0.6)(-1)$

$= 0 + 0.6 = 0.6$

$\cos(x + 540°)$

$\equiv \cos x° \cos 540° - \sin x° \sin 540°$

$= (-0.8)(-1) - (0.6)(0)$

$= 0.8 - 0 = 0.8$

14 a One example is sufficient to disprove a statement. Let $A = 60°,\ B = 0°$

$\sec(A + B) = \sec(60° + 0°)$

$= \sec 60° = \frac{1}{\cos 60°} = 2$

$\sec A = \sec 60° = \frac{1}{\cos 60°} = 2$

$\sec B = \sec 0° = \frac{1}{\cos 0°} = 1$

So $\sec A + \sec B = 2 + 1 = 3$

So $\sec(60° + 0°) \ne \sec 60° + \sec 0°$

$\Rightarrow \quad \sin(A + B) \equiv \sec A + \sec B$ is not true for all values of $A,\ B$.

b LHS $\equiv \tan\theta + \cot\theta$

$\equiv \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$

$\equiv \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$

$\equiv \frac{1}{\frac{1}{2}\sin 2\theta}$

Using $\sin^2\theta + \cos^2\theta \equiv 1$, and $\sin 2\theta \equiv 2\sin\theta\cos\theta$

So LHS $\equiv \frac{2}{\sin 2\theta}$

$\equiv 2\operatorname{cosec} 2\theta$

$\equiv$ RHS

15 a Using $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$ with $\theta = \dfrac{\pi}{8}$

$$\Rightarrow \tan\frac{\pi}{4} = \frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$$

Let $t = \tan\dfrac{\pi}{8}$

So $1 = \dfrac{2t}{1-t^2}$

$$\Rightarrow 1-t^2 = 2t$$

$$\Rightarrow t^2+2t-1=0$$

$$\Rightarrow t = \frac{-2\pm\sqrt{8}}{2} = \frac{-2\pm2\sqrt{2}}{2}$$

$$= -1\pm\sqrt{2}$$

As $\dfrac{\pi}{8}$ is acute, $\tan\dfrac{\pi}{8}$

is positive, so $\tan\dfrac{\pi}{8} = \sqrt{2}-1$

b $\tan\dfrac{3\pi}{8} = \tan\left(\dfrac{\pi}{4}+\dfrac{\pi}{8}\right) = \dfrac{\tan\frac{\pi}{4}+\tan\frac{\pi}{8}}{1-\tan\frac{\pi}{4}\tan\frac{\pi}{8}}$

$$= \frac{1+\left(\sqrt{2}-1\right)}{1-\left(\sqrt{2}-1\right)} = \frac{\sqrt{2}}{2-\sqrt{2}}$$

$$= \frac{\sqrt{2}\left(2+\sqrt{2}\right)}{\left(2-\sqrt{2}\right)\left(2+\sqrt{2}\right)}$$

$$= \frac{\sqrt{2}}{2}\left(2+\sqrt{2}\right) = \sqrt{2}+1$$

16 a Let $\sin x - \sqrt{3}\cos x \equiv R\sin(x-\alpha)$

$$\equiv R\sin x\cos\alpha - R\cos x\sin\alpha$$

$R>0,\ 0<\alpha<90°$

Compare $\sin x$: $R\cos\alpha = 1$ (1)

Compare $\cos x$: $R\sin\alpha = \sqrt{3}$ (2)

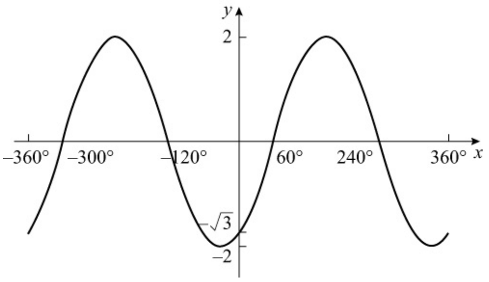
Divide (2) by (1): $\tan\alpha = \sqrt{3}$

$\Rightarrow \alpha = 60°$

$R^2 = \left(\sqrt{3}\right)^2 + 1^2 = 4 \Rightarrow R = 2$

So $\sin x - \sqrt{3}\cos x \equiv 2\sin(x-60°)$

b Sketch $y = 2\sin(x-60°)$ by first translating $y=\sin x$ by 60° to the right and then stretching the result in the y direction by scale factor 2.



Graph meets y-axis when $x=0$,

i.e. $y = 2\sin(-60°) = -\sqrt{3}$, at $\left(0,-\sqrt{3}\right)$

Graph meets x-axis when $y=0$,

i.e. $(-300°,\ 0),\ (-120°,\ 0),$

$(60°,\ 0),\ 240°,\ 0)$

17 a Let $7\cos2\theta + 24\sin2\theta \equiv R\cos(2\theta-\alpha)$

$$\equiv R\cos2\theta\cos\alpha + R\sin2\theta\sin\alpha$$

$R>0,\ 0<\alpha<\dfrac{\pi}{2}$

Compare $\cos2\theta : R\cos\alpha = 7$ (1)

Compare $\sin2\theta : R\sin\alpha = 24$ (2)

Divide (2) by (1) : $\tan\alpha = \dfrac{24}{7}$

$\Rightarrow \alpha = 1.29$ (2 d.p.)

$R^2 = 24^2+7^2 \Rightarrow R = 25$

So $7\cos2\theta + 24\sin2\theta \equiv 25\cos(2\theta-1.29)$

b $14\cos2\theta + 48\sin\theta\cos\theta$

$$\equiv 14\left(\frac{1+\cos2\theta}{2}\right) + 24(2\sin\theta\cos\theta)$$

$$\equiv 7(1+\cos2\theta) + 24\sin2\theta$$

$$\equiv 7+7\cos2\theta+24\sin2\theta$$

The maximum value of $7\cos2\theta + 24\sin2\theta$ is 25 (using (a) with $\cos(2\theta-1.29)=1$)

So maximum value of $7+7\cos2\theta+24\sin2\theta = 7+25 = 32$

17 c Using the answer to part a:
Solve $25\cos(2\theta-1.29)=12.5$

$$\cos(2\theta-1.29)=\frac{1}{2}$$

$$2\theta-1.29=-\frac{\pi}{3},\ \frac{\pi}{3}$$

$$\theta=0.119902...,\ 1.167099...$$

$$\theta=0.12,\ 1.17\text{ (2 d.p.)}$$

18 a Let $1.5\sin 2x+2\cos 2x\equiv R\sin(2x+\alpha)$

$$\equiv R\sin 2x\cos\alpha+R\cos 2x\sin\alpha$$

$$R>0,\ 0<\alpha<\frac{\pi}{2}$$

Compare $\sin 2x: R\cos\alpha=1.5$ (1)

Compare $\cos 2x: R\sin\alpha=2$ (2)

Divide (2) by (1) : $\tan\alpha=\dfrac{4}{3}$

$$\Rightarrow\ \alpha=0.927\text{ (3 d.p.)}$$

$$R^2=2^2+1.5^2\ \Rightarrow\ R=2.5$$

b $3\sin x\cos x+4\cos^2 x$

$$\equiv\frac{3}{2}(2\sin x\cos x)+4\left(\frac{1+\cos 2x}{2}\right)$$

$$\equiv\frac{3}{2}\sin 2x+2+2\cos 2x$$

$$\equiv\frac{3}{2}\sin 2x+2\cos 2x+2$$

c From part (a) $1.5\sin 2x+2\cos 2x$

$$\equiv 2.5\sin(2x+0.927)$$

So maximum value of

$1.5\sin 2x+2\cos 2x=2.5\times 1=2.5$

So maximum value of

$3\sin x\cos x+4\cos^2 x=2.5+2=4.5$

19 a $\sin^2\dfrac{\theta}{2}=2\sin\theta$

$$\frac{1-\cos\theta}{2}=2\sin\theta$$

$$1-\cos\theta=4\sin\theta$$

$$4\sin\theta+\cos\theta=1$$

Let $4\sin\theta+\cos\theta=R\sin(\theta+\alpha)$

$$=R\sin\theta\cos\alpha+R\cos\theta\sin\alpha$$

So $R\cos\alpha=4$ and $R\sin\alpha=1$

$$\frac{R\sin\alpha}{R\cos\alpha}=\tan\alpha=\frac{1}{4}$$

$$\alpha=\tan^{-1}\left(\frac{1}{4}\right)=\tan^{-1}0.25=14.04\text{ (2 d.p.)}$$

$$R^2=4^2+1^2=\sqrt{17}$$

$$4\sin\theta+\cos\theta=\sqrt{17}\sin(\theta+14.04°)=1$$

b $\sqrt{17}\sin(\theta+14.04°)=1$, for $0°\le\theta\le 360°$

$$\sin(\theta+14.04°)=\frac{1}{\sqrt{17}}=0.24\text{ (2 d.p.)}$$

$\theta+14.04°=\sin^{-1}0.24=14.04°$, for

$14.04°\le\theta+14.04°\le 374.04°$

$$\theta+14.04°=14.04°, 165.96°, 374.04°$$

$$\theta=0°, 151.9°, 360°$$

20 a $2\cos\theta=1+3\sin\theta$

So $2\cos\theta-3\sin\theta=1$

Let $2\cos\theta-3\sin\theta=R\cos(\theta+\alpha)$

$$=R\cos\theta\cos\alpha-R\sin\theta\sin\alpha$$

So $R\cos\alpha=2$ and $R\sin\alpha=3$

$$\frac{R\sin\alpha}{R\cos\alpha}=\tan\alpha=\frac{3}{2}$$

$$\alpha=\tan^{-1}\left(\frac{3}{2}\right)=56.3°\text{ (1 d.p.)}$$

$$R^2=2^2+3^2=13$$

$$R=\sqrt{13}$$

So

$$2\cos\theta-3\sin\theta=\sqrt{13}\cos(\theta+56.3°)=1$$

20 b $\sqrt{13}\cos(\theta+56.3°)=1$, for $0°\le\theta\le 360°$

$\cos(\theta+56.3°)=\frac{1}{\sqrt{13}}$,

for $56.3°\le\theta+56.3°\le 416.3°$

$\theta+56.3°=73.9°, 286.1°$ (1 d.p.)

$\theta=17.6°, 229.8°$ (1 d.p.)

21 a $\text{LHS} \equiv \frac{1}{\cos\theta}\times\frac{1}{\sin\theta}\equiv\frac{1}{\frac{1}{2}\sin 2\theta}$

$\equiv\frac{2}{\sin 2\theta}\equiv 2\operatorname{cosec}2\theta\equiv$ RHS

b $\text{LHS} \equiv \frac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x}-\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}$

$\equiv\frac{1+\tan x}{1-\tan x}-\frac{1-\tan x}{1+\tan x}$

$\equiv\frac{(1+\tan x)^2-(1-\tan x)^2}{(1-\tan x)(1+\tan x)}$

$\equiv\frac{(1+2\tan x+\tan^2 x)}{1-\tan^2 x}$

$-\frac{(1-2\tan x+\tan^2 x)}{1-\tan^2 x}$

$\equiv\frac{4\tan x}{1-\tan^2 x}$

$\equiv 2\left(\frac{2\tan x}{1-\tan^2 x}\right)$

$\equiv 2\tan 2x\equiv$ RHS

c $\text{LHS} \equiv(\sin x\cos y+\cos x\sin y)$

$\times(\sin x\cos y-\cos x\sin y)$

$\equiv\sin^2 x\cos^2 y-\cos^2 x\sin^2 y$

$\equiv(1-\cos^2 x)\cos^2 y$

$-\cos^2 x(1-\cos^2 y)$

$\equiv\cos^2 y-\cos^2 x\cos^2 y$

$-\cos^2 x+\cos^2 x\cos^2 y$

$\equiv\cos^2 y-\cos^2 x=$ RHS

d $\text{LHS} \equiv 1+2\cos 2\theta+(2\cos^2 2\theta-1)$

$\equiv 2\cos 2\theta+2\cos^2 2\theta$

$\equiv 2\cos 2\theta(1+\cos 2\theta)$

$\equiv 2\cos 2\theta(2\cos^2\theta)$

$\equiv 4\cos^2\theta\cos 2\theta=\text{RHS}$

22 a $\text{LHS} \equiv\frac{1-\cos 2x}{1+\cos 2x}\equiv\frac{1-(1-2\sin^2 x)}{1+(2\cos^2 x-1)}$

$\equiv\frac{2\sin^2 x}{2\cos^2 x}\equiv\tan^2 x=\text{RHS}$

b $\tan^2 x=3$

$\tan x=\pm\sqrt{3}$, for $-\pi\le x\le\pi$

$\tan x=\sqrt{3}\Rightarrow x=\frac{\pi}{3},-\frac{2\pi}{3}$

$\tan x=-\sqrt{3}\Rightarrow x=-\frac{\pi}{3},\frac{2\pi}{3}$

$x=-\frac{2\pi}{3},-\frac{\pi}{3},\frac{\pi}{3},\frac{2\pi}{3}$

23 a $\text{LHS} \equiv\cos^4 2\theta-\sin^4 2\theta$

$\equiv(\cos^2 2\theta-\sin^2 2\theta)(\cos^2 2\theta+\sin^2 2\theta)$

$\equiv(\cos^2 2\theta-\sin^2 2\theta)(1)$

$\equiv\cos 4\theta\equiv$ RHS

b $\cos 4\theta=\frac{1}{2}$, for $0°\le 4\theta\le 720°$

$4\theta=60°, 300°, 420°, 660°$

$\theta=15°, 75°, 105°, 165°$

24 a $\text{LHS} \equiv\frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta}$

$\equiv\frac{2\sin^2\theta}{2\sin\theta\cos\theta}$

$\equiv\frac{\sin\theta}{\cos\theta}\equiv\tan\theta=$ RHS

b When $\theta=180°$, $\sin 2\theta=\sin 360°=0$ and $2-2\cos 360°=2-2=0$ therefore $\theta=180°$ is a solution of the equation $\sin 2\theta=2-2\cos 2\theta$

24 c Rearrange $\sin 2\theta = 2 - 2\cos 2\theta$ to give

$$\frac{2(1-\cos 2\theta)}{\sin 2\theta} = 1$$

Using the identity in part (a) gives

$$2\tan\theta = 1$$

$$\Rightarrow \tan\theta = \frac{1}{2}, \text{ for } 0 < \theta < 360°$$

$$\theta = 26.6°, 206.6° \text{ (1 d.p.)}$$

25 a Set $2\cos x - \sqrt{5}\sin x \equiv R\cos(x+\alpha)$

$$\equiv R\cos x\cos\alpha - R\sin x\sin\alpha$$

So $R\cos\alpha = 2$ and $R\sin\alpha = \sqrt{5}$

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{\sqrt{5}}{2}$$

$$\alpha = \tan^{-1}\left(\frac{\sqrt{5}}{2}\right) = 0.841 \text{ (3 d.p.)}$$

$$R^2 = 2^2 + \left(\sqrt{5}\right)^2 = 9$$

$$R = 3$$

$$2\cos x - \sqrt{5}\sin x \equiv 3\cos(x + 0.841)$$

b $3\cos(x+0.841) = -1$,

for $0.841 \le x + 0.841 < 2\pi + 0.841$

$$\cos(x+0.841) = -\frac{1}{3}$$

$$x + 0.841 = 1.911, 4.372$$

$$x = 1.07, 3.53 \text{ (2 d.p.)}$$

26 a Set $1.4\sin\theta - 5.6\cos\theta \equiv R\sin(\theta - \alpha)$

$$\equiv R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$$

So $R\cos\alpha = 1.4$ and $R\sin\alpha = 5.6$

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{5.6}{1.4}$$

$$\alpha = \tan^{-1} 4 == 75.964° \text{ (3 d.p.)}$$

$$R^2 = 1.4^2 + 5.6^2 = 33.32$$

$$R = 5.772 \text{ (3 d.p.)}$$

b The maximum value of $5.772\sin(\theta - 75.964)°$ is when $\sin(\theta - 75.964)° = 1$. So the maximum value is 5.772 and it occurs when $\theta - 75.964° = 90°$, $\theta = 165.964°$

c $12 - 5.6\cos\left(\frac{360t}{365}\right)° + 1.4\sin\left(\frac{360t}{365}\right)°$

$$\equiv 12 + 5.772\sin\left(\frac{360t}{365} - 75.964\right)°$$

The minimum number of daylight hours is when $\sin\left(\frac{360t}{365} - 75.964\right)° = -1$

So minimum is $12 - 5.772 = 6.228$ hours

d $\sin\left(\frac{360t}{365} - 75.964\right)° = -1$

$$\frac{360t}{365} - 75.964 = 270°$$

$$t = 351 \text{ days}$$

27 a Let $12\sin x + 5\cos x \equiv R\sin(x+\alpha)$

$$\equiv R\sin x\cos\alpha + R\cos x\sin\alpha$$

So $R\cos\alpha = 12$ and $R\sin\alpha = 5$

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{5}{12}$$

$$\alpha = \tan^{-1}\left(\frac{5}{12}\right) = 22.6° \text{ (1 d.p.)}$$

$$R^2 = 12^2 + 5^2 = 169$$

$$R = 13$$

So $12\sin x + 5\cos x = 13\sin(x + 22.6°)$

b $$\text{v}(x) = \frac{50}{12\sin\left(\frac{2x}{5}\right)° + 5\cos\left(\frac{2x}{5}\right)°}$$

$$= \frac{50}{13\sin\left(\frac{2x}{5} + 22.6°\right)}$$

The minimum value of v is when

$$\sin\left(\frac{2x}{5} + 22.6\right)° = 1$$

So $\frac{50}{13} = 3.85$ m/s (2 d.p.)

27 c $\sin\left(\frac{2x}{5}+22.6°\right)=1$, for

$$22.6° \le \frac{2x}{5}+22.6° \le 166.6°$$

$$\frac{2x}{5}+22.6°=90°$$

$$x=168.5 \text{ minutes}$$

Challenge

1 a Write $\cos 2\theta$ as $\cos(3\theta-\theta)$ and write $\cos 4\theta$ as $\cos(3\theta+\theta)$.

Then, using $\cos(A \pm B) = \cos A\cos B \mp \sin A\cos B$,

$$\cos 2\theta = \cos 3\theta\cos\theta + \sin 3\theta\sin\theta$$

$$\cos 4\theta = \cos 3\theta\cos\theta - \sin 3\theta\sin\theta$$

$$\Rightarrow \quad \cos 2\theta + \cos 4\theta = 2\cos 3\theta\cos\theta$$

Similarly, using $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$,

$$\sin 2\theta = \sin 3\theta\cos\theta - \cos 3\theta\sin\theta$$

$$\sin 4\theta = \sin 3\theta\cos\theta + \cos 3\theta\sin\theta$$

$$\Rightarrow \sin 2\theta - \sin 4\theta = -2\cos 3\theta\sin\theta$$

Therefore,

$$\frac{\cos 2\theta+\cos 4\theta}{\sin 2\theta-\sin 4\theta}=\frac{2\cos 3\theta\cos\theta}{-2\cos 3\theta\sin\theta}$$

$$=-\frac{\cos\theta}{\sin\theta}$$

$$=-\cot\theta \qquad \text{as required.}$$

$$\equiv\frac{2\cos\left(\frac{6\theta}{2}\right)\cos\left(\frac{2\theta}{2}\right)}{2\cos\left(\frac{6\theta}{2}\right)\sin\left(\frac{-2\theta}{2}\right)}$$

$$\equiv\frac{2\cos 3\theta\cos\theta}{2\cos 3\theta\sin(-\theta)}$$

$$\equiv\frac{\cos\theta}{\sin(-\theta)}\equiv-\cot\theta$$

b LHS $\equiv \cos x+2\cos 3x+\cos 5x$

$$\equiv \cos 5x+\cos x+2\cos 3x$$

$$\equiv 2\cos\left(\frac{6x}{2}\right)\cos\left(\frac{4x}{2}\right)+2\cos 3x$$

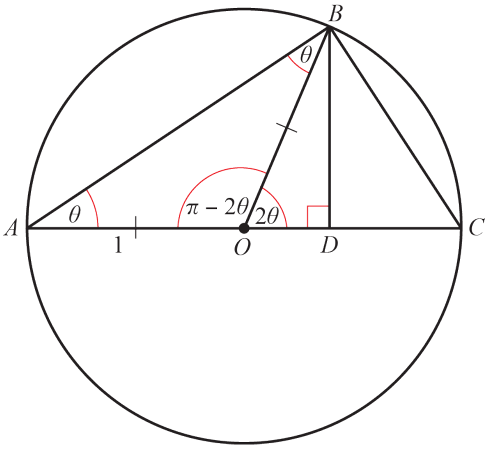
$$\equiv 2\cos 3x\cos 2x+2\cos 3x$$

$$\equiv 2\cos 3x(\cos 2x+1)$$

$$\equiv 2\cos 3x(2\cos^2 x)$$

$$\equiv 4\cos^2 x\cos 3x \equiv \text{ RHS}$$

2 a As $\angle OAB=\angle OBA \Rightarrow \angle AOB=\pi-2\theta$, so $\angle BOD=2\theta$



$OB=1$

$OD=\cos 2\theta$

$BD=\sin 2\theta$

$AB=2\cos\theta$

$$\sin\theta=\frac{BD}{AB}=\frac{BD}{2\cos\theta}$$

So $BD=2\sin\theta\cos\theta$

But $BD=\sin 2\theta$

So $\sin 2\theta\equiv 2\sin\theta\cos\theta$

b $AB=2\cos\theta$

$AD=(2\cos\theta)\cos\theta=2\cos^2\theta$

$OD=2\cos^2\theta-1$

From part **a**, $OD=\cos 2\theta$

So $\cos 2\theta\equiv 2\cos^2\theta-1$

Parametric equations 8A

1 a $x = t - 2$

so $t = x + 2 \quad (1)$

$y = t^2 + 1 \quad (2)$

Substitute (1) into (2):

$y = (x+2)^2 + 1$

$= x^2 + 4x + 4 + 1$

$\therefore\ y = x^2 + 4x + 5$

$x = t - 2,\ -4 \leq t \leq 4$

So the domain of f(x) is $-6 \leq x \leq 2$.

$y = t^2 + 1,\ -4 \leq t \leq 4$

So the range of f(x) is $1 \leq y \leq 17$.

b $x = 5 - t$

so $t = 5 - x \quad (1)$

$y = t^2 - 1 \quad (2)$

Substitute (1) into (2):

$y = (5 - x)^2 - 1$

$= 25 - 10x + x^2 - 1$

$\therefore\ y = x^2 - 10x + 24$

$x = 5 - t,\ t \in \mathbb{R}$

So the domain of f(x) is $x \in \mathbb{R}$.

$y = t^2 - 1,\ t \in \mathbb{R}$

So the range of f(x) is $y \geq -1$.

c $x = \frac{1}{t}$

so $t = \frac{1}{x} \quad (1)$

$y = 3 - t \quad (2)$

Substitute (1) into (2):

$y = 3 - \frac{1}{x}$

$x = \frac{1}{t},\ t \neq 0$

So the domain of f(x) is $x \neq 0$.

$y = 3 - t,\ t \neq 0$

Range of f(x) is $y \neq 3$.

d $x = 2t + 1$

so $t = \frac{x-1}{2} \quad (1)$

$y = \frac{1}{t} \quad (2)$

Substitute (1) into (2):

$$y = \frac{1}{\frac{x-1}{2}}$$

$$y = \frac{2}{x-1}$$

$x = 2t + 1,\ t > 0$

So the domain of f(x) is $x > 1$.

$y = \frac{1}{t},\ t > 0$

So the range of f(x) is $y > 0$.

e $x = \frac{1}{t-2}$

so $t - 2 = \frac{1}{x}$

$t = 2 + \frac{1}{x} \quad (1)$

$y = t^2 \quad (2)$

Substitute (1) into (2):

$$y = \left(2 + \frac{1}{x}\right)^2$$

$$y = \left(\frac{2x+1}{x}\right)^2$$

$x = \frac{1}{t-2},\ t > 2$

So the domain of f(x) is $x > 0$.

$y = t^2,\ t > 2$

So the range of f(x) is $y > 4$.

1 f $x = \dfrac{1}{t+1}$

so $t + 1 = \dfrac{1}{x}$

$$t = \frac{1}{x} - 1 \qquad (1)$$

$$y = \frac{1}{t-2} \qquad (2)$$

Substitute (1) into (2):

$$y = \frac{1}{\frac{1}{x} - 1 - 2}$$

$$= \frac{1}{\frac{1}{x} - 3}$$

$$= \frac{1}{\frac{1-3x}{x}}$$

$$\therefore\ y = \frac{x}{1-3x}$$

$x = \dfrac{1}{t+1},\ t > 2$

So the domain of f(x) is $0 < x < \dfrac{1}{3}$

$y = \dfrac{1}{t-2},\ t > 2,\ t > 2$

So the range of f(x) is $y > 0$.

2 a i $x = 2\ln(5-t)$

$\dfrac{1}{2}x = \ln(5-t)$

$e^{\frac{1}{2}x} = 5 - t$

So $t = 5 - e^{\frac{1}{2}x}$

Substitute $t = 5 - e^{\frac{1}{2}x}$ into $y = t^2 - 5$:

$$y = (5 - e^{\frac{1}{2}x})^2 - 5$$

$$= 25 - 10e^{\frac{1}{2}x} + e^x - 5$$

$$= 20 - 10e^{\frac{1}{2}x} + e^x$$

$x = 2\ln(5-t),\ t < 4$

When $t = 4,\ x = 2\ln 1 = 0$

and as t increases $2\ln(5-t)$ decreases.

So the range of the parametric function for x is $x > 0$.

Hence the Cartesian equation is

$y = 20 - 10e^{\frac{1}{2}x} + e^x,\ x > 0$

ii $y = t^2 - 5,\ t < 4$

$y = t^2 - 5$ is a quadratic function with minimum value −5 at $t = 0$.

So the range of the parametric function for y is $y \geq -5$.

Hence the range of f(x) is $y \geq -5$.

b i $x = \ln(t+3)$

$e^x = t + 3$

$e^x - 3 = t$

Substitute $t = e^x - 3$ into $y = \dfrac{1}{t+5}$:

$$y = \frac{1}{e^x - 3 + 5} = \frac{1}{e^x + 2}$$

$x = \ln(t+3),\ t > -2$

When $t = -2,\ x = \ln 1 = 0$

and as t increases, $\ln(t+3)$ increases.

So the range of the parametric function for x is $x > 0$.

Hence the Cartesian equation is

$y = \dfrac{1}{e^x + 2},\ x > 0$

2 b ii $y = \dfrac{1}{t+5},\ t > -2$

When $t = -2,\ y = \dfrac{1}{3}$

and as t increases, $\dfrac{1}{t+5}$ decreases towards zero.

So the range of the parametric function for y is $0 < y < \dfrac{1}{3}$

Hence the range of f(x) is $0 < y < \dfrac{1}{3}$

c i $x = \mathrm{e}^t$

So $y = \mathrm{e}^{3t} = (\mathrm{e}^t)^3 = x^3$

(Note that since y is a power of x there is no need to substitute for t.)

$x = \mathrm{e}^t,\ t \in \mathbb{R}$

The range of the parametric function for x is $x > 0$.

Hence the Cartesian equation is

$y = x^3,\ x > 0$

ii $y = \mathrm{e}^{3t},\ t \in \mathbb{R}$

The range of the parametric function for y is $y > 0$.

Hence the range of f(x) is $y > 0$.

3 a $x = \sqrt{t}$

so $x^2 = t$

Substitute $t = x^2$ into $y = t(9-t)$:

$$y = x^2(9-x^2)$$
$$= 9x^2 - x^4$$

$x = \sqrt{t},\ 0 \le t \le 5$

The range of the parametric function for x is $0 \le x \le \sqrt{5}$.

Hence the Cartesian equation is

$y = 9x^2 - x^4,\ 0 \le x \le \sqrt{5}$

$y = t(9-t),\ 0 \le t \le 5$

When $t = 0$, $y = 0$;

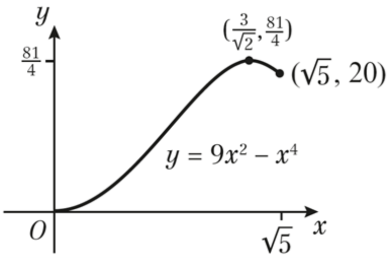
when $t = 5$, $y = 20$;

and $y = t(9-t)$ is a quadratic function with maximum value $\dfrac{81}{4}$ at $t = \dfrac{9}{2}$

So the range of the parametric function for y is $0 \le y \le \dfrac{81}{4}$

Hence the range of f(x) is $0 \le y \le \dfrac{81}{4}$

b



4 a i $x = 2t^2 - 3$

$x + 3 = 2t^2$

$\frac{x+3}{2} = t^2$

$\pm\sqrt{\frac{x+3}{2}} = t$

Take the positive root since $t > 0$.

Substitute $t = \sqrt{\frac{x+3}{2}}$ into $y = 9 - t^2$:

$y = 9 - \left(\sqrt{\frac{x+3}{2}}\right)^2 = 9 - \frac{x+3}{2}$

$= \frac{18 - x - 3}{2} = \frac{15 - x}{2}$

The Cartesian equation is $y = \frac{15}{2} - \frac{1}{2}x$

ii $x = 2t^2 - 3,\ t > 0$

$2t^2 - 3$ is a quadratic function with minimum value -3 at $t = 0$.

The range of the parametric function for x is $x > -3$.

Hence the domain of f(x) is $x > -3$.

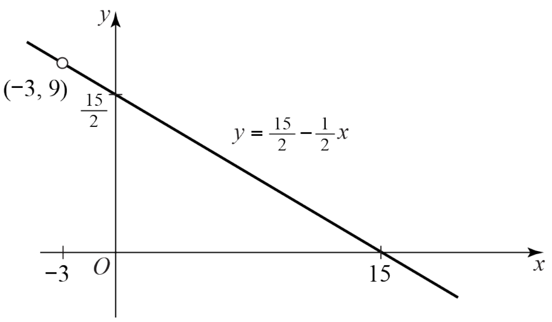
$y = 9 - t^2,\ t > 0$

$y = 9 - t^2$ is a quadratic function with maximum value 9 at $t = 0$.

So the range of the parametric function for y is $y < 9$.

Hence the range of f(x) is $y < 9$.

iii



b i $x = 3t - 1$

$x + 1 = 3t$

$\frac{x+1}{3} = t$

Substitute $t = \frac{x+1}{3}$ into $y = (t-1)(t+2)$:

$y = \left(\frac{x+1}{3} - 1\right)\left(\frac{x+1}{3} + 2\right)$

$= \left(\frac{x+1-3}{3}\right)\left(\frac{x+1+6}{3}\right)$

$= \left(\frac{x-2}{3}\right)\left(\frac{x+7}{3}\right)$

The Cartesian equation is

$y = \frac{1}{9}(x-2)(x+7)$

ii $x = 3t - 1,\ -4 < t < 4$

When $t = -4$, $x = -13$;

when $t = 4$, $x = 11$.

The range of the parametric function for x is $-13 < x < 11$.

So the domain of f(x) is $-13 < x < 11$.

$y = (t-1)(t+2),\ -4 < t < 4$

When $t = -4$, $y = 10$;

when $t = 4$, $y = 18$;

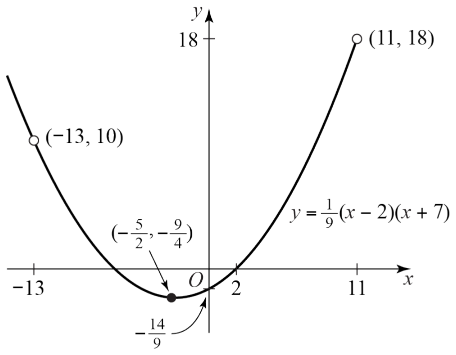
and $(t-1)(t+2)$ is a quadratic function with minimum value $-\frac{9}{4}$ at $t = -0.5$.

The range of the parametric function for y is $-\frac{9}{4} \le y < 18$.

Hence the range of f(x) is $-\frac{9}{4} \le y < 18$.

Note: Due to symmetry, the minimum value of y occurs midway between the roots $t = 1$ and $t = -2$, i.e. at $t = -0.5$.

4 b iii



c i $x = t + 1$

$x - 1 = t$

Substitute $t = x - 1$ into $y = \frac{1}{t-1}$:

$$y = \frac{1}{x-1-1} = \frac{1}{x-2}$$

The Cartesian equation is

$$y = \frac{1}{x-2}$$

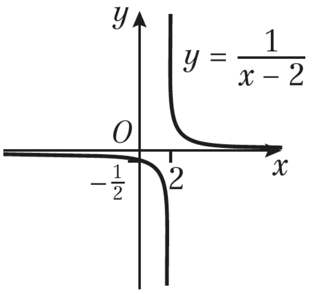
ii $x = t + 1,\ t \in \mathbb{R},\ t \neq 1$

So the domain of f(*x*) is $x \in \mathbb{R},\ x \neq 2$.

$y = \frac{1}{t-1},\ t \in \mathbb{R},\ t \neq 1$

So the range of f(*x*) is $y \in \mathbb{R},\ y \neq 0$.

iii



d i $x = \sqrt{t} - 1$

$x + 1 = \sqrt{t}$

$(x+1)^2 = t$

Substitute $t = (x+1)^2$ into $y = 3\sqrt{t}$:

$y = 3\sqrt{(x+1)^2} = 3(x+1)$

The Cartesian equation is $y = 3x + 3$

ii $x = \sqrt{t} - 1,\ t > 0$

When $t = 0$, $x = -1$

and as *t* increases $\sqrt{t} - 1$ increases.

The range of the parametric function for *x* is $x > -1$.

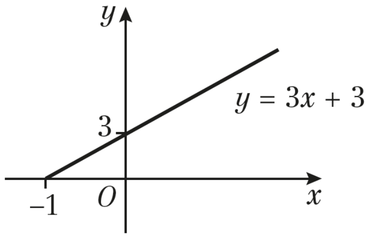
So the domain of f(*x*) is $x > -1$.

$y = 3\sqrt{t},\ t > 0$

The range of the parametric function for *y* is $y > 0$.

So the range of f(*x*) is $y > 0$.

d iii



e i $x = \ln(4 - t)$

$e^x = 4 - t$

$t = 4 - e^x$

Substitute $t = 4 - e^x$ into $y = t - 2$:

$y = 4 - e^x - 2 = 2 - e^x$

The Cartesian equation is $y = 2 - e^x$

ii $x = \ln(4 - t),\ t < 3$

When $t = 3$, $x = \ln 1 = 0$

and as *t* decreases $\ln(4 - t)$ increases.

So the domain of f(*x*) is $x > 0$.

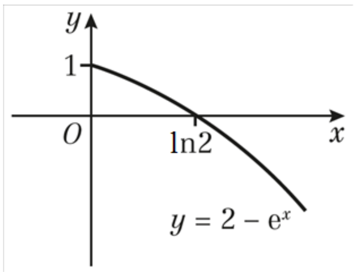
$y = t - 2,\ t < 3$

When $t = 3$, $y = 1$

and as *t* decreases $t - 2$ decreases.

So the range of f(*x*) is $y < 1$.

4 e iii



5 a C_1: $x = 1 + 2t$

$$\Rightarrow \frac{x-1}{2} = t$$

Substitute $t = \frac{x-1}{2}$ into $y = 2 + 3t$:

$$y = 2 + 3\left(\frac{x-1}{2}\right)$$

$$= \frac{4+3x-3}{2} = \frac{3x+1}{2}$$

So the Cartesian equation of C_1 is

$$y = \frac{3}{2}x + \frac{1}{2}$$

C_2: $x = \frac{1}{2t-3}$

$$2t - 3 = \frac{1}{x}$$

$$2t = 3 + \frac{1}{x} = \frac{3x+1}{x}$$

$$\therefore\ t = \frac{3x+1}{2x}$$

and $y = \frac{t}{2t-3} = t\left(\frac{1}{2t-3}\right)$

Substitute $t = \frac{3x+1}{2x}$ and $x = \frac{1}{2t-3}$

into $y = t\left(\frac{1}{2t-3}\right)$:

$$y = \left(\frac{3x+1}{2x}\right)x = \frac{3x+1}{2}$$

So the Cartesian equation of C_2 is

$$y = \frac{3}{2}x + \frac{1}{2}$$

Therefore C_1 and C_2 are segments of the same line $y = \frac{3}{2}x + \frac{1}{2}$

5 b For the length of each segment find the domain and range of C_1 and C_2.

For C_1: $x = 1 + 2t,\ 2 < t < 5$

When $t = 2, x = 5$;
when $t = 5, x = 11$.
The range of the parametric function for x is $5 < x < 11$,
so the domain of C_1 is $5 < x < 11$.

$y = 2 + 3t,\ 2 < t < 5$
When $t = 2, y = 8$;
when $t = 5, y = 17$.
The range of the parametric function for y is $8 < y < 17$,
so the range of C_1 is $8 < y < 17$.

The endpoints of C_1 have coordinates (5, 8) and (11, 17).

$$\therefore \text{length of } C_1 = \sqrt{(11-5)^2 + (17-8)^2} = \sqrt{36+81} = \sqrt{117} = 3\sqrt{13}$$

For C_2: $x = \dfrac{1}{2t-3},\ 2 < t < 3$

When $t = 2, x = 1$;
when $t = 3,\ x = \dfrac{1}{3}$.
The range of the parametric function for x is $\dfrac{1}{3} < x < 1$,
so the domain of C_2 is $\dfrac{1}{3} < x < 1$.

$y = \dfrac{t}{2t-3},\ 2 < t < 3$

When $t = 2, y = 2$;
when $t = 3, y = 1$.
The range of the parametric function for y is $1 < y < 2$,
so the range of C_2 is $1 < y < 2$.
The endpoints of C_2 have coordinates \` $\left(\dfrac{1}{3}, 1\right)$ and (1, 2).

$$\therefore \text{length of } C_2 = \sqrt{\left(1-\frac{1}{3}\right)^2 + (2-1)^2} = \sqrt{\frac{4}{9}+1} = \sqrt{\frac{4+9}{9}} = \frac{\sqrt{13}}{3}$$

6 a $x = \dfrac{3}{t} + 2,\ t \neq 0$

The range of the parametric function for x is $x \neq 2$.
(This is also the domain of the Cartesian equation $y = f(x)$.)

$y = 2t - 3 - t^2,\ t \neq 0$
When $t = 0, y = -3$;
$2t - 3 - t^2$ is a quadratic function with maximum value -2 at $t = 1$.
The range of the parametric function for y is $y \leq -2,\ y \neq -3$.
(This is also the range of the Cartesian equation $y = f(x)$.)

Note: To find the maximum point of the quadratic $y = 2t - 3 - t^2$,

either solve $\dfrac{dy}{dt} = 0$

$$2 - 2t = 0$$
$$2 = 2t$$
$$t = 1$$
$$\therefore y = 2(1) - 3 - (1)^2 = -2$$

or complete the square

$$y = -\left((t-1)^2 - 1 + 3\right) = -\left((t-1)^2 + 2\right) = -(t-1)^2 - 2$$

6 b $x = \frac{3}{t} + 2$

$$x - 2 = \frac{3}{t}$$

$$t = \frac{3}{x-2}$$

Substitute $t = \frac{3}{x-2}$ into $y = 2t - 3 - t^2$:

$$y = 2\left(\frac{3}{x-2}\right) - 3 - \left(\frac{3}{x-2}\right)^2$$

$$= \frac{6(x-2) - 3(x-2)^2 - 3^2}{(x-2)^2}$$

$$= -3\left(\frac{-2(x-2) + (x-2)^2 + 3}{(x-2)^2}\right)$$

$$= -3\left(\frac{-2x + 4 + x^2 - 4x + 4 + 3}{(x-2)^2}\right)$$

$$= \frac{-3(x^2 - 6x + 11)}{(x-2)^2}$$

This is a Cartesian equation in the form

$y = \frac{A(x^2 + bx + c)}{(x-2)^2}$ with

$A = -3$, $b = -6$ and $c = 11$.

7 a $x = \ln(t+3),\ t > -2$

$$e^x = t + 3$$

$$e^x - 3 = t$$

Substitute $t = e^x - 3$ into $y = \frac{1}{t+5}$:

$$y = \frac{1}{e^x - 3 + 5} = \frac{1}{e^x + 2}$$

When $t = -2,\ x = \ln 1 = 0$

and as t increases $\ln(t+3)$ increases.

The range of the parametric function for x is $x > 0$,

so the domain of f(x) is $x > 0$.

Therefore the Cartesian equation is

$y = \frac{1}{e^x + 2},\ x > k$ where $k = 0$.

b $y = \frac{1}{t+5},\ t > -2$

When $t = -2,\ y = \frac{1}{3}$

and as t increases, $\frac{1}{t+5}$ decreases towards zero.

The range of the parametric function for y is $0 < y < \frac{1}{3}$

so the range of f(x) is $0 < y < \frac{1}{3}$

8 a $x = 3\sqrt{t}$

$$\frac{x}{3} = \sqrt{t}$$

$$\frac{x^2}{9} = t$$

Substitute $t = \frac{x^2}{9}$ into $y = t^3 - 2t$:

$$y = \left(\frac{x^2}{9}\right)^3 - 2\left(\frac{x^2}{9}\right) = \frac{x^6}{729} - \frac{2x^2}{9}$$

The Cartesian equation is

$$y = \frac{x^6}{729} - \frac{2x^2}{9}$$

$x = 3\sqrt{t},\ 0 \le t \le 2$

When $t = 0$, $x = 0$;

when $t = 2,\ x = 3\sqrt{2}$.

The range of the parametric function for x is $0 \le x \le 3\sqrt{2}$

so the domain of f(x) is $0 \le x \le 3\sqrt{2}$.

b $\frac{dy}{dt} = 3t^2 - 2$

$\frac{dy}{dt} = 0$ when $3t^2 - 2 = 0$

$$3t^2 = 2$$

$$t^2 = \frac{2}{3}$$

$$t = \sqrt{\frac{2}{3}} \quad (\text{as } 0 \le t \le 2)$$

8 c $\frac{d^2y}{dt^2} = 6t$

When $t = \sqrt{\frac{2}{3}}$, $\frac{d^2y}{dt^2} = 6\left(\sqrt{\frac{2}{3}}\right) > 0$

So $t = \sqrt{\frac{2}{3}}$ gives a minimum point of the parametric function for y.
The minimum value of y is

$$\left(\sqrt{\frac{2}{3}}\right)^3 - 2\left(\sqrt{\frac{2}{3}}\right)$$

$$= \frac{2}{3}\sqrt{\frac{2}{3}} - \frac{6}{3}\sqrt{\frac{2}{3}} = -\frac{4}{3}\frac{\sqrt{2}}{\sqrt{3}} = -\frac{4\sqrt{6}}{9}$$

When $t = 0$, $y = 0$;
when $t = 2$, $y = 4$.
The range of the parametric function for y is $-\frac{4\sqrt{6}}{9} \leq y \leq 4$.
Therefore the range of f(x) is

$$-\frac{4\sqrt{6}}{9} \leq f(x) \leq 4.$$

9 a $x = t^3 - t = t(t^2 - 1)$

$\Rightarrow x^2 = t^2(t^2 - 1)^2$ (1)

$y = 4 - t^2 \Rightarrow t^2 = 4 - y$ (2)

Substitute (2) into (1):

$x^2 = (4 - y)(4 - y - 1)^2$

$x^2 = (4 - y)(3 - y)^2$

This is in the form $x^2 = (a - y)(b - y)^2$ with $a = 4$ and $b = 3$.

b $y = 4 - t^2$, $t \in \mathbb{R}$

This is a quadratic function of t, and (by symmetry) the maximum value of y occurs at $t = 0$, where $y = 4$.
So 4 is the maximum y-coordinate.

Challenge

a Squaring the parametric functions gives

$$x^2 = \left(\frac{1-t^2}{1+t^2}\right)^2 \qquad (1)$$

$$y^2 = \left(\frac{2t}{1+t^2}\right)^2 \qquad (2)$$

Add (1) and (2):

$$x^2 + y^2 = \left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2$$

$$= \frac{(1-t^2)^2 + 4t^2}{(1+t^2)^2}$$

$$= \frac{1 - 2t^2 + t^4 + 4t^2}{(1+t^2)^2}$$

$$= \frac{1 + 2t^2 + t^4}{(1+t^2)^2}$$

$$= \frac{(1+t^2)^2}{(1+t^2)^2} = 1$$

So a Cartesian equation for curve C is $x^2 + y^2 = 1$.

b $x^2 + y^2 = 1$

$\Rightarrow (x-0)^2 + (y-0)^2 = 1$

Curve C is the equation of a circle with centre (0, 0) and radius 1.

Parametric equations 8B

1 a $x = 2\sin t - 1$

So $\sin t = \dfrac{x+1}{2}$ $\quad$ (1)

$y = 5\cos t + 4$

$\cos t = \dfrac{y-4}{5}$ $\quad$ (2)

Substitute (1) and (2) into $\sin^2 t + \cos^2 t \equiv 1$:

$$\left(\frac{x+1}{2}\right)^2 + \left(\frac{y-4}{5}\right)^2 = 1$$

$$\frac{(x+1)^2}{4} + \frac{(y-4)^2}{25} = 1$$

$$25(x+1)^2 + 4(y-4)^2 = 100$$

b $y = \sin 2t$

$= 2\sin t\cos t$

So, since $x = \cos t$,

$y = 2x\sin t$ $\quad$ (1)

$\sin^2 t + \cos^2 t \equiv 1$

$\sin^2 t \equiv 1 - \cos^2 t = 1 - x^2$

$\sin t = \sqrt{1-x^2}$ $\quad$ (2)

Substitute (2) into (1):

$y = 2x\sqrt{1-x^2}$

or $y^2 = 4x^2(1-x^2)$

c $y = 2\cos 2t$

$= 2(2\cos^2 t - 1)$

So, since $x = \cos t$,

$y = 2(2x^2 - 1)$

$y = 4x^2 - 2$

d $y = \tan 2t$

So $y = \dfrac{2\tan t}{1-\tan^2 t}$ $\quad$ (1)

$\sin^2 t + \cos^2 t \equiv 1$

$\cos^2 t \equiv 1 - \sin^2 t = 1 - x^2$

$\cos t = \sqrt{1-x^2}$ $\quad$ (2)

Substitute (2) and $x = \sin t$ into (1):

$$y = \frac{2\dfrac{\sin t}{\cos t}}{1 - \dfrac{\sin^2 t}{\cos^2 t}} = \frac{\dfrac{2x}{\sqrt{1-x^2}}}{1 - \dfrac{x^2}{1-x^2}} = \frac{\dfrac{2x}{\sqrt{1-x^2}}}{\dfrac{1-2x^2}{1-x^2}}$$

$$= \frac{2x(1-x^2)}{(1-2x^2)\sqrt{1-x^2}}$$

Hence $y = \dfrac{2x\sqrt{1-x^2}}{1-2x^2}$

e $x = \cos t + 2$

$\cos t = x - 2$ $\quad$ (1)

$y = \sec t = \dfrac{4}{\cos t}$

$\cos t = \dfrac{4}{y}$ $\quad$ (2)

Substitute (1) into (2):

$x - 2 = \dfrac{4}{y}$

$y = \dfrac{4}{x-2}$

f $x = 3\cot t$

$\cot t = \dfrac{x}{3}$ $\quad$ (1)

$\operatorname{cosec} t = y$ $\quad$ (2)

Substitute (1) and (2) into $1 + \cot^2 t \equiv \operatorname{cosec}^2 t$:

$$1 + \left(\frac{x}{3}\right)^2 = y^2$$

$$y^2 = 1 + \frac{x^2}{9}$$

2 **a** $x = \sin t - 5$

$\Rightarrow \sin t = x + 5 \qquad (1)$

$y = \cos t + 2$

$\Rightarrow \cos t = y - 2 \qquad (2)$

Substitute (1) and (2) into

$\sin^2 t + \cos^2 t \equiv 1$:

$(x+5)^2 + (y-2)^2 = 1$

b This is a circle with centre (−5, 2) and radius 1

c One full revolution around the circle is obtained for an interval of *t* corresponding to one period of both parametric equations $y = \cos t + 2$ and $x = \sin t - 5$.

So $0 \leq t \leq 2\pi$ is a suitable domain.

3 $x = 4\sin t + 3$

$4\sin t = x - 3$

$\therefore \sin t = \frac{x-3}{4} \qquad (1)$

$y = 4\cos t - 1$

$4\cos t = y + 1$

$\therefore \cos t = \frac{y+1}{4} \qquad (2)$

Substitute (1) and (2) into

$\sin^2 t + \cos^2 t = 1$:

$$\left(\frac{x-3}{4}\right)^2 + \left(\frac{y+1}{4}\right)^2 = 1$$

$$\frac{(x-3)^2}{4^2} + \frac{(y+1)^2}{4^2} = 1$$

$$\frac{(x-3)^2}{16} + \frac{(y+1)^2}{16} = 1$$

$(x-3)^2 + (y+1)^2 = 16$

So the radius of the circle is 4 and the centre is (3, −1).

4 $x = \cos t - 2$

$\Rightarrow \cos t = x + 2 \qquad (1)$

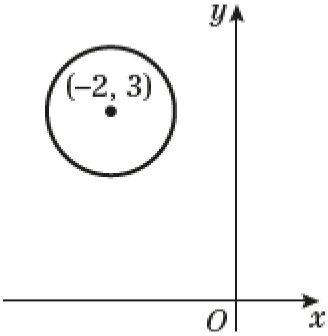
$y = \sin t + 3$

$\Rightarrow \sin t = y - 3 \qquad (2)$

Substitute (1) and (2) into $\sin^2 t + \cos^2 t \equiv 1$:

$(x+2)^2 + (y-3)^2 = 1$

This is a circle with centre (−2, 3) and radius 1:



5 **a** $y = \sin\left(t + \frac{\pi}{4}\right)$

$$= \sin t \cos\frac{\pi}{4} + \cos t \sin\frac{\pi}{4}$$

$$= \frac{\sqrt{2}}{2}\sin t + \frac{\sqrt{2}}{2}\cos t$$

$$y = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}\cos t \qquad (1)$$

(since $x = \sin t$)

$\sin^2 t + \cos^2 t \equiv 1$

$\cos^2 t \equiv 1 - \sin^2 t = 1 - x^2$

$\therefore \cos t = \sqrt{1-x^2} \qquad (2)$

Substitute (2) into (1):

$$y = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}\sqrt{1-x^2}$$

$$\text{or } y = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2(1-x^2)}}{2}$$

$x = \sin t,\ -\frac{\pi}{2} < t < \frac{\pi}{2}$

$\Rightarrow \ -1 < x < 1$

5 b $x = 3\cos t$

$$\Rightarrow \cos t = \frac{x}{3}$$

$$y = 2\cos\left(t + \frac{\pi}{6}\right)$$

$$= 2\cos t\cos\frac{\pi}{6} - 2\sin t\sin\frac{\pi}{6}$$

$$= 2\cos t \times \frac{\sqrt{3}}{2} - 2\sin t \times \frac{1}{2}$$

$$= \sqrt{3}\cos t - \sin t$$

So $y = \frac{\sqrt{3}}{3}x - \sin t$ (1)

$$\sin^2 t + \cos^2 t \equiv 1$$

$$\sin^2 t \equiv 1 - \cos^2 t = 1 - \left(\frac{x}{3}\right)^2$$

$\sin t = \sqrt{1 - \left(\frac{x}{3}\right)^2}$ (2)

Substitute (2) into (1):

$$y = \frac{\sqrt{3}}{3}x - \sqrt{1 - \left(\frac{x}{3}\right)^2}$$

$$= \frac{\sqrt{3}}{3}x - \sqrt{\frac{9 - x^2}{9}}$$

$$\therefore\ y = \frac{\sqrt{3}}{3}x - \frac{\sqrt{9 - x^2}}{3}$$

$$x = 3\cos t,\quad 0 < t < \frac{\pi}{3}$$

$$\Rightarrow\ \frac{3}{2} < x < 3$$

c $y = 3\sin(t + \pi)$

$$= 3\sin t\cos\pi + 3\cos t\sin\pi$$

$$= 3\sin t \times (-1) + 3\cos t \times 0$$

$$= -3\sin t$$

Since $x = \sin t$,

$y = -3x$

$$x = \sin t,\quad 0 < t < 2\pi$$

$$\Rightarrow -1 < x < 1$$

6 a $x = 8\cos t$

$$\cos t = \frac{x}{8}$$

So $y = \frac{1}{4}\sec^2 t = \frac{1}{4\cos^2 t}$

$$= \frac{1}{4\left(\frac{x}{8}\right)^2} = \frac{1}{4} \times \frac{64}{x^2} = \frac{16}{x^2}$$

Therefore a Cartesian equation for C is

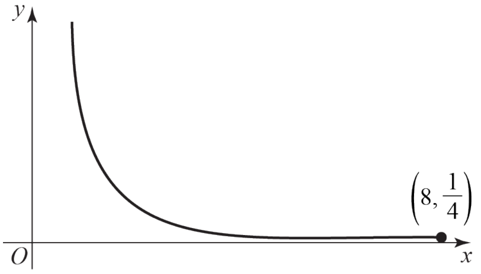
$$y = \frac{16}{x^2}$$

$$x = 8\cos t,\quad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

$$\Rightarrow\ 0 < x < 8$$

b For $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$ the range of the parametric equation $x = 8\cos t$ is $0 \leq x \leq 8$, so the domain of $y = \mathrm{f}(x)$ is $0 \leq x \leq 8$.

The range of the parametric equation $y = \frac{1}{4}\sec^2 t$ is $y \geq \frac{1}{4}$, so the range of $y = \mathrm{f}(x)$ is $y \geq \frac{1}{4}$



7 $x = 3\cot^2 2t$

$$\cot^2 2t = \frac{x}{3}$$

$$\frac{x}{3} = \frac{\cos^2 2t}{\sin^2 2t} = \frac{1-\sin^2 2t}{\sin^2 2t} = \frac{1}{\sin^2 2t} - 1$$

$$\frac{x}{3} + 1 = \frac{1}{\sin^2 2t}$$

$$\frac{x+3}{3} = \frac{1}{\sin^2 2t}$$

$$\sin^2 2t = \frac{3}{x+3}$$

$$\therefore\ y = 3\sin^2 2t = 3\times\frac{3}{x+3} = \frac{9}{x+3}$$

For $0 < t \le \frac{\pi}{4}$ the range of the parametric function $x = 3\cot^2 2t$ is $x \ge 0$, so the domain of f(x) is $x \ge 0$.

8 **a** $x = \frac{1}{3}\sin t$

$$\Rightarrow \sin t = 3x$$

$$y = \sin 3t = \sin(t + 2t)$$

$$= \sin t\cos 2t + \cos t\sin 2t$$

$$= \sin t(1 - 2\sin^2 t) + \cos t(2\sin t\cos t)$$

$$= \sin t(1 - 2\sin^2 t) + 2\sin t(1 - \sin^2 t)$$

$$= 3x(1 - 2\times 9x^2) + 6x(1 - 9x^2)$$

$$= 3x - 54x^3 + 6x - 54x^3$$

$$= 9x - 108x^3$$

$$= 9x(1 - 12x^2)$$

So the Cartesian equation of the curve is $y = 9x(1-12x^2)$, which is in the form $y = ax(1-bx^2)$ with a = 9 and b = 12.

b For $0 < t < \frac{\pi}{2}$ the range of the parametric function $x = \frac{1}{3}\sin t$ is $0 < x < \frac{1}{3}$

so the domain of y = f(x) is $0 < x < \frac{1}{3}$

For $0 < t < \frac{\pi}{2}$ the range of the parametric function $y = \sin 3t$ is $-1 < y < 1$

so the range of y = f(x) is $-1 < y < 1$.

9 $x = 2\cos t \Rightarrow \cos t = \frac{x}{2}$

$$y = \sin\left(t - \frac{\pi}{6}\right)$$

$$= \sin t\cos\frac{\pi}{6} - \cos t\sin\frac{\pi}{6}$$

$$= \frac{\sqrt{3}}{2}\sin t - \frac{1}{2}\cos t$$

$$\therefore\ y = \frac{\sqrt{3}}{2}\sin t - \frac{1}{4}x \qquad (1)$$

$$\sin^2 t + \cos^2 t \equiv 1$$

$$\sin^2 t \equiv 1 - \cos^2 t = 1 - \frac{x^2}{4}$$

$$\therefore\ \sin t = \sqrt{1 - \frac{x^2}{4}} \qquad (2)$$

Substitute (2) into (1):

$$y = \frac{\sqrt{3}}{2}\sqrt{1 - \frac{x^2}{4}} - \frac{1}{4}x$$

$$= \frac{1}{2}\sqrt{\frac{12 - 3x^2}{4}} - \frac{1}{4}x$$

So the Cartesian equation is

$$y = \frac{1}{4}\left(\sqrt{12 - 3x^2} - x\right)$$

For 0 < t < π, the range of the parametric function $x = 2\cos t$ is $-2 < x < 2$, so the domain of y = f(x) is $-2 < x < 2$.

10 a $y = 5\sin t$

So $\sin t = \dfrac{y}{5}$

$$\sin^2 t = \frac{y^2}{25}$$

$$x = \tan^2 t + 5$$

$$\tan^2 t = x - 5$$

$$\frac{\sin^2 t}{\cos^2 t} = x - 5$$

$$\frac{\sin^2 t}{1 - \sin^2 t} = x - 5$$

$$\frac{1}{x-5} = \frac{1}{\sin^2 t} - 1$$

$$\frac{1}{x-5} + 1 = \frac{1}{\frac{y^2}{25}}$$

$$\frac{x-4}{x-5} = \frac{25}{y^2}$$

$$\therefore\ y^2 = 25\left(\frac{x-5}{x-4}\right) = 25\left(1 - \frac{1}{x-4}\right)$$

b For $0 < t < \dfrac{\pi}{2}$, the range of the parametric function $x = \tan^2 t + 5$ is $x > 5$,
so the domain of the curve is $x > 5$.
The range of the parametric function $y = 5\sin t$ is $0 < y < 5$,
so the range of the curve is $0 < y < 5$.

11
$$\begin{aligned} y &= 3\sin(t - \pi) \\ &= 3\sin t\cos\pi - 3\cos t\sin\pi \\ &= 3\sin t \times (-1) - 3\cos t \times 0 \\ &= -3\sin t \end{aligned}$$

So $\sin t = -\dfrac{y}{3}$

$$\sin^2 t + \cos^2 t \equiv 1$$

$$\cos^2 t \equiv 1 - \sin^2 t$$

$$\cos t = \sqrt{1 - \sin^2 t} = \sqrt{1 - \frac{y^2}{9}}$$

$$\therefore\ x = \tan t = \frac{\sin t}{\cos t} = \frac{-\frac{y}{3}}{\sqrt{1 - \frac{y^2}{9}}} = \frac{-y}{\sqrt{9 - y^2}}$$

Therefore $x = -\dfrac{y}{\sqrt{9 - y^2}}$ is a Cartesian equation for C.

Challenge

$$x = \frac{1}{2}\cos 2t$$

$$= \frac{1}{2}(2\cos^2 t - 1)$$

$$\therefore \frac{2x+1}{2} = \cos^2 t$$

$$\cos t = \sqrt{\frac{2x+1}{2}}$$

But also

$$x = \frac{1}{2}\cos 2t$$

$$= \frac{1}{2}(1 - 2\sin^2 t)$$

$$\therefore \sin^2 t = \frac{1-2x}{2}$$

$$\sin t = \sqrt{\frac{1-2x}{2}}$$

Therefore

$$y = \sin\left(t + \frac{\pi}{6}\right)$$

$$y = \sin t \cos\frac{\pi}{6} + \cos t \sin\frac{\pi}{6}$$

$$y = \frac{\sqrt{3}}{2}\sin t + \frac{1}{2}\cos t$$

$$y = \frac{\sqrt{3}}{2}\sqrt{\frac{1-2x}{2}} + \frac{1}{2}\sqrt{\frac{2x+1}{2}}$$

$$y = \sqrt{\frac{3-6x}{8}} + \sqrt{\frac{2x+1}{8}}$$

$$y^2 = \frac{1}{4}\left(\sqrt{3-12x^2} - 2x + 2\right)$$

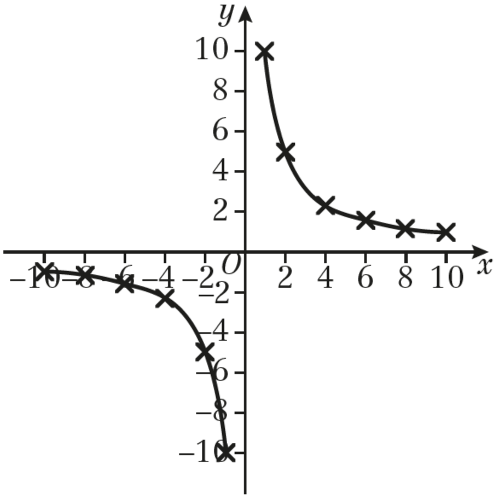
$$4y^2 + 2x - 2 = \sqrt{3-12x^2}$$

$$\left(4y^2 + 2x - 2\right)^2 + 12x^2 - 3 = 0$$

Parametric equations 8C

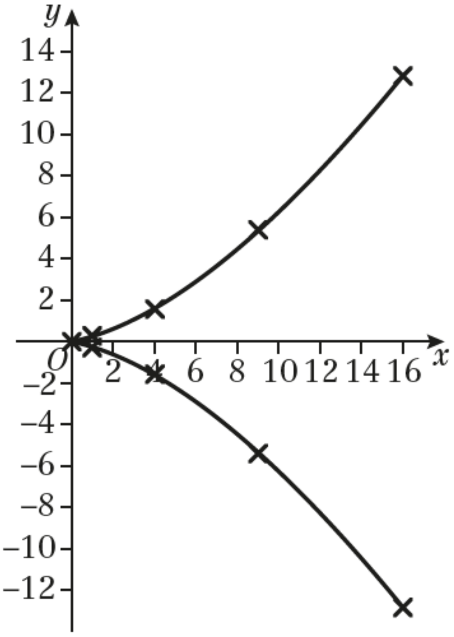
1

t	-5	-4	-3	-2	-1	-0.5	0.5	1	2	3	4	5
$x = 2t$	10	-8	-6	-4	-2	-1	1	2	4	6	8	10
$y = \frac{5}{t}$	-1	-1.25	-1.67	-2.5	-5	-10	10	5	2.5	1.67	1.25	1



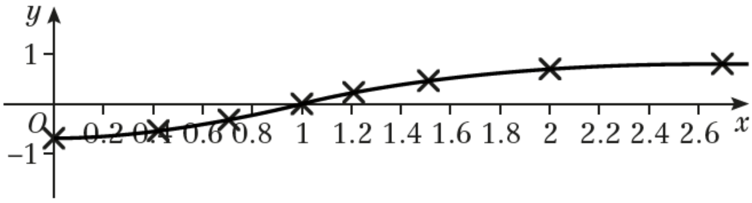
2

t	-4	-3	-2	-1	0	1	2	3	4
$x = t^2$	16	9	4	1	0	1	4	9	16
$y = \frac{t^3}{5}$	-12.8	-5.4	-1.6	-0.2	0	0.2	1.6	5.4	12.8



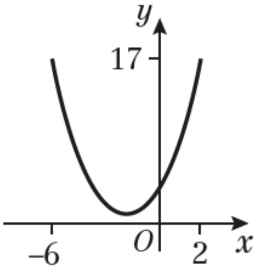
3

t	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	$-\frac{\pi}{12}$	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$x = \tan t + 1$	0	0.423	0.732	1	1.268	1.577	2	2.732
$y = \sin t$	−0.707	−0.5	−0.259	0	0.259	0.5	0.707	0.866



4 a

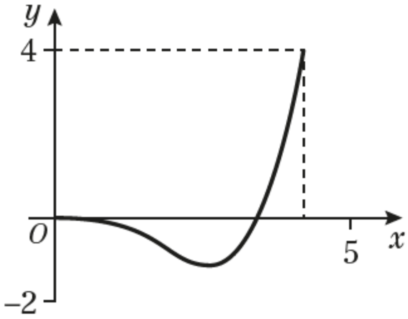
t	−4	−3	−2	−1	0	1	2	3	4
$x = t - 2$	−6	−5	−4	−3	−2	−1	0	1	2
$y = t^2 + 1$	17	10	5	2	1	2	5	10	17



Note that the curve is a parabola with minimum point having coordinates (−2, 1).

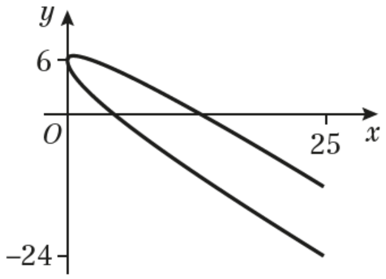
b

t	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
$x = 3\sqrt{t}$	0	1.50	2.12	2.60	3.00	3.35	3.67	3.97	4.24
$y = t^3 - 2t$	0	−0.48	−0.88	−1.08	−1.00	−0.55	0.38	1.86	4.00



4 c

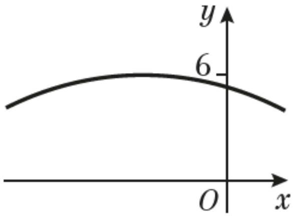
t	-5	-4	-3	-2	-1	0	1	2	3	4	5
$x=t^2$	25	16	9	4	1	0	1	4	9	16	25
$y=(2-t)(t+3)$	-14	-6	0	4	6	6	4	0	-6	-14	-24



Note that the curve crosses the x-axis at $x=4$ and $x=9$ and touches the y-axis at $y=6$.

d

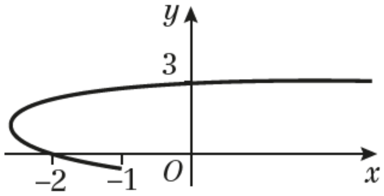
t	$-\frac{\pi}{4}$	$-\frac{3\pi}{16}$	$-\frac{\pi}{8}$	$-\frac{\pi}{16}$	0	$\frac{\pi}{16}$	$\frac{\pi}{8}$	$\frac{3\pi}{16}$	$\frac{\pi}{4}$
$x=2\sin t-1$	-2.41	-2.11	-1.77	-1.39	-1.00	-0.61	-0.23	0.11	0.41
$y=5\cos t+1$	4.54	5.16	5.62	5.90	6.00	5.90	5.62	5.16	4.54



Note the symmetry in the curve about the line $x=-1$, with maximum value $y=6$.

e

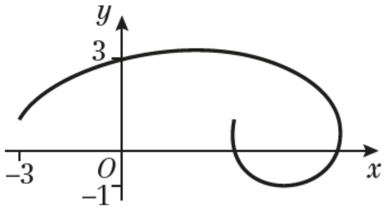
t	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	$-\frac{\pi}{12}$	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$x=\sec^2 t-3$	-1.00	-1.67	-1.93	-2.00	-1.93	-1.67	-1.00	1.00	11.93	∞
$y=2\sin t+1$	0.41	0	0.48	1.00	1.52	2.00	2.41	2.73	2.93	3.00



Note that as $y\to 3$ (y approaches 3), $x\to\infty$ (x tends to infinity, that is, gets very large without bound). The line $y=3$ is an asymptote of the curve.

4 f

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$x = t - 3\cos t$	−3.00	−1.34	1.57	4.48	6.14	6.05	4.71	3.38	3.28
$y = 1 + 2\sin t$	1.00	2.41	3.00	2.41	1.00	−0.41	−1.00	−0.41	1.00



5 a $x = 3 - t \Rightarrow t = 3 - x \quad (1)$

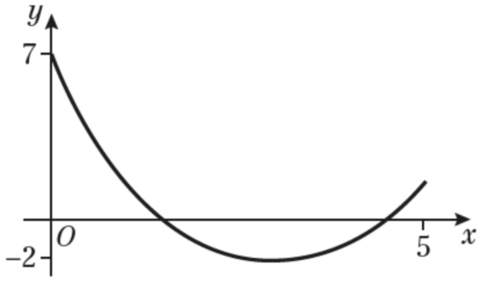
Substitute (1) into
$y = t^2 - 2$

$y = (3 - x)^2 - 2$
or $y = x^2 - 6x + 7$

b

t	−2	−1	0	1	2	3
$x = 3 - t$	5	4	3	2	1	0
$y = t^2 - 2$	2	−1	−2	−1	2	7

The curve is quadratic with a minimum value of $y = -2$ that occurs when $x = 3$.



6 a $x = 9\cos t - 2 \Rightarrow \frac{x+2}{9} = \cos t$ (1)

$y = 9\sin t + 1 \Rightarrow \frac{y-1}{9} = \sin t$ (2)

Substitute (1) and (2) into
$\cos^2 t + \sin^2 t = 1$

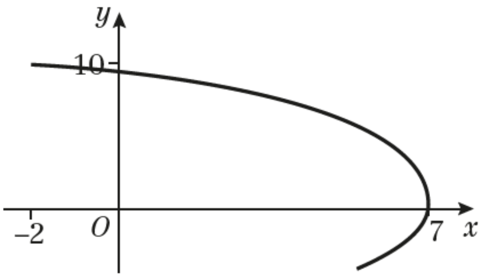
$$\left(\frac{x+2}{9}\right)^2 + \left(\frac{y-1}{9}\right)^2 = 1$$

$$(x+2)^2 + (y-1)^2 = 81$$

So $a = 2$, $b = -1$ and $c = 81$.
The curve is a circle, centre $(-2, 1)$ and with radius 9 units.

b

t	$-\frac{\pi}{6}$	$-\frac{\pi}{12}$	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$x = 9\cos t - 2$	5.79	6.69	7.00	6.69	5.79	4.36	2.50	0.33	−2.00
$y = 9\sin t + 1$	−3.50	−1.33	1.00	3.33	5.50	7.36	8.79	9.69	10.00



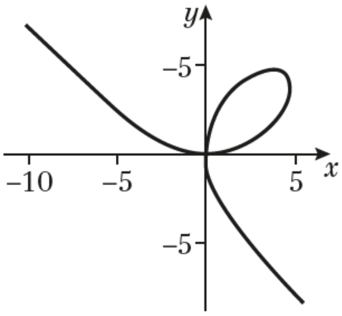
c $r = 9$

$$\theta = \frac{\pi}{6} + \frac{\pi}{2} = \frac{2\pi}{3}$$

Arc length $= r\theta = 9 \times \frac{2\pi}{3} = 6\pi$

Challenge

t	-4	-3	-2	-1.2	-0.8	0	1	2	3	4
$x = \dfrac{9t}{1+t^3}$	0.57	1.04	2.57	14.84	−14.75	0	4.50	2	0.96	0.55
$y = \dfrac{9t^2}{1+t^3}$	−2.29	−3.12	−5.14	−17.80	11.80	0	4.50	4	2.89	2.22



As t increases from $t = 0$, the function f(x) creates, in an anticlockwise direction, a small loop in the first quadrant.

To consider the behaviour of the function for $t < 0$, note that the parametric equations are not defined at $t = -1$. It is, therefore, important to investigate the behaviour of the curve as t approaches this value from above and from below. Choose some values of t that are close to −1 and on either side of it. Then observe the behaviour of the curve as t moves away from $t = -1$ in both directions. In particular, check what happens as $t \to -\infty$.

As t approaches −1 from the positive direction the curve heads off to infinity in the second quadrant, and as it approaches −1 from the negative direction it heads off to infinity in the fourth quadrant.

Note that any set of t values can be selected and that they do not need to be spaced equidistantly. In sketching more complicated curves, it is often important to consider additional values of t. Select positions such as $t = -1$ or regions, such as $0 \le t \le 4$, where you notice that the curve is not moving in the same direction.

Parametric equations 8D

1 a The curve meets the x-axis when $y = 0$

$y = 6 - t$

$0 = 6 - t$

So $t = 6$

Substitute $t = 6$ into the parametric equation for x:

$x = 5 + t$

$x = 5 + 6 = 11$

The coordinates are (11, 0).

b The curve meets the x-axis when $y = 0$

$y = 2t - 6$

$0 = 2t - 6$

$6 = 2t$

So $t = 3$

Substitute $t = 3$ into the parametric equation for x:

$x = 2t + 1$

$x = 2 \times 3 + 1 = 7$

The coordinates are (7, 0).

c The curve meets the x-axis when $y = 0$

$y = (1 - t)(t + 3)$

$0 = (1 - t)(t + 3)$

So $t = 1$ or $t = -3$

Substitute each value of t into the parametric equation for x:

$x = t^2$

When $t = 1$, $x = 1^2 = 1$

When $t = -3$, $x = (-3)^2 = 9$

The coordinates are (1, 0) and (9, 0).

d The curve meets the x-axis when $y = 0$

$y = (t - 1)(2t - 1)$

$0 = (t - 1)(2t - 1)$

So $t = 1$ or $t = \frac{1}{2}$

Substitute each value of t into the parametric equation for x:

$x = \frac{1}{t}$

When $t = 1$, $x = \frac{1}{1} = 1$

When $t = \frac{1}{2}$, $x = \frac{1}{\frac{1}{2}} = 2$

The coordinates are (1, 0) and (2, 0).

e The curve meets the x-axis when $y = 0$

$y = t - 9$

$t - 9 = 0$

So $t = 9$

Substitute $t = 9$ into the parametric equation for x:

$x = \frac{2t}{1 + t}$

$x = \frac{2(9)}{1 + (9)} = \frac{18}{10} = \frac{9}{5}$

The coordinates are $\left(\frac{9}{5}, 0\right)$.

2 a The curve meets the y-axis when $x = 0$

$x = 2t$

$0 = 2t$

So $t = 0$

Substitute $t = 0$ into the parametric equation for y:

$y = t^2 - 5$

$y = 0^2 - 5 = -5$

The coordinates are (0, −5).

b The curve meets the y-axis when $x = 0$

$x = 3t - 4$

$0 = 3t - 4$

$3t = 4$

So $t = \frac{4}{3}$

Substitute $t = \frac{4}{3}$ into the parametric equation for y:

$y = \frac{1}{t^2} = \left(\frac{1}{t}\right)^2$

$y = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$

The coordinates are $\left(0, \frac{9}{16}\right)$.

2 **c** The curve meets the y-axis when $x = 0$

$x = t^2 + 2t - 3$

$0 = t^2 + 2t - 3$

$0 = (t-1)(t+3)$

So $t = 1$ or $t = -3$

Substitute each value of t into the parametric equation for y:

$y = t(t-1)$

When $t = 1$, $y = 1(1-1) = 0$

When $t = -3$, $y = -3(-3-1) = 12$

The coordinates are (0, 0) and (0, 12).

d The curve meets the y-axis when $x = 0$

$x = 27 - t^3$

$0 = 27 - t^3$

$t^3 = 27$

So $t = 3$

Substitute $t = 3$ into the parametric equation for y:

$$y = \frac{1}{t-1}$$

$$y = \frac{1}{3-1} = \frac{1}{2}$$

The coordinates are $\left(0, \frac{1}{2}\right)$.

e The curve meets the y-axis when $x = 0$

$$x = \frac{t-1}{t+1}$$

$$0 = \frac{t-1}{t+1}$$

$0 = t - 1$

So $t = 1$

Substitute $t = 1$ into the parametric equation for y:

$$y = \frac{2t}{t^2+1}$$

$$y = \frac{2\times 1}{1^2+1} = 1$$

The coordinates are (0, 1).

3 At point (4, 0), $x = 4$ and $y = 0$

Hence

$4at^2 = 4$ (1)

$a(2t-1) = 0$ (2)

Solving equation (2) for t:

$2t - 1 = 0$

$2t = 1$

$$t = \frac{1}{2}$$

Substitute into equation (1):

$$4a\left(\frac{1}{2}\right)^2 = 4$$

$$4a \times \frac{1}{4} = 4$$

$a = 4$

So the value of a is 4.

4 At point (0, −5), $x = 0$ and $y = -5$

Hence

$b(2t-3) = 0$ (1)

$b(1-t^2) = -5$ (2)

Solving equation (1) for t:

$2t - 3 = 0$

$2t = 3$

$$t = \frac{3}{2}$$

Substitute into equation (2):

$$b\left(1 - \left(\frac{3}{2}\right)^2\right) = -5$$

$$b\left(1 - \frac{9}{4}\right) = -5$$

$$b\left(-\frac{5}{4}\right) = -5$$

$b = 4$

So the value of b is 4.

5 Substitute $x = 3t + 2$ and $y = 1 - t$ into $y + x = 2$:

$$(1-t)+(3t+2)=2$$
$$1-t+3t+2=2$$
$$2t+3=2$$
$$2t=-1$$
$$t=-\frac{1}{2}$$

Substitute $t=-\frac{1}{2}$ into the parametric equations:

$$x=3t+2=3\left(-\frac{1}{2}\right)+2=-\frac{3}{2}+2=\frac{1}{2}$$
$$y=1-t=1-\left(-\frac{1}{2}\right)=1+\frac{1}{2}=\frac{3}{2}$$

The coordinates of the point of intersection are $\left(\frac{1}{2},\frac{3}{2}\right)$.

6 Substitute $x=t^2$ and $y=2t$ into $4x-2y-15=0$:

$$4(t^2)-2(2t)-15=0$$
$$4t^2-4t-15=0$$
$$(2t+3)(2t-5)=0$$

So $2t+3=0$ or $2t-5=0$

$$t=-\frac{3}{2} \text{ or } t=\frac{5}{2}$$

Substitute $t=-\frac{3}{2}$ into the parametric equations:

$$x=t^2=\left(-\frac{3}{2}\right)^2=\frac{9}{4}$$
$$y=2t=2\left(-\frac{3}{2}\right)=-3$$

Substitute $t=\frac{5}{2}$ into the parametric equations:

$$x=t^2=\left(\frac{5}{2}\right)^2=\frac{25}{4}$$
$$y=2t=2\left(\frac{5}{2}\right)=5$$

The coordinates of the points of intersection are $\left(\frac{9}{4},-3\right)$ and $\left(\frac{25}{4},5\right)$.

7 Substitute $x=t^2$ and $y=2t$ into $x^2+y^2-9x+4=0$:

$$(t^2)^2+(2t)^2-9(t^2)+4=0$$
$$t^4+4t^2-9t^2+4=0$$
$$t^4-5t^2+4=0$$
$$(t^2-4)(t^2-1)=0$$

So $t^2-4=0$ or $t^2-1=0$

$t^2=4$ or $t^2=1$

$t=\pm2$ or $t=\pm1$

Substitute $t=\pm2$ into the parametric equations:

$x=(\pm2)^2=4$

$y=2\times(\pm2)=\pm4$

Substitute $t=\pm1$ into the parametric equations:

$x=(\pm1)^2=1$

$y=2\times(\pm1)=\pm2$

The coordinates of the points of intersection are (4, 4), (4, −4), (1, 2) and (1, −2).

8 a The curve meets the x-axis when $y = 0$

$y=\cos t,\ 0<t<\pi$

$0=\cos t$

$$t=\frac{\pi}{2}$$

Substitute $t=\frac{\pi}{2}$ into the parametric equation for x:

$x=t^2-1$

$$x=\left(\frac{\pi}{2}\right)^2-1=\frac{\pi^2}{4}-1$$

Coordinates on the x-axis are $\left(\frac{\pi^2}{4}-1,0\right)$.

The curve meets the y-axis when $x = 0$

$x=t^2-1,\ 0<t<\pi$

$0=t^2-1$

$1=t^2$

$t=1$ (as $t=-1$ is outside the domain of t)

Substitute $t = 1$ into the parametric equation for y:

$y=\cos t$

$y=\cos 1$

Coordinates on the y-axis are $(0,\cos 1)$.

8 b The curve meets the x-axis when $y = 0$

$$y = 2\cos t + 1,\ \pi < t < 2\pi$$
$$0 = 2\cos t + 1$$
$$\cos t = -\frac{1}{2}$$
$$t = \frac{4\pi}{3}$$

Substitute $t = \frac{4\pi}{3}$ into the parametric equation for x:

$$x = \sin 2t$$
$$x = \sin\left(2\left(\frac{4\pi}{3}\right)\right) = \sin\frac{8\pi}{3} = \frac{\sqrt{3}}{2}$$

Coordinates on the x-axis are $\left(\frac{\sqrt{3}}{2}, 0\right)$.

The curve meets the y-axis when $x = 0$

$$x = \sin 2t,\ \pi < t < 2\pi$$
$$0 = \sin 2t,\ 2\pi < 2t < 4\pi$$
$$\therefore 2t = 3\pi$$
$$t = \frac{3\pi}{2}$$

Substitute $t = \frac{3\pi}{2}$ into the parametric equation for y:

$$y = 2\cos t + 1$$
$$y = 2\cos\left(\frac{3\pi}{2}\right) + 1 = 1$$

Coordinates on the y-axis are (0, 1).

c The curve meets the x-axis when $y = 0$

$$y = \sin t - \cos t,\ 0 < t < \frac{\pi}{2}$$
$$0 = \sin t - \cos t$$
$$\cos t = \sin t$$
$$\tan t = 1$$
$$t = \frac{\pi}{4}$$

Substitute $t = \frac{\pi}{4}$ into the parametric equation for x:

$$x = \tan t$$
$$x = \tan\frac{\pi}{4} = 1$$

Coordinates on the x-axis are (1, 0).

The curve meets the y-axis when $x = 0$

$$0 = \tan t$$

There are no solutions in the domain of t.

So the curve does not meet the y-axis in the given domain of t.

9 a The curve meets the x-axis when $y = 0$

$$y = \ln t$$
$$0 = \ln t$$
$$t = e^0 = 1$$

Substitute $t = 1$ into the parametric equation for x:

$$x = e^t + 5$$
$$x = e^1 + 5 = e + 5$$

Coordinates on the x-axis are (e + 5, 0).

The curve meets the y-axis when $x = 0$

$$0 = e^t + 5$$
$$e^t = -5$$

This equation has no solutions since $e^t > 0$ always.

So the curve does not meet the y-axis.

9 **b** The curve meets the x-axis when $y = 0$

$y = t^2 - 64$

$0 = t^2 - 64$

$t^2 = 64$

$t = 8$ (since $t > 0$)

Substitute $t = 8$ into the parametric equation for x:

$x = \ln t$

$x = \ln 8$

Coordinates on the x-axis are $(\ln 8, 0)$.

The curve meets the y-axis when $x = 0$

$x = \ln t$

$0 = \ln t$

$t = e^0 = 1$

Substitute $t = 1$ into the parametric equation for y:

$y = t^2 - 64$

$y = 1^2 - 64 = -63$

Coordinates on the y-axis are $(0, -63)$.

c The curve meets the x-axis when $y = 0$

$y = 2e^t - 1$

$0 = 2e^t - 1$

$1 = 2e^t$

$e^t = \frac{1}{2}$

Substitute $e^t = \frac{1}{2}$ into the parametric equation for x:

$x = e^{2t} + 1 = (e^t)^2 + 1$

$x = \left(\frac{1}{2}\right)^2 + 1 = \frac{5}{4}$

Coordinates on the x-axis are $\left(\frac{5}{4}, 0\right)$.

The curve meets the y-axis when $x = 0$

$0 = e^{2t} + 1$

$e^{2t} = -1$

This equation has no solutions since $e^{2t} > 0$ always.

So the curve does not meet the y-axis.

10 Substitute $x = t^2$ and $y = t$ into $y = -3x + 2$:

$t = -3t^2 + 2$

$3t^2 + t - 2 = 0$

$(3t - 2)(t + 1) = 0$

So $t = \frac{2}{3}$ or $t = -1$

Substitute $t = \frac{2}{3}$ into the parametric equations:

$x = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$

$y = \frac{2}{3}$

Substitute $t = -1$ into the parametric equations:

$x = (-1)^2 = 1$

$y = -1$

The coordinates of the points of intersection are $\left(\frac{4}{9}, \frac{2}{3}\right)$ and $(1, -1)$.

11 Substitute $x = \ln(t - 1)$ and $y = \ln(2t - 5)$ into $y = x - \ln 3$:

$\ln(2t - 5) = \ln(t - 1) - \ln 3$

$\ln(2t - 5) = \ln\left(\frac{t - 1}{3}\right)$

$2t - 5 = \frac{t - 1}{3}$

$6t - 15 = t - 1$

$5t = 14$

so $t = \frac{14}{5}$

Substitute $t = \frac{14}{5}$ into the parametric equations:

$x = \ln\left(\frac{14}{5} - 1\right) = \ln\left(\frac{9}{5}\right)$

$y = \ln\left(\frac{28}{5} - 5\right) = \ln\left(\frac{3}{5}\right)$

The coordinates of the points of intersection are $\left(\ln\frac{9}{5}, \ln\frac{3}{5}\right)$.

12 a The curve intersects the x-axis when $y = 0$

$$y = 4\sin 2t + 2,\ -\frac{\pi}{2} < t < \frac{\pi}{2}$$

$$0 = 4\sin 2t + 2$$

$$\sin 2t = -\frac{1}{2}$$

$$2t = -\frac{5\pi}{6}, -\frac{\pi}{6}$$

$$t = -\frac{5\pi}{12}, -\frac{\pi}{12}$$

Substitute each value of t into the parametric equation $x = 6\cos t$:

$$x = 6\cos\left(-\frac{5\pi}{12}\right) = 6\cos\left(\frac{5\pi}{12}\right)$$

$$x = 6\cos\left(-\frac{\pi}{12}\right) = 6\cos\left(\frac{\pi}{12}\right)$$

The coordinates are

$\left(6\cos\left(\frac{\pi}{12}\right), 0\right)$ and $\left(6\cos\left(\frac{5\pi}{12}\right), 0\right)$.

12 b Substitute the parametric equation $y = 4\sin 2t + 2$ into $y = 4$:

$$4\sin 2t + 2 = 4$$
$$4\sin 2t = 2$$
$$\sin 2t = \frac{1}{2}$$
$$2t = \frac{\pi}{6}, \frac{5\pi}{6}$$
$$t = \frac{\pi}{12}, \frac{5\pi}{12} \text{ (in the domain } -\frac{\pi}{2} < t < \frac{\pi}{2}\text{)}$$

c Substitute each value of t into the parametric equations:

When $t = \frac{\pi}{12}$,

$$x = 6\cos\frac{\pi}{12}$$
$$y = 4\sin\left(2\left(\frac{\pi}{12}\right)\right) + 2 = 4 \times \frac{1}{2} + 2 = 4$$

When $t = \frac{5\pi}{12}$,

$$x = 6\cos\frac{5\pi}{12}$$
$$y = 4\sin\left(2\left(\frac{5\pi}{12}\right)\right) + 2 = 4 \times \frac{1}{2} + 2 = 4.$$

Coordinates are

$$\left(6\cos\frac{\pi}{12}, 4\right) \text{ and } \left(6\cos\frac{5\pi}{12}, 4\right).$$

13 To find any intersections between the line and the curve, substitute the parametric equations $x = 2t$ and $y = 4t(t-1)$ into $y = 2x - 5$:

$$4t(t-1) = 2(2t) - 5$$
$$4t^2 - 4t = 4t - 5$$
$$4t^2 - 8t + 5 = 0$$

The discriminant of this quadratic equation is

$$b^2 - 4ac = (-8)^2 - 4 \times 4 \times 5$$
$$= 64 - 80 = -16 < 0$$

So the quadratic equation has no real roots. Hence the line does not intersect the curve.

14 a To find intersections between the line and the curve, substitute the parametric equation $y = \cos 2t + 1$ into $y = k$:

$$\cos 2t + 1 = k$$
$$\cos 2t = k - 1$$

Since $-1 \le \cos 2t \le 1$,

$$-1 \le k - 1 \le 1$$
$$\text{so } 0 \le k \le 2$$

b First find a Cartesian equation for the curve:

$$y = \cos 2t + 1$$
$$= (1 - 2\sin^2 t) + 1$$
$$= 2 - 2\sin^2 t$$

Since $x = \sin t$,

$$y = 2 - 2x^2$$

Substitute $y = k$ into this Cartesian equation:

$$k = 2 - 2x^2$$
$$2x^2 + (k - 2) = 0$$

If $y = k$ is a tangent to the curve, then it touches the curve at one point, so the discriminant of the quadratic is 0.

$$b^2 - 4ac = 0$$
$$0^2 - 4 \times 2 \times (k - 2) = 0$$
$$-8(k - 2) = 0$$
$$k - 2 = 0$$
$$\therefore k = 2$$

15 a At the point A, $t = \ln 2$

$$x = e^{2t} = e^{2\ln 2} = e^{\ln 2^2} = 2^2 = 4$$
$$y = e^t - 1 = e^{\ln 2} - 1 = 2 - 1 = 1$$

$\therefore$ coordinates of A are (4, 1).

At the point B, $t = \ln 3$

$$x = e^{2t} = e^{2\ln 3} = e^{\ln 3^2} = 3^2 = 9$$
$$y = e^t - 1 = e^{\ln 3} - 1 = 3 - 1 = 2$$

$\therefore$ coordinates of B are (9, 2).

b Points A and B lie on the line l, so the gradient of l can be found from the coordinates of A and B:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 1}{9 - 4} = \frac{1}{5}$$

15 c Using the gradient and the coordinates of *A*, the equation of *l* is given by

$$y-y_1=m(x-x_1)$$

$$y-1=\frac{1}{5}(x-4)$$

$$5y-5=x-4$$

$$x-5y+1=0$$

$$x-5y+1=0$$

16 At the point *A*, $t=\frac{\pi}{6}$

$$x=\sin t=\sin\frac{\pi}{6}=\frac{1}{2}$$

$$y=\cos t=\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$$

Coordinates of *A* are $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$.

At the point *B*, $t=\frac{\pi}{2}$

$$x=\sin t=\sin\frac{\pi}{2}=1$$

$$y=\cos t=\cos\frac{\pi}{2}=0$$

Coordinates of *B* are (1, 0).

As the line *l* passes through *A* and *B*, the gradient of *l* can be found from the coordinates of *A* and *B*.

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{0-\frac{\sqrt{3}}{2}}{1-\frac{1}{2}}=\frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}}=-\sqrt{3}$$

So, using the coordinates of *B*, the equation of *l* is given by

$$y-y_1=m(x-x_1)$$

$$y-0=-\sqrt{3}(x-1)$$

$$y=-\sqrt{3}x+\sqrt{3}$$

$$\sqrt{3}x+y-\sqrt{3}=0$$

17 a At *A* (on the *y*-axis), $x=0$

$$x=\frac{t-1}{t}$$

$$0=\frac{t-1}{t}$$

$$0=t-1$$

So $t=1$

Substitute $t=1$ into the parametric equation for *y*:

$$y=t-4$$

$$y=1-4=-3$$

Coordinates of *A* are (0, −3).

At *B* (on the *x*-axis), $y=0$

$$y=t-4$$

$$0=t-4$$

$$t=4$$

Substitute $t=4$ into the parametric equation for *x*:

$$x=\frac{t-1}{t}$$

$$x=\frac{4-1}{4}=\frac{3}{4}$$

Coordinates of *B* are $\left(\frac{3}{4},0\right)$.

17 b Find the gradient of l_1 using the coordinates of A and B:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-(-3)}{\frac{3}{4}-0} = \frac{3}{\frac{3}{4}} = 4$$

Because the lines l_2 and l_3 are parallel to l_1, they have the same gradient as l_1 and so have equations

l_1: $y = 4x + c_2$

l_2: $y = 4x + c_3$

Substitute the parametric equations for x and y into $y = 4x + c$:

$$t - 4 = 4\left(\frac{t-1}{t}\right) + c$$

$$t^2 - 4t = 4t - 4 + ct$$

$$t^2 - (8+c)t + 4 = 0 \qquad (1)$$

The lines l_2 and l_3 are tangents to the curve when the discriminant of (1) equals 0.

$$(8+c)^2 - 4\times1\times4 = 0$$

$$64 + 16c + c^2 - 16 = 0$$

$$c^2 + 16c + 48 = 0$$

$$(c+4)(c+12) = 0$$

$$c = -4 \text{ or } c = -12$$

Taking $c_2 = -4$ and $c_3 = -12$, equations for l_2 and l_3 are $y = 4x - 4$ and $y = 4x - 12$.

c Substituting $c = -4$ into (1) gives

$$t^2 - 4t + 4 = 0$$

$$(t-2)^2 = 0$$

$$t = 2$$

At $t = 2$,

$$x = \frac{t-1}{t} = \frac{2-1}{2} = \frac{1}{2}$$

$$y = t - 4 = 2 - 4 = -2$$

So l_2 meets the curve at $\left(\frac{1}{2}, -2\right)$.

Substituting $c = -12$ into (1) gives

$$t^2 + 4t + 4 = 0$$

$$(t+2)^2 = 0$$

$$t = -2$$

At $t = -2$,

$$x = \frac{-2-1}{-2} = \frac{3}{2}$$

$$y = -2 - 4 = -6$$

So l_3 meets the curve at $\left(\frac{3}{2}, -6\right)$.

Challenge

Find a Cartesian equation for C_1:
$x = e^{2t} \Rightarrow 2t = \ln x$
Substitute into the parametric equation for y:
$y = 2t + 1$
$\therefore\ y = \ln x + 1 \qquad (1)$

Find a Cartesian equations for C_2:
$x = e^{t} \Rightarrow t = \ln x$
Substitute into the parametric equation for y:
$y = 1 + t^2$
$\therefore\ y = 1 + (\ln x)^2 \qquad (2)$

Now solve equations (1) and (2).
Substituting (1) into (2) gives
$\ln x + 1 = 1 + (\ln x)^2$
$(\ln x)^2 - \ln x = 0$
$\ln x(\ln x - 1) = 0$
so $\ln x = 0$ or $\ln x = 1$
$x = e^0 = 1$ or $x = e^1 = e$

Substitute these x-values into either (1) or (2):
When $x = 1$, $y = \ln 1 + 1 = 0 + 1 = 1$
When $x = e$, $y = \ln e + 1 = 1 + 1 = 2$

$\therefore$ the coordinates of the points of intersection are (1, 1) and (e, 2).

Parametric equations 8E

1 **a** Substitute $x = 75$ into

$$x = 0.9t$$
$$75 = 0.9t$$
$$t = \frac{10 \times 75}{9}$$
$$= \frac{250}{3} = 83.3\,\text{seconds}$$

b $y = -3.2t$

$$y = -3.2\left(\frac{250}{3}\right)$$
$$= -\frac{800}{3} = -267 \text{ (3 s.f.)}$$

The boat has been moved off course by 267 m southwards.

c $x = 0.9t \Rightarrow \frac{10}{9}x = t \qquad (1)$

Substitute (1) into

$$y = -3.2t$$
$$y = -3.2\left(\frac{10}{9}x\right)$$
$$= -\frac{32}{9}x$$

This equation is in the form $y = mx\ (+\ c)$ and so is a straight line.

d Total distance travelled is

$$\sqrt{75^2 + \left(\frac{800}{3}\right)^2} = \frac{\sqrt{690625}}{3}$$
$$= 277\,\text{m (3 s.f.)}$$

$$\text{Speed} = \frac{\text{distance}}{\text{time}}$$
$$= \frac{\frac{\sqrt{690625}}{3}}{\frac{250}{3}} = \frac{\sqrt{690625}}{250}$$
$$= 3.324...$$

The speed is 3.32 m s^{-1} (3 s.f.).

2 **a** Substitute $t = 0$ into

$$y = -9.1t + 3000$$
$$y = -9.1(0) + 3000$$
$$= 3000$$

The initial height is 3000 m.

b Substitute $y = 0$ into

$$y = -9.1t + 3000$$
$$0 = -9.1t + 3000$$
$$9.1t = 3000$$
$$t = \frac{3000}{9.1} = 329.67\,\text{minutes}$$

$t = 0$ is the start of the descent.

When $t \geq 300$, $y < 0$, so the plane would be underground or below sea level.

c Substitute $t = \frac{30\,000}{91}$ into

$$x = 80t$$
$$x = 80\left(\frac{30\,000}{91}\right) = 26\,373$$

The horizontal distance travelled during the descent is 26 400 m (3 s.f.).

3 $x = 10\sqrt{3}t\,\text{m} \qquad (1)$

$y = (-4.9t^2 + 10t)\,\text{m} \qquad (2)$

$0 \leq t \leq k$ seconds

a Substitute $y = 0$ into (2).

$$y = -4.9t^2 + 10t$$
$$0 = -t(4.9t - 10)$$

Either $t = 0$ (when the ball is kicked) or

$$t = \frac{100}{49} = 2.04$$

Substitute $t = \frac{100}{49}$ into (1).

$$x = 10\sqrt{3}t$$
$$= 10\sqrt{3}\left(\frac{100}{49}\right) = 35.34...$$

The distance is 35.3 m (3 s.f.).

3 **b** Substitute $y = 1.5$ into (2).

$$y = -4.9t^2 + 10t$$
$$1.5 = -4.9t^2 + 10t$$
$$4.9t^2 - 10t + 1.5 = 0$$
$$t = \frac{10 \pm \sqrt{100 - 4(4.9)(1.5)}}{2(4.9)}$$

Either $t = 1.88$ or $t = 0.163$
On the descent, $t = 1.88$ seconds (3 s.f.).

Substitute $y = 2.5$ into (2).

$$4.9t^2 - 10t + 2.5 = 0$$
$$t = \frac{10 \pm \sqrt{100 - 4(4.9)(2.5)}}{2(4.9)}$$

Either $t = 1.75$ or $t = 0.292$
On the descent, $t = 1.75$ seconds (3 s.f.).

The range of time is $1.75 \le t \le 1.88$ seconds (3 s.f.).

c The closest distance is at the lower end of the range of times, so substitute $t = 1.75$ into (1).

$$x = 10\sqrt{3}t$$
$$x = 10\sqrt{3} \times 1.75 = 30.3$$

The closest distance is 30.3 m (3 s.f.)

4 $x = 2t\,\text{m}$ (1)
$y = (-4.9t^2 + 10t)\,\text{m}$ (2)

a Substitute $y = 0$ into (2).

$$y = -4.9t^2 + 10t$$
$$0 = -t(4.9t - 10)$$

Either $t = 0$ (at the start) or $t = \frac{100}{49}$

The time taken is $t = \frac{100}{49}$ seconds.

b Substitute $t = \frac{100}{49}$ into (1).

$$x = 2t$$
$$x = 2\left(\frac{100}{49}\right) = \frac{200}{49}\,\text{m}$$

c $x = 2t \Rightarrow t = \frac{x}{2}$ (3)

Substitute (3) into (2).

$$y = -4.9\left(\frac{x}{2}\right)^2 + 10\left(\frac{x}{2}\right)$$
$$= -\frac{4.9}{4}x^2 + 5x$$
$$= -\frac{49}{40}x^2 + 5x \qquad (4)$$

Therefore, the path is a quadratic curve.

d For (maximum) height use either (2) or (4).

From (2):

$$y = -4.9t^2 + 10t$$
$$\frac{dy}{dt} = -9.8t + 10$$
$$0 = -9.8t + 10$$
$$t = \frac{10}{9.8} = \frac{100}{98}\text{ seconds}$$

Substitute $t = \frac{100}{98}$ into (2).

$$y = -4.9\left(\frac{100}{98}\right)^2 + 10\left(\frac{100}{98}\right) = \frac{250}{49}$$

The maximum height is $\frac{250}{49}$ m.

5 $x = 12\sin t$ (1)
$y = 12 - 12\cos t$ (2)
$t \geq 0$ minutes

a Rearrange (1) and (2).

$$x = 12\sin t \Rightarrow \frac{x}{12} = \sin t \quad (3)$$

$$y = 12 - 12\cos t \Rightarrow \cos t = \frac{12-y}{12} \quad (4)$$

Substitute (3) and (4) into
$\sin^2 t + \cos^2 t = 1$

$$\left(\frac{x}{12}\right)^2 + \left(\frac{12-y}{12}\right)^2 = 1$$

$$x^2 + (12-y)^2 = 12^2$$

or

$$x^2 + (-(y-12))^2 = 12^2$$

$$x^2 + (-1)^2(y-12)^2 = 12^2$$

$$x^2 + (y-12)^2 = 12^2$$

The motion is a circle, centre (0, 12) and radius 12 m.

b The car is at its maximum height when $t = \pi$ minutes.

Substitute $t = \pi$ into (2).
$y = 12 - 12\cos\pi = 24$

The maximum height is 24 m.

c Time (t) = 2π minutes

Distance (d) = $2\pi r$
$= 2\pi \times 12$
$= 24\pi$ m

$$\text{Average speed} = \frac{d}{t} = \frac{24\pi}{2\pi} = 12$$

The average speed is 12 m min^{-1}.

6 $x = t - 4\sin t$ (1)
$y = 1 - 2\cos t$ (2)

$$-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

a Substitute $t = -\frac{\pi}{2}$ into (1).
$x = t - 4\sin t$

$$x = -\frac{\pi}{2} - 4\sin\left(-\frac{\pi}{2}\right) = 4 - \frac{\pi}{2}$$

Substitute $t = \frac{\pi}{2}$ into (1).

$$x = \frac{\pi}{2} - 4\sin\frac{\pi}{2} = \frac{\pi}{2} - 4$$

Length of opening

$$= 4 - \frac{\pi}{2} - \left(\frac{\pi}{2} - 4\right)$$

$= 8 - \pi = 4.86$ (3 s.f.)

b Substitute $t = \frac{\pi}{2}$ into (2).
$y = 1 - 2\cos t$

$$y = 1 - 2\cos\left(\frac{\pi}{2}\right) = 1$$

y is a minimum when $t = 0$ (by symmetry) so substitute $t = 0$ into (2).
$y = 1 - 2\cos(0) = -1$

Depth $= 1 - (-1) = 2$

7 $$x = \frac{t^2 - 3t + 2}{t} \quad (1)$$
$y = 2t$ (2)
$t > 0$ seconds

a Substitute $t = 0.5$ into (1) and then (2).

$$x = \frac{(0.5)^2 - 3(0.5) + 2}{0.5} = 1.5$$

$y = 2 \times 0.5 = 1$

Distance from origin

$$= \sqrt{\left(\frac{3}{2}\right)^2 + 1^2} = \frac{\sqrt{13}}{2}$$

7 **b** Substitute $x = 0$ into (1).

$$x = \frac{t^2 - 3t + 2}{t}$$
$$0 = \frac{t^2 - 3t + 2}{t}$$
$$= t^2 - 3t + 2$$
$$= (t-1)(t-2)$$

Either $t = 1$ or $t = 2$

Substitute $t = 1$ into (2).

$$y = 2t$$
$$y = 2(1) = 2$$

Substitute $t = 2$ into (2).

$$y = 2t$$
$$y = 2(2) = 4$$

Coordinates are $(0, 2)$ and $(0, 4)$.

c Substitute (1) and (2) into $y = 2x + 10$

$$2t = 2\left(\frac{t^2 - 3t + 2}{t}\right) + 10$$
$$2t^2 = 2\left(t^2 - 3t + 2\right) + 10t$$
$$2t^2 = 2t^2 - 6t + 4 + 10t$$
$$-4 = 4t$$
$$-1 = t$$

But $t > 0$, therefore the two particles never meet.

8 $x = 18t$ (1)

$y = -4.9t^2 + 4t + 10$ (2)

$0 \le t \le k$

a Substitute $t = 0$ into (2).

$$y = -4.9(0)^2 + 4(0) + 10 = 10$$

The initial height is 10 m.

b Substitute $y = 0$ into (2).

$$y = -4.9t^2 + 4t + 10$$
$$0 = -4.9t^2 + 4t + 10$$
$$0 = 4.9t^2 - 4t - 10$$
$$t = \frac{4 \pm \sqrt{16 - 4(4.9)(-10)}}{2(4.9)}$$
$$= \frac{4 \pm \sqrt{212}}{9.8}$$

Either $t = 1.89...$ or $t = -1.08...$

But $t \ge 0$ (and $t \le k$), so $k = 1.89$ (3 s.f.).

The time taken is 1.89 seconds.

c Substitute $t = 1.89...$ into (1).

$$x = 18t$$
$$x = 18\left(\frac{4 + \sqrt{212}}{9.8}\right) = 34.1$$

The horizontal distance travelled is 34.1 m (3 s.f.).

8 d Rearrange (1).

$$x = 18t \Rightarrow t = \frac{x}{18} \qquad (3)$$

Substitute (3) into (2).

$$y = -4.9\left(\frac{x}{18}\right)^2 + 4\left(\frac{x}{18}\right) + 10$$
$$= -\frac{4.9}{324}x^2 + \frac{4}{18}x + 10$$
$$= -\frac{49}{3240}x^2 + \frac{2}{9}x + 10 \qquad (4)$$

Since (4) is a quadratic equation, the ski jumper's path is a parabola.

For (maximum) height use (2) or (4).

Choosing (2), find $\frac{\mathrm{d}y}{\mathrm{d}t}$.

$$y = -4.9t^2 + 4t + 10$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -9.8t + 4$$
$$0 = -9.8t + 4$$
$$9.8t = 4$$
$$t = \frac{4}{9.8} = \frac{20}{49}$$

Substitute $t = \frac{20}{49}$ into (2).

$$y = -4.9t^2 + 4t + 10$$
$$y = -4.9\left(\frac{20}{49}\right)^2 + 4\left(\frac{20}{49}\right) + 10$$
$$= \frac{530}{49} = 10.81...$$

The maximum height above ground is 10.8 m (3 s.f.).

9 $x = 50\tan t$ m (1)

$y = 20\sin 2t$ m (2)

$0 < t \le \frac{\pi}{2}$

a Maximum value of y is when

$$\sin 2t = 1$$
$$2t = \frac{\pi}{2}$$
$$t = \frac{\pi}{4}$$

b Substitute $t = \frac{\pi}{4}$ into (1) and (2).

$$x = 50\tan\left(\frac{\pi}{4}\right) = 50 \text{ m}$$
$$y = 20\sin 2\left(\frac{\pi}{4}\right) = 20 \text{ m}$$

Coordinates are $(50, 20)$.

c Substitute $t = 1$ into (1) and (2).

$$x = 50\tan 1 = 77.87... \text{ m}$$
$$y = 20\sin 2 = 18.18... \text{ m}$$

Coordinates are $(77.9, 18.2)$ (3 s.f.).

Note that $\frac{\pi}{4} < 1 < \frac{\pi}{2}$, and that the sine function is continuous between $\sin 2\left(\frac{\pi}{4}\right) = 1$ and $\sin 2\left(\frac{\pi}{2}\right) = 0$.

So, $\sin 2\left(\frac{\pi}{4}\right) > \sin 2(1) > \sin 2\left(\frac{\pi}{2}\right)$

or $1 > \sin 2(1) > 0$.

Therefore, around $t = 1$, as t increases, y decreases. The cyclist is descending.

10 $x = 5 + \ln t$ (1)

$y = 5\sin 2t$ (2)

$0 \le t \le \frac{\pi}{2}$

a Substitute $t = \frac{\pi}{6}$ into (1) and (2).

$$x = 5 + \ln\frac{\pi}{6} = 4.35 \text{ (3 s.f.)}$$
$$y = 5\sin 2\left(\frac{\pi}{6}\right)$$
$$= \frac{5\sqrt{3}}{2} = 4.33 \text{ (3 s.f.)}$$

Coordinates are $(4.35, 4.33)$.

10 b Maximum value of y occurs when

$\sin 2t = 1$

Substitute $\sin 2t = 1$ into (2).

$y = 5\sin 2t$

$y = 5(1) = 5$

Maximum height $= 5 \times 5 = 25$ m

c Find t when y is a maximum.

$\sin 2t = 1$

$2t = \frac{\pi}{2}$ (only one solution to consider)

$t = \frac{\pi}{4}$

When $t = \frac{\pi}{4}$ (at maximum height):

$x = 5 + \ln t = 5 + \ln\frac{\pi}{4}$

When $t = \frac{\pi}{2}$ (at end of descent):

$x = 5 + \ln\frac{\pi}{2}$

Horizontal distance covered

$$= 5 + \ln\frac{\pi}{2} - \left(5 + \ln\frac{\pi}{4}\right)$$

$$= \ln\frac{\pi}{2} - \ln\frac{\pi}{4}$$

$$= \ln\left(\frac{\frac{\pi}{2}}{\frac{\pi}{4}}\right) = \ln 2$$

Distance $= 5\ln 2 = 3.47$ m (3 s.f.).

d Gradient $= \frac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{0 - 25}{5\ln 2}$$

$$= \frac{-5}{\ln 2}$$

$= -7.21$ (3 s.f.)

Parametric equations, Mixed exercise 8

1 **a** At A, $y = 0 \Rightarrow 3\sin t = 0 \Rightarrow \sin t = 0$

So $t = 0$ or $t = \pi$

Substitute $t = 0$ and $t = \pi$ into

$x = 4\cos t$

$t = 0 \;\Rightarrow\; x = 4\cos 0 = 4 \times 1 = 4$

$t = \pi \;\Rightarrow\; x = 4\cos\pi = 4 \times (-1) = -4$

The coordinates of A are $(4, 0)$.

At B, $x = 0 \Rightarrow 4\cos t = 0 \Rightarrow \cos t = 0$

So $t = \frac{\pi}{2}$ or $t = \frac{3\pi}{2}$

Substitute $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ into

$y = 3\sin t$

$t = \frac{\pi}{2} \;\Rightarrow\; y = 3\sin\frac{\pi}{2} = 3 \times 1 = 3$

$t = \frac{3\pi}{2} \;\Rightarrow\; y = 3\sin\frac{3\pi}{2} = 3 \times -1 = -3$

The coordinates of B are $(0, 3)$.

b At C, $t = \frac{\pi}{6}$

$x = 4\cos\frac{\pi}{6} = 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3}$

$y = 3\sin\frac{\pi}{6} = 3 \times \frac{1}{2} = \frac{3}{2}$

The coordinates of C are $\left(2\sqrt{3}, \frac{3}{2}\right)$.

c $x = 4\cos t \Rightarrow \frac{x}{4} = \cos t \qquad (1)$

$y = 3\sin t \Rightarrow \frac{y}{3} = \sin t \qquad (2)$

Substitute (1) and (2) into

$\cos^2 t + \sin^2 t = 1$

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

$9x^2 + 16y^2 = 144$

2 Substitute $t = 0$ into

$x = \cos t$ and $y = \frac{1}{2}\sin 2t$

$x = \cos 0 = 1$

$y = \frac{1}{2}\sin(2 \times 0) = \frac{1}{2}\sin 0 = \frac{1}{2} \times 0 = 0$

So when $t = 0$, $(x, y) = (1, 0)$.

Substitute $t = \frac{\pi}{2}$ into

$x = \cos t$ and $y = \frac{1}{2}\sin 2t$

$x = \cos\frac{\pi}{2} = 0$

$y = \frac{1}{2}\sin\left(2 \times \frac{\pi}{2}\right) = \frac{1}{2}\sin\pi = \frac{1}{2} \times 0 = 0$

So when $t = \frac{\pi}{2}$, $(x, y) = (0, 0)$.

Substitute $t = \pi$ into

$x = \cos t$ and $y = \frac{1}{2}\sin 2t$

$x = \cos\pi = -1$

$y = \frac{1}{2}\sin 2\pi = \frac{1}{2} \times 0 = 0$

So when $t = \pi$, $(x, y) = (-1, 0)$.

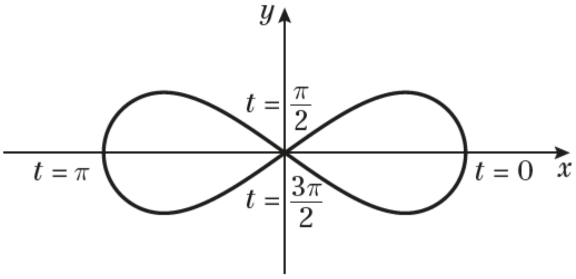
Substitute $t = 3\pi$ into

$x = \cos t$ and $y = \frac{1}{2}\sin 2t$

$x = \cos\frac{3\pi}{2} = 0$

$y = \frac{1}{2}\sin\left(2 \times \frac{3\pi}{2}\right) = \frac{1}{2}\sin 3\pi = \frac{1}{2} \times 0 = 0$

So when $t = \frac{3\pi}{2}$, $(x, y) = (0, 0)$.



3 a $x = e^{2t+1} + 1 \quad (1)$

$x - 1 = e^{2t+1}$

$\ln(x-1) = 2t + 1$

$\ln(x-1) - 1 = 2t$

$\frac{1}{2}\ln(x-1) - \frac{1}{2} = t \quad (2)$

$y = t + \ln 2 \quad (3)$

Substitute (2) into (3).

$$y = \frac{1}{2}\ln(x-1) - \frac{1}{2} + \ln 2$$
$$= \ln(x-1)^{\frac{1}{2}} + \ln 2 - \frac{1}{2}$$
$$= \ln\left(2\sqrt{x-1}\right) - \frac{1}{2}$$

The Cartesian equation of the curve is

$y = \ln\left(2\sqrt{x-1}\right) - \frac{1}{2}$

Substitute $t = 1$ into (1).

$x = e^3 + 1$

Since x is an increasing function and $t > 1$,

$x > e^3 + 1$

So $k = e^3 + 1$

b The range of f(x) is the range of y = q(t) so substitute $t = 1$ into (3).

$y = 1 + \ln 2$

Since y is an increasing function and $t > 1$,

$y > 1 + \ln 2$

The range of f(x) is $y > 1 + \ln 2$.

4 $x = \frac{1}{2t+1} \quad (1)$

$2t + 1 = \frac{1}{x}$

$2t = \frac{1}{x} - 1$

$t = \frac{1}{2x} - \frac{1}{2} \quad (2)$

$y = 2\ln\left(t + \frac{1}{2}\right) \quad (3)$

Substitute (2) into (3).

$$y = 2\ln\left(\frac{1}{2x} - \frac{1}{2} + \frac{1}{2}\right)$$
$$= 2\ln\left(\frac{1}{2x}\right)$$
$$= -2\ln(2x)$$
$$= -2(\ln 2 + \ln x)$$
$$= -\ln 4 - 2\ln x$$

The Cartesian equation of the curve is

$y = -\ln 4 - 2\ln x$

The domain of f(x) is the range of x = p(t) so substitute $t = \frac{1}{2}$ into (1).

$$x = \frac{1}{2\left(\frac{1}{2}\right) + 1} = \frac{1}{2}$$

As $t \to \infty$, $x \to 0$.

So the domain is $0 < x < \frac{1}{2}$.

The range of f(x) is the range of y = q(t) so substitute $t = \frac{1}{2}$ into (3).

$$y = 2\ln\left(\frac{1}{2} + \frac{1}{2}\right) = 0$$

As $t \to \infty$, $y \to \infty$

So the range is $y > 0$.

5 **a** $x = \sin t \qquad (1)$

$y = \cos^2 t \qquad (2)$

Substitute (1) and (2) into

$\cos^2 t = 1 - 2\sin^2 t$

$y = 1 - 2x^2$

A Cartesian equation of the curve is

$y = 1 - 2x^2$

b Substitute $y = 0$ into

$$y = 1 - 2x^2$$
$$0 = 1 - 2x^2$$
$$2x^2 = 1$$
$$x^2 = \frac{1}{2}$$
$$x = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{1}}{\sqrt{2}} = \pm\frac{1}{\sqrt{2}}$$
$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

So the curve meets the x-axis at

$\left(\frac{\sqrt{2}}{2}, 0\right)$ and $\left(-\frac{\sqrt{2}}{2}, 0\right)$.

So a and b are $\frac{\sqrt{2}}{2}$ and $-\frac{\sqrt{2}}{2}$.

6 $x = \frac{1}{1+t} \qquad (1)$

$x(1+t) = 1$

$1 + t = \frac{1}{x}$

So $t = \frac{1}{x} - 1 \qquad (2)$

Substitute (2) into

$$y = \frac{1}{(1+t)(1-t)}$$
$$y = \frac{1}{\left(1 + \frac{1}{x} - 1\right)\left(1 - \left(\frac{1}{x} - 1\right)\right)}$$
$$= \frac{1}{\left(\frac{1}{x}\right)\left(2 - \frac{1}{x}\right)}$$
$$= \frac{x^2}{x^2\left(\frac{1}{x}\right)\left(2 - \frac{1}{x}\right)}$$
$$= \frac{x^2}{2x - 1}$$

So the Cartesian equation of the curve is $y = \frac{x^2}{2x-1}$

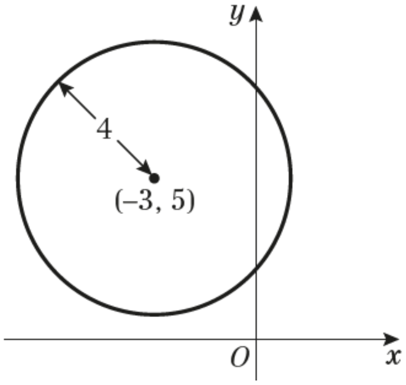
7 **a** $x = 4\sin t - 3 \Rightarrow \sin t = \frac{x+3}{4} \qquad (1)$

$y = 4\cos t + 5 \Rightarrow \cos t = \frac{y-5}{4} \qquad (2)$

Substitute (1) and (2) into

$$\sin^2 t + \cos^2 t = 1$$
$$\left(\frac{x+3}{4}\right)^2 + \left(\frac{y-5}{4}\right)^2 = 1$$
$$\frac{(x+3)^2}{16} + \frac{(y-5)^2}{16} = 1$$
$$(x+3)^2 + (y-5)^2 = 16$$

7 **b** The circle $(x+3)^2+(y-5)^2=4^2$ has centre $(-3,5)$ and radius 4.



c Substitute $x=0$ into

$$(x+3)^2+(y-5)^2=16$$
$$(0+3)^2+(y-5)^2=16$$
$$3^2+(y-5)^2=16$$
$$9+(y-5)^2=16$$
$$(y-5)^2=7$$
$$y-5=\pm\sqrt{7}$$
$$y=5\pm\sqrt{7}$$

The points of intersectin of the circle and the y-axis are at $\left(0,\,5+\sqrt{7}\right)$ and $\left(0,\,5-\sqrt{7}\right)$.

8 **a**
$$x=\frac{2-3t}{1+t} \quad (1)$$
$$x+xt=2-3t$$
$$xt+3t=2-x$$
$$t\left(x+3\right)=2-x$$
$$t=\frac{2-x}{x+3} \quad (2)$$

$$y=\frac{3+2t}{1+t} \quad (3)$$

Substitute (2) into (3).

$$y=\frac{3+2\left(\frac{2-x}{x+3}\right)}{1+\left(\frac{2-x}{x+3}\right)}$$
$$=\frac{\frac{3\left(x+3\right)+2\left(2-x\right)}{x+3}}{\frac{x+3+2-x}{x+3}}$$
$$=\frac{3\left(x+3\right)+2\left(2-x\right)}{x+3+2-x}$$
$$=\frac{3x+9+4-2x}{5}$$
$$=\frac{x+13}{5}$$
$$=\frac{1}{5}x+\frac{13}{5}$$

This is in the form $y=mx+c$, therefore the curve C is a straight line.

8 b Substitute $t = 0$ into (1) and (2).

$$x = \frac{2-3(0)}{1+0} = 2$$

$$y = \frac{3+2(0)}{1+0} = 3$$

Coordinates are $(2, 3)$.

Substitute $t = 4$ into (1) and (2).

$$x = \frac{2-3(4)}{1+4} = -2$$

$$y = \frac{3+2(4)}{1+4} = \frac{11}{5}$$

Coordinates are $\left(-2, \frac{11}{5}\right)$

$$\text{Length} = \sqrt{(2-(-2))^2 + \left(3-\frac{11}{5}\right)^2}$$

$$\sqrt{(4)^2 + \left(\frac{4}{5}\right)^2}$$

$$= \sqrt{\frac{416}{25}}$$

$$= \frac{4\sqrt{26}}{5}$$

9 a $x = t^2 - 2$

$x + 2 = t^2$

$\pm\sqrt{x+2} = t$

But $0 \le t \le 2$ so choose the positive value.

$t = \sqrt{x+2}$ (1)

Substitute (1) into

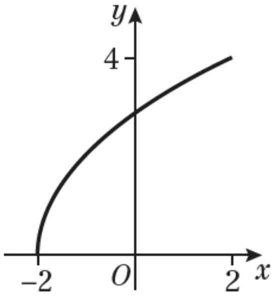
$y = 2t$

$y = 2\sqrt{x+2}$

b Domain of f(x) is $-2 \le x \le 2$.

Range of f(x) is $0 \le y \le 4$.

c



10 a $x = 2\cos t \Rightarrow \frac{x}{2} = \cos t$ (1)

$y = 2\sin t - 5 \Rightarrow \frac{y+5}{2} = \sin t$ (2)

Substitute (1) and (2) into

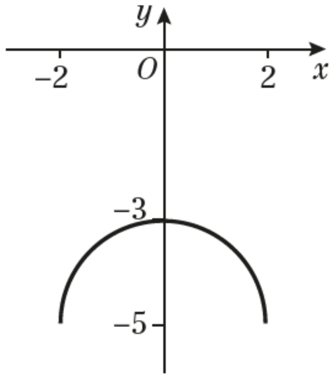
$\cos^2 t + \sin^2 t = 1$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y+5}{2}\right)^2 = 1$$

$x^2 + (y+5)^2 = 4$

So the curve C forms part of a circle of radius 2 and centre (0, −5).

b



c Since $0 \le t \le \pi$, the curve C forms half of the circle.

Arc length $= r\theta = 2\pi$

11 a $x = t - 2 \Rightarrow x + 2 = t$ (1)

Substitute (1) into

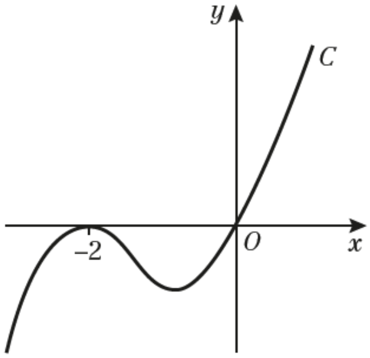
$y = t^3 - 2t^2$

$$\begin{aligned} y &= (x+2)^3 - 2(x+2)^2 \\ &= (x+2)^2(x+2-2) \\ &= x(x^2+4x+4) \\ &= x^3+4x^2+4x \end{aligned}$$

The Cartesian equation of C is

$y = x^3 + 4x^2 + 4x$

b



12 $x = t - 3$ (1)

$y = 4 - t^2$ (2)

Substitute (1) and (2) into

$y = 4x + 20$

$4 - t^2 = 4(t-3) + 20$

$0 = t^2 + 4t + 4$

Use the discriminant:

$$\begin{aligned} \text{‘}b^2 - 4ac\text{’} &= 4^2 - 4(1)(4) \\ &= 0 \end{aligned}$$

Therefore, the line and the curve have only one point of intersection, that is, they touch. So the line is a tangent to the curve.

13 a $x = 2\ln t$ (1)

$x = 5$ (2)

Substitute (2) into (1).

$5 = 2\ln t$

$\frac{5}{2} = \ln t$

$e^{\frac{5}{2}} = t$ (3)

Substitute (3) into

$y = t^2 - 1$

$$\begin{aligned} y &= \left(e^{\frac{5}{2}}\right)^2 - 1 \\ &= e^5 - 1 \end{aligned}$$

The coordinates of the point of intersection of the line $x = 5$ and the curve are $(5, e^5 - 1)$.

b $y = t^2 - 1, \quad t > 0$

(In this domain) the function is increasing. So the range is $y > -1$.

Therefore, $k > -1$.

14 a At A, $x = 0$

$x = 1 + 2t$ (1)

$0 = 1 + 2t$

$-\frac{1}{2} = t$

So substitute $t = -\frac{1}{2}$ into

$y = 4^t - 1$

$y = 4^{-\frac{1}{2}} - 1 = -\frac{1}{2}$

The coordinates of A are $\left(0, -\frac{1}{2}\right)$.

At B, $y = 0$

$y = 4^t - 1$

$0 = 4^t - 1$

$1 = 4^t$

$t = 0$

So substitute $t = 0$ into (1).

$x = 1 + 2t$

$x = 1 + 2(0) = 1$

The coordinates of B are $(1, 0)$.

b $m_l = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{0 - \left(-\frac{1}{2}\right)}{1 - 0} = \frac{1}{2}$

Substitute $(1, 0)$ and $m = \frac{1}{2}$ into

$y - y_1 = m(x - x_1)$

$y - 0 = \frac{1}{2}(x - 1)$

$2y = x - 1$

$x - 2y - 1 = 0$

15 At A, $x = 0$

$x = \ln t - \ln\left(\frac{\pi}{2}\right)$ (1)

$0 = \ln t - \ln\left(\frac{\pi}{2}\right)$

$\ln\left(\frac{\pi}{2}\right) = \ln t$

$\frac{\pi}{2} = t$

$y = \sin t$ (2)

Substitute $t = \frac{\pi}{2}$ into (2).

$y = \sin\frac{\pi}{2} = 1$

The coordinates of A are $(0, 1)$.

At B, $y = 0$

$y = \sin t$

$0 = \sin t$, $0 < t < 2\pi$

$\pi = t$

Substitute $t = \pi$ into (1).

$x = \ln \pi - \ln\left(\frac{\pi}{2}\right)$

$= \ln \pi - (\ln \pi - \ln 2) = \ln 2$

The coordinates of B are $(\ln 2, 0)$.

$m_l = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{0 - 1}{\ln 2 - 0} = -\frac{1}{\ln 2}$

Substitute $(0, 1)$ and $m = -\frac{1}{\ln 2}$ into

$y - y_1 = m(x - x_1)$

$y - 1 = -\frac{1}{\ln 2}(x - 0)$

$y \ln 2 - \ln 2 = -x$

$x + y \ln 2 - \ln 2 = 0$

16 a $x = 80t \quad (1)$

$\frac{x}{80} = t \quad (2)$

$y = 3000 - 30t \quad (3)$

Substitute (2) into (3).

$$y = 3000 - 30\left(\frac{x}{80}\right)$$
$$= 3000 - \frac{3}{8}x$$

This is in the form $y = mx + c$, therefore the plane's descent is a straight line.

b Substitute $y = 30$ into (3).

$30 = 3000 - 30t$
$1 = 100 - t$
$t = 99$

So $k = 99$

c Substitute $t = 99$ into (1).

$x = 80(99) = 7920$

During this portion of its descent the plane travels 7920 m horizontally and descends vertically by 3000 − 30 = 2970 m.

$$\text{Distance travelled} = \sqrt{7920^2 + 2970^2}$$
$$= 8458.56\text{ m (2 d.p.)}$$

17 a Substitute $y = 0$ into

$y = 1.5 - 4.9t^2 + 50\sqrt{2}t \quad (1)$

$0 = 1.5 - 4.9t^2 + 50\sqrt{2}t$

$4.9t^2 - 50\sqrt{2}t - 1.5 = 0$

$$t = \frac{50\sqrt{2} \pm \sqrt{2(50)^2 - 4(4.9)(-1.5)}}{2(4.9)}$$

$= 14.45\ldots$ or $t = -0.021\ldots$

$t = 14.45$ (since $t > 0$)

$x = 50\sqrt{2}t \quad (2)$

Substitute $t = 14.45$ into (2).

$x = 50\sqrt{2}(14.45\ldots)$

$= 1022\ldots$

The furthest horizontal distance is 1022 m (4 s.f.).

b Substitute $x = 1000$ into (2).

$1000 = 50\sqrt{2}t$

$$t = \frac{1000}{50\sqrt{2}}$$
$$= \frac{20}{\sqrt{2}} = 10\sqrt{2}$$

Substitute $t = 10\sqrt{2}$ into (1).

$$y = 1.5 - 4.9\left(10\sqrt{2}\right)^2 + 50\sqrt{2}\left(10\sqrt{2}\right)$$
$$= 1.5 - 490 \times 2 + 500 \times 2$$
$$= 1.5 + 20$$
$$= 21.5$$

$21.5 > 10$, so the arrow is too high to hit the castle wall.

c Substitute $y = 10$ into (1).

$10 = 1.5 - 4.9t^2 + 50\sqrt{2}t$

$4.9t^2 - 50\sqrt{2}t + 8.5 = 0$

$$t = \frac{50\sqrt{2} \pm \sqrt{2(50)^2 - 4(4.9)(8.5)}}{2(4.9)}$$

$= 14.309\ldots$ (or $t = 0.1212\ldots$)

$x = 50\sqrt{2}(14.309\ldots) = 1011.799\ldots$

The archer needs to move back by 11.8 m (3 s.f.).

18 a $y = 244t(4 - t), \quad 0 < t < 4 \quad (1)$

y is a parabola with midpoint (which corresponds to the maximum height) at $t = 2$ hours.

Substitute $t = 2$ into (1).

$y = 244(2)(4 - 2) = 976\text{ m}$

b The mountaineer completes her walk at sea level, when $y = 0$.

$y = 0$ when $t = 4$ (and when $t = 0$).

Substitute $t = 4$ into

$x = 300\sqrt{t}$

$x = 300\sqrt{4} = 600\text{ m}$

The horizontal distance is 600 m.

19 a The curve is symmetrical in the x-axis. Its highest point is at the midpoint, at which $t = \pi$.

At $t = \pi$:

$$\begin{aligned} y &= -\cos t \\ &= -\cos \pi \\ &= 1 \end{aligned}$$

The height of the bridge is 10 m.

b At $t = \frac{\pi}{2}$:

$$\begin{aligned} x &= \frac{4t}{\pi} - 2\sin t \\ &= \frac{4}{\pi} \times \frac{\pi}{2} - 2\sin\frac{\pi}{2} \\ &= 0 \end{aligned}$$

At $t = \frac{3\pi}{2}$:

$$x = \frac{4}{\pi} \times \frac{3\pi}{2} - 2\sin\frac{3\pi}{2} = 8$$

The maximum width is 80 – 0 = 80 m.

20 a $y = 10(t-1)^2$ (1)

Substitute $t = 0$ into (1).

$$y = 10(0-1)^2 = 10$$

The cyclist’s initial height is 10 m.

b Substitute $y = 0$ into (1).

$$\begin{aligned} 0 &= 10(t-1)^2 \\ 0 &= (t-1)^2 \\ t &= 1 \end{aligned}$$

The cyclist is at her lowest height after 1 second.

c Substitute $t = 1.3$ into (1).

$$y = 10(1.3-1)^2 = 0.9$$

The cyclist leaves the ramp at height 0.9 m.

Challenge

a If the particles collide at time t seconds, their x- and y- positions must both be the same at this time.

$$x_A = \frac{2}{t} \qquad (1)$$

$$x_B = 5 - 2t \qquad (2)$$

Set their x-positions equal.

$$\frac{2}{t} = 5 - 2t$$

$$2 = 5t - 2t^2$$

$$2t^2 - 5t + 2 = 0$$

$$(2t-1)(t-2) = 0$$

Either $t = \frac{1}{2}$ or $t = 2$

$$y_A = 3t + 1 \qquad (3)$$

$$y_B = 2t^2 + 2k - 1 \qquad (4)$$

Set their y-positions equal and rearrange for k.

$$2t^2 + 2k - 1 = 3t + 1$$

$$2k = 2 + 3t - 2t^2$$

$$k = \frac{2 + 3t - 2t^2}{2} \qquad (5)$$

Substitute $t = \frac{1}{2}$ into (5).

$$k = \frac{2 + 3\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right)^2}{2} = \frac{3}{2}$$

Substitute $t = 2$ into (5).

$$k = \frac{2 + 3(2) - 2(2)^2}{2} = 0$$

Since $k > 0$, $k = \frac{3}{2}$

b $k = \frac{3}{2}$ when $t = \frac{1}{2}$

Substitute $t = \frac{1}{2}$ into (2) and (3).

$$x = 5 - 2\left(\frac{1}{2}\right) = 4$$

$$y = 3\left(\frac{1}{2}\right) + 1 = \frac{5}{2}$$

The coordinates of the point of collision are $\left(4, \frac{5}{2}\right)$.

Note that you can check your answer by using (1) and (4).

Review Exercise 2

1 Crosses y-axis when $x = 0$ at $\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$

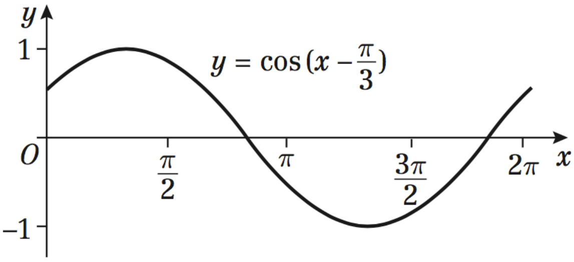
Crosses x-axis when $\sin\left(x + \frac{3\pi}{4}\right) = 0$

$$x + \frac{3\pi}{4} = -\pi, 0, \pi, 2\pi$$

$$x = -\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}$$

So coordinates are $\left(0, \frac{1}{\sqrt{2}}\right), \left(-\frac{7\pi}{4}, 0\right), \left(-\frac{3\pi}{4}, 0\right), \left(\frac{\pi}{4}, 0\right), \left(\frac{5\pi}{4}, 0\right)$

2 **a** $y = \cos\left(x - \frac{\pi}{3}\right)$ is $y = \cos x$ translated by the vector $\begin{pmatrix} \frac{\pi}{3} \\ 0 \end{pmatrix}$



b Crosses y-axis when $y = \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$

Crosses x-axis when $\cos\left(x - \frac{\pi}{3}\right) = 0$

$$x - \frac{\pi}{3} = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$x = \frac{5\pi}{6}, \frac{11\pi}{6}$$

So coordinates are $\left(0, \frac{1}{2}\right), \left(\frac{5\pi}{6}, 0\right), \left(\frac{11\pi}{6}, 0\right)$

2 **c** $\cos\left(x - \frac{\pi}{3}\right) = -0.27, \ 0 \leq x \leq 2\pi$

$$\cos^{-1}(-0.27) = 1.844 \text{ (3 d.p.)}$$

$$\Rightarrow x - \frac{\pi}{3} \approx 1.844 \text{ and } x - \frac{\pi}{3} \approx 2\pi - 1.844$$

$$\Rightarrow x = 2.89, 5.49 \text{ (2 d.p.)}$$

3 a Let C be the midpoint of the line AB, then
AOC is a right-angled triangle and $AC = 3\,\text{cm}$, so

$$\sin\frac{\theta}{2} = \frac{3}{5} = 0.6 \Rightarrow \frac{\theta}{2} = 0.6435...$$

$\theta = 1.287$ radians (3 d.p.)

b Use $l = r\theta$
So arc $AB = 5 \times 1.287 = 6.44\,\text{cm}$ (3 s.f.)

4 As ABC is equilateral, $BC = AC = 8\,\text{cm}$
$BP = AB - AP = 8 - 6 = 2\,\text{cm}$
$QC = BP = 2\,\text{cm}$

$\angle BAC = \frac{\pi}{3}$, $PQ = 6 \times \frac{\pi}{3} = 2\pi = 6.28\,\text{cm}$ (2 d.p.)

So perimeter = $BC + BP + PQ + QC = 18.28\,\text{cm}$ (2 d.p.)

Exact answer $12 + 2\pi$ cm

5 a $\frac{1}{2}(r + 10)^2\theta - \frac{1}{2}r^2\theta = 40$

$$\Rightarrow 20r\theta + 100\theta = 80$$
$$\Rightarrow r\theta + 5\theta = 4$$
$$\Rightarrow r = \frac{4}{\theta} - 5$$

b $r = \frac{4}{\theta} - 5 = 6\theta$

$$\Rightarrow 4 - 5\theta = 6\theta^2$$
$$\Rightarrow 6\theta^2 + 5\theta - 4 = 0$$
$$\Rightarrow (3\theta + 4)(2\theta - 1) = 0$$
$$\Rightarrow \theta = -\frac{4}{3} \text{ or } \frac{1}{2}$$

But θ cannot be negative, so $\theta = \frac{1}{2}$, $r = 3$

So perimeter = $20 + r\theta + (10 + r)\theta = 20 + \frac{3}{2} + \frac{13}{2} = 28\,\text{cm}$

6 a arc $BD = 10 \times 0.6 = 6\,\text{cm}$

b Area of triangle $ABC = \frac{1}{2}(13 \times 10)\sin 0.6 = 65 \times 0.567 = 36.7\,\text{cm}^2$ (1 d.p.)

Area of sector $ABD = \frac{1}{2}10^2 \times 0.6 = 30\,\text{cm}^2$

Area of shaded area $BCD = 36.7 - 30 = 6.7\,\text{cm}^2$ (1 d.p.)

7 **a** $\angle OED = 90^\circ$ because BC is parallel to ED

So $r = \dfrac{10}{\cos 0.7} = 13.07\,\text{cm}$ (2 d.p.)

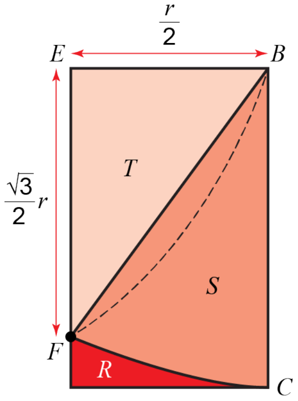
Area of sector $OAB = \dfrac{1}{2}r^2 \times 1.4 = 119.7\,\text{cm}^2$ (1 d.p.)

b $BC = AC = r\tan 0.7$

$$\begin{aligned}\text{So perimeter} &= 2r\tan 0.7 + r \times 1.4\\ &= (2 \times 13.07 \times 0.842) + (13.07 \times 1.4) = 40.3\,\text{cm}\end{aligned}$$

8 Split each half of the rectangle as shown.

EFB is a right-angled triangle, and by Pythagoras' theorem $EF = \frac{\sqrt{3}}{2}r$



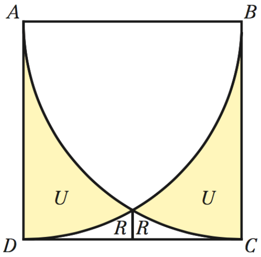
Let $\angle EBF = \theta,$ so $\tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$

So $\angle FBC = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$

Area $S = \frac{1}{2}r^2\frac{\pi}{6} = \frac{\pi}{12}r^2$

Area $T = \frac{1}{2} \times \frac{\sqrt{3}}{2}r \times \frac{1}{2}r = \frac{\sqrt{3}}{8}r^2$

$\Rightarrow$ Area $R = \frac{1}{2}r^2 -$ Area $S -$ Area $T = \left(\frac{1}{2} - \frac{\sqrt{3}}{8} - \frac{\pi}{12}\right)r^2$



Area of sector $ACB = \frac{1}{2}r^2\frac{\pi}{2} = \frac{\pi}{4}r^2$

Area U = Area $ABCD -$ Area sector $ACB - 2R$

$$= r^2 - \frac{\pi}{4}r^2 - 2\left(\frac{1}{2} - \frac{\sqrt{3}}{8} - \frac{\pi}{12}\right)r^2$$

$$= r^2\left(\frac{\sqrt{3}}{4} - \frac{\pi}{12}\right)$$

So area $U = r^2 - \frac{\pi}{4}r^2 - 2R$

$$= \left(1 - \frac{\pi}{4} - 1 + \frac{\sqrt{3}}{4} + \frac{\pi}{6}\right)r^2$$

$$= r^2\left(\frac{\sqrt{3}}{4} - \frac{\pi}{12}\right) = \frac{r^2}{12}\left(3\sqrt{3} - \pi\right)$$

So shaded area $= 2U = \frac{r^2}{6}\left(3\sqrt{3} - \pi\right)$

9 a
$$\begin{aligned}3\sin^2 x + 7\cos x + 3 &= 3(1-\cos^2 x) + 7\cos x + 3\\ &= -3\cos^2 x + 7\cos x + 6\\ &= 3\cos^2 x - 7\cos x - 6\end{aligned}$$

b $3\cos^2 x - 7\cos x - 6 = 0$

$(3\cos x + 2)(\cos x - 3) = 0$

$\cos x = -\dfrac{2}{3}$ or 3

$\cos x$ cannot be 3

so $\cos x = -\dfrac{2}{3}$

$x = 2.30,\ 2\pi - 2.30 = 2,30,\ 3.98$ (2 d.p.)

10 a For small values of θ:

$\sin 4\theta \approx 4\theta$

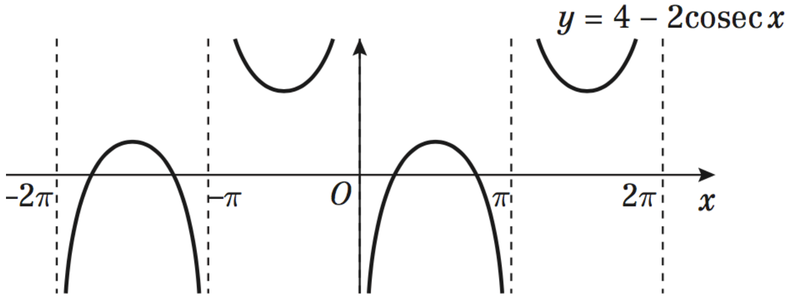
$\cos 4\theta \approx 1 - \tfrac{1}{2}(4\theta)^2 \approx 1 - 8\theta^2,$

$\tan 3\theta \approx 3\theta$

$$\begin{aligned}\sin 4\theta - \cos 4\theta + \tan 3\theta &\approx 4\theta - \left(1 - 8\theta^2\right) + 3\theta\\ &\approx 8\theta^2 + 7\theta - 1\end{aligned}$$

b -1

11 a $y = 4 - 2\operatorname{cosec} x$ is $y = \operatorname{cosec} x$ stretched by a scale factor 2 in the y-direction, then reflected in the x-axis and then translated by the vector $\begin{pmatrix}0\\4\end{pmatrix}$



b The minima in the graph occur when cosec $x = -1$ and $y = 6$. The maxima occur when cosec $x = 1$ and $y = 2$. So there are no solutions for $2 < k < 6$.

12 a The graph is a translation of $y = \sec\theta$ by α.

So $\alpha = \dfrac{\pi}{3}$

b As the curve passes through (0, 4)

$4 = k\sec\dfrac{\pi}{3} \Rightarrow k = 4\cos\dfrac{\pi}{3} = 2$

12 c $-2\sqrt{2} = 2\sec\left(\theta - \frac{\pi}{3}\right)$

$$\Rightarrow \cos\left(\theta - \frac{\pi}{3}\right) = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta - \frac{\pi}{3} = -\frac{5\pi}{4}, -\frac{3\pi}{4}$$

$$\Rightarrow \theta = -\frac{11\pi}{12}, -\frac{5\pi}{12}$$

13 a $\frac{\cos x}{1-\sin x} + \frac{1-\sin x}{\cos x} \equiv \frac{\cos^2 x + (1-\sin x)^2}{\cos x(1-\sin x)}$

$$\equiv \frac{\cos^2 x + 1 - 2\sin x + \sin^2 x}{\cos x(1-\sin x)}$$

$$\equiv \frac{2-2\sin x}{\cos x(1-\sin x)}$$

$$\equiv \frac{2}{\cos x}$$

$$\equiv 2\sec x$$

b By part a the equation becomes

$$2\sec x = -2\sqrt{2}$$

$$\Rightarrow \sec x = -\sqrt{2}$$

$$\Rightarrow \cos x = -\frac{1}{\sqrt{2}}$$

$$x = \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}$$

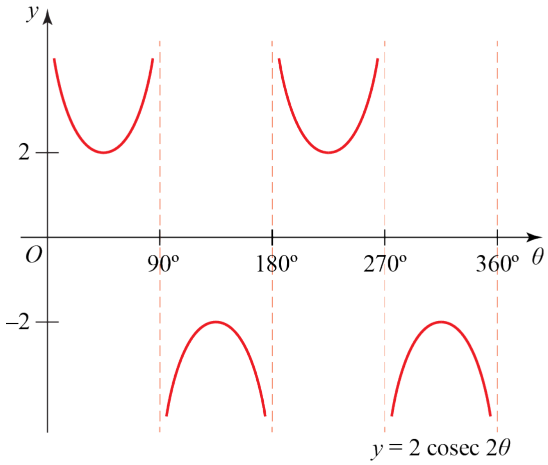
14 a $\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}$

$= \frac{1}{\sin\theta\cos\theta}$ (using $\cos^2\theta + \sin^2\theta \equiv 1$)

$= \frac{1}{\frac{1}{2}\sin 2\theta}$ (using double-angle formula $\sin 2\theta \equiv 2\sin\theta\cos\theta$)

$= 2\operatorname{cosec} 2\theta$

14b The graph of $y = 2\operatorname{cosec} 2\theta$ is a stretch of the graph of $y = \operatorname{cosec}\theta$ by a scale factor of $\frac{1}{2}$ in the horizontal direction and then a stretch by a factor of 2 in the vertical direction.



c By part a the equation becomes

$2\operatorname{cosec} 2\theta = 3$

$\Rightarrow \operatorname{cosec} 2\theta = \dfrac{3}{2}$

$\Rightarrow \sin 2\theta = \dfrac{2}{3}$, in the interval $0 \le 2\theta \le 720°$

Calculator value is $2\theta = 41.81°$ (2 d.p.)

Solutions are $2\theta = 41.81°, 180° - 41.81°, 360° + 41.81°, 540° - 41.81°$

So the solution set is: 20.9°, 69.1°, 200.9°, 249.1°

15 a Note the angle $BDC = \theta$

$$\cos\theta = \frac{BC}{10} \Rightarrow BC = 10\cos\theta$$

$$\sin\theta = \frac{BC}{BD} \Rightarrow BD = \frac{BC}{\sin\theta} = \frac{10\cos\theta}{\sin\theta} = 10\cot\theta$$

b $10\cot\theta = \dfrac{10}{\sqrt{3}}$

$\Rightarrow \cot\theta = \dfrac{1}{\sqrt{3}},\ \theta = \dfrac{\pi}{3}$

From the triangle BCD, $\cos\theta = \dfrac{DC}{BD}$

$\Rightarrow DC = BD\cos\theta$

So $DC = 10\cot\theta\cos\theta$

$$= 10\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{2}\right)$$

$$= \frac{5}{\sqrt{3}}$$

16 a $\sin^2\theta+\cos^2\theta\equiv 1$

$$\Rightarrow \frac{\sin^2\theta}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}\equiv\frac{1}{\cos^2\theta} \qquad \text{(dividing by } \cos^2\theta\text{)}$$

$$\Rightarrow \tan^2\theta+1\equiv\sec^2\theta$$

b $2\tan^2\theta+\sec\theta=1$

$$\Rightarrow 2\sec^2\theta-2+\sec\theta=1$$

$$\Rightarrow 2\sec^2\theta+\sec\theta-3=0$$

$$\Rightarrow (2\sec\theta+3)(\sec\theta-1)=0$$

$$\Rightarrow \sec\theta=-\frac{3}{2},\ \sec\theta=1$$

$$\Rightarrow \cos\theta=-\frac{2}{3},\ \cos\theta=1$$

Solutions are $131.8°, 360°-131.8°, 0°$

So solution set is: $0.0°, 131.8°, 228.2°$ (1 d.p.)

17 a $a=\dfrac{1}{\sin x}=\dfrac{1}{\frac{1}{2}b}=\dfrac{2}{b}$

b

$$\frac{4-b^2}{a^2-1}=\frac{4-b^2}{\left(\frac{2}{b}\right)^2-1}$$

$$=\frac{4-b^2}{\frac{4}{b^2}-1}=\frac{4-b^2}{\frac{4-b^2}{b^2}}=(4-b^2)\times\frac{b^2}{4-b^2}$$

$$=b^2$$

An alternative approach is to first substitute the trigonometric functions for a and b

$$\frac{4-b^2}{a^2-1}=\frac{4-4\sin^2 x}{\operatorname{cosec}^2 x-1}$$

$$=\frac{4(1-\sin^2 x)}{\cot^2 x}$$

$$=\frac{4\cos^2 x}{\cot^2 x}$$

$$=4\sin^2 x=b^2$$

18 a $y=\arcsin x$

$$\Rightarrow\ \sin y=x$$

$$x=\cos\left(\tfrac{\pi}{2}-y\right)$$

Using $\sin\theta=\cos\left(\frac{\pi}{2}-\theta\right)$

$$\Rightarrow \tfrac{\pi}{2}-y=\arccos x$$

b $\arcsin x+\arccos x=y+\frac{\pi}{2}-y$

$$=\tfrac{\pi}{2}$$

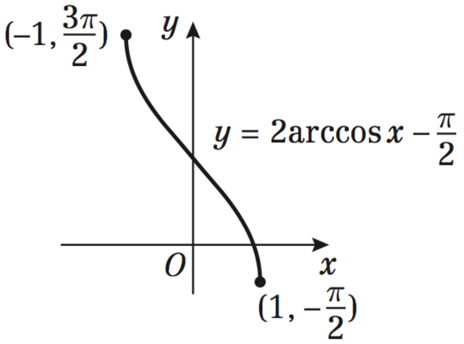
19 a $\arccos\frac{1}{x} = p \Rightarrow \cos p = \frac{1}{x}$

Use Pythagoras' theorem to show that opposite side of the right-angle triangle with angle p is $\sqrt{x^2 - 1}$

So $\sin p = \frac{\sqrt{x^2-1}}{x} \Rightarrow p = \arcsin\frac{\sqrt{x^2-1}}{x}$

b If $0 \le x < 1$ then $x^2 - 1$ is negative and you cannot take the square root of a negative number.

20 a $y = 2\arccos x - \frac{\pi}{2}$ is $y = \arccos x$ stretched by a scale factor of 2 in the y-direction and then translated by $-\frac{\pi}{2}$ in the vertical direction



b $2\arccos x - \frac{\pi}{2} = 0$

$\Rightarrow \arccos x = \frac{\pi}{4}$

$\Rightarrow x = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

Coordinates are $\left(\frac{1}{\sqrt{2}}, 0\right)$

21 $\tan\left(x + \frac{\pi}{6}\right) = \frac{1}{6} \Rightarrow \frac{\tan x + \frac{\sqrt{3}}{3}}{1 - \frac{\sqrt{3}}{3}\tan x} = \frac{1}{6}$ [using the addition formula for tan $(A + B)$]

$$6\tan x + 2\sqrt{3} = 1 - \frac{\sqrt{3}}{3}\tan x$$

$$\left(\frac{18+\sqrt{3}}{3}\right)\tan x = 1 - 2\sqrt{3}$$

$$\tan x = \frac{3\left(1-2\sqrt{3}\right)\left(18-\sqrt{3}\right)}{\left(18+\sqrt{3}\right)\left(18-\sqrt{3}\right)}$$

$$= \frac{72 - 111\sqrt{3}}{321}$$

22 a $\sin(x+30°)=2\sin(x+60°)$

So $\sin x\cos 30°+\cos x\sin 30°=2\left(\sin x\cos 60°-\cos x\sin 60°\right)$ (using the addition formulae for sin)

$$\frac{\sqrt{3}}{2}\sin x+\frac{1}{2}\cos x=2\left(\frac{1}{2}\sin x-\frac{\sqrt{3}}{2}\cos x\right)$$

$\sqrt{3}\sin x+\cos x=2\sin x-2\sqrt{3}\cos x$ (multiplying both sides by 2)

$$\left(-2+\sqrt{3}\right)\sin x=\left(-1-2\sqrt{3}\right)\cos x$$

$$\text{So }\tan x=\frac{-1-2\sqrt{3}}{-2+\sqrt{3}}$$
$$=\frac{\left(-1-2\sqrt{3}\right)\left(-2-\sqrt{3}\right)}{\left(-2+\sqrt{3}\right)\left(-2-\sqrt{3}\right)}$$
$$=\frac{2+6+4\sqrt{3}+\sqrt{3}}{4-3}$$
$$=8+5\sqrt{3}$$

b $$\tan(x+60°)=\frac{\tan x+\tan 60}{1-\tan x\tan 60}$$
$$=\frac{8+5\sqrt{3}+\sqrt{3}}{1-\left(8+5\sqrt{3}\right)\sqrt{3}}$$
$$=\frac{8+6\sqrt{3}}{-14-8\sqrt{3}}$$
$$=\frac{\left(4+3\sqrt{3}\right)\left(-7+4\sqrt{3}\right)}{\left(-7-4\sqrt{3}\right)\left(-7+4\sqrt{3}\right)}$$
$$=\frac{36-28-21\sqrt{3}+16\sqrt{3}}{49-48}$$
$$=8-5\sqrt{3}$$

23 a $$\sin 165°=\sin(120°+45°)$$
$$=\sin 120°\cos 45°+\cos 120°\sin 45°$$
$$=\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}+\frac{-1}{2}\times\frac{1}{\sqrt{2}}$$
$$=\frac{\sqrt{3}-1}{2\sqrt{2}}$$
$$=\frac{\sqrt{6}-\sqrt{2}}{4}$$

23 b
$$\begin{aligned}\operatorname{cosec}165° &= \frac{1}{\sin 165°}\\ &= \frac{4}{\left(\sqrt{6}-\sqrt{2}\right)} \times \frac{\left(\sqrt{6}+\sqrt{2}\right)}{\left(\sqrt{6}+\sqrt{2}\right)}\\ &= \frac{4\left(\sqrt{6}+\sqrt{2}\right)}{6-2}\\ &= \sqrt{6}+\sqrt{2}\end{aligned}$$

24 a $\cos A = \frac{3}{4}$

Using Pythagoras' theorem and noting that sin A is negative as A is in the fourth quadrant, this gives

$\sin A = -\frac{\sqrt{7}}{4}$

Using the double-angle formula for sin gives

$\sin 2A = 2\sin A \cos A = 2\left(-\frac{\sqrt{7}}{4}\right)\left(\frac{3}{4}\right) = -\frac{3\sqrt{7}}{8}$

b $\cos 2A = 2\cos^2 A - 1 = \frac{1}{8}$

$\Rightarrow \tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{\left(-\frac{3\sqrt{7}}{8}\right)}{\left(\frac{1}{8}\right)} = -3\sqrt{7}$

25 a $\cos 2x + \sin x = 1$

$\Rightarrow 1 - 2\sin^2 x + \sin x = 1$ (using double-angle formula for $\cos 2x$)

$\Rightarrow 2\sin^2 x - \sin x = 0$

$\Rightarrow \sin x(2\sin x - 1) = 0$

$\Rightarrow \sin x = 0, \sin x = \frac{1}{2}$

Solutions in the given interval are: $-180°, 0°, 30°, 150°, 180°$

b $\sin x(\cos x + \operatorname{cosec} x) = 2\cos^2 x$

$\Rightarrow \sin x \cos x + 1 = 2\cos^2 x$

$\Rightarrow \text{in}\, x \cos x = 2\cos^2 x - 1$

$\Rightarrow \frac{1}{2}\sin 2x = \cos 2x$ (using the double-angle formulae for $\sin 2x$ and $\cos 2x$)

$\Rightarrow \tan 2x = 2,$ for $-360° \le 2x \le 360°$

So $2x = 63.43° - 360°, 63.43° - 180°, 63.43°, 63.43° + 180°$

Solution set: $-148.3°, -58.3°, 31.7°, 121.7°$ (1 d.p.)

26 a $R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$

So $R\cos\alpha = 3$, $R\sin\alpha = 2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 2^2 = 9 + 4 = 13$

$\Rightarrow R = \sqrt{13}$ (as $\cos^2\alpha + \sin^2\alpha \equiv 1$)

$\tan\alpha = \frac{2}{3} \Rightarrow \alpha = 0.588$ (3 d.p.)

b $R^4 = \left(\sqrt{13}\right)^4 = 169$ since the maximum value the sin function can take is 1

c $\sqrt{13}\sin(x+0.588) = 1$

$\sin(x+0.5880) = \frac{1}{\sqrt{13}} = 0.27735\ldots$

$x + 0.588 = \pi - 0.281, 2\pi + 0.281$

$x = 2.273, 5.976$ (3 d.p.)

27 a LHS $\equiv \cot\theta - \tan\theta$

$$\equiv \frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}$$

$$\equiv \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}$$

$$\equiv \frac{\cos 2\theta}{\frac{1}{2}\sin 2\theta}$$ (using the double angle formulae for $\sin 2\theta$ and $\cos 2\theta$)

$\equiv 2\cot 2\theta \equiv$ RHS

b $2\cot 2\theta = 5 \Rightarrow \cot 2\theta = \frac{5}{2} \Rightarrow \tan 2\theta = \frac{2}{5}$, for $-2\pi < 2\theta < 2\pi$

So $2\theta = 0.3805 - 2\pi, 0.3805 - \pi, 0.3805, 0.3805 + \pi$

Solution set: $-2.95, -1.38, 0.190, 1.76$ (3 s.f.)

28 a LHS $\equiv \cos 3\theta$

$\equiv \cos(2\theta + \theta)$

$\equiv \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$\equiv (\cos^2\theta - \sin^2\theta)\cos\theta - \left(2\sin\theta\cos\theta\right)\sin\theta$

$\equiv \cos^3\theta - 3\sin^2\theta\cos\theta$

$\equiv \cos^3\theta - 3(1-\cos^2\theta)\cos\theta$

$\equiv 4\cos^3\theta - 3\cos\theta \equiv$ RHS

b From part a $\cos 3\theta = 4\frac{2\sqrt{2}}{27} - \sqrt{2} = -\frac{19\sqrt{2}}{27}$

So $\sec 3\theta = -\frac{27}{19\sqrt{2}} = -\frac{27\sqrt{2}}{38}$

29 $\sin^4\theta = \left(\sin^2\theta\right)\left(\sin^2\theta\right)$

Use the double-angle formula to write $\sin^2\theta$ in terms of $\cos 2\theta$

$$\cos 2\theta = 1-\sin^2\theta \Rightarrow \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Now substitute the expression for $\sin^2\theta$ and expand the brackets

$$\text{So } \sin^4\theta = \left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1-\cos 2\theta}{2}\right)$$

$$= \frac{1}{4}\left(1-2\cos 2\theta+\cos^2 2\theta\right)$$

Again use the double-angle formula to write $\cos^2 2\theta$ in terms of $\cos 4\theta$

$$\text{So } \sin^4\theta = \frac{1}{4}\left(1-2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)$$

$$= \frac{3}{8}-\frac{1}{2}\cos 2\theta+\frac{1}{8}\cos 4\theta$$

30 a $R\sin(\theta+\alpha) = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha = 6\sin\theta+2\cos\theta$

So $R\cos\alpha = 6$, $R\sin\alpha = 2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 6^2+2^2 = 36+4 = 40$

$\Rightarrow R = \sqrt{40}$ (as $\cos^2\alpha+\sin^2\alpha \equiv 1$)

$\tan\alpha = \frac{2}{6} \Rightarrow \alpha = 0.32175\ldots = 0.32$ (2 d.p.)

So $6\sin\theta+2\cos\theta \approx \sqrt{40}\sin(\theta+0.32)$

b i $\sqrt{40}$, since the maximum value the sin function can take is 1

ii Maximum occurs when $\sin(\theta+0.322) = 1$

$\Rightarrow \theta+0.322 = \frac{\pi}{2}$

$\Rightarrow \theta = 1.25$ (2 d.p.)

Note that you should use a value of α to 3 decimal places in the model and then give your answers to 2 decimal places.

c $T = 9+\sqrt{40}\sin\left(\frac{\pi t}{12}+0.322\right)$

So minimum value of T is $9-\sqrt{40} = 2.68$ °C (2 d.p.)

Occurs when $\sin\left(\frac{\pi t}{12}+0.322\right) = -1$

$\Rightarrow \frac{\pi t}{12}+0.322 = \frac{3\pi}{2}$

$\Rightarrow t = 16.77$ hours

30 d $9+\sqrt{40}\sin\left(\frac{\pi t}{12}+0.322\right)=14$

$\Rightarrow \sqrt{40}\sin\left(\frac{\pi t}{12}+0.322\right)=5$

$\Rightarrow \sin\left(\frac{\pi t}{12}+0.322\right)=\frac{5}{\sqrt{40}}$

$\Rightarrow \frac{\pi t}{12}+0.322=0.9117, 2.2299$

$\Rightarrow t=2.25, 7.29$ (2 d.p.)

0.25 h ≈ 15 minutes and 0.29 h ≈ 17 minutes

So times are 11:15 am and 4:17 pm

31 a As $\frac{4}{t}\neq 0, x\neq 1$

The equation for y can be rewritten as

$y=\left(t-\frac{3}{2}\right)^2-\frac{5}{4}$

So $y \geq -1.25$

b $t=\frac{4}{1-x}$

So $y=\left(\frac{4}{1-x}\right)^2-3\left(\frac{4}{1-x}\right)+1$

$=\frac{16}{(1-x)^2}-\frac{12(1-x)}{(1-x)^2}+\frac{(1-x)^2}{(1-x)^2}$

$=\frac{16-12+12x+1-2x+x^2}{(1-x)^2}$

$=\frac{x^2+10x+5}{(1-x)^2}$

So $a=1,\ b=10,\ c=5$

32 a $x=\ln(t+2)\Rightarrow \mathrm{e}^x=t+2\Rightarrow t=\mathrm{e}^x-2$

$y=\frac{3t}{t+3}=\frac{3\mathrm{e}^x-6}{\mathrm{e}^x+1}$

$t>4\Rightarrow \mathrm{e}^x-2>4\Rightarrow \mathrm{e}^x>6\Rightarrow x>\ln 6$

So the solution is $y=\frac{3\mathrm{e}^x-6}{\mathrm{e}^x+1}, x>\ln 6$

32 b When x → ∞, y → 3

When $x = \ln 6, y = \dfrac{3\,\mathrm{e}^{\ln 6} - 6}{\mathrm{e}^{\ln 6} + 1} = \dfrac{(3\times 6) - 6}{6+1} = \dfrac{12}{7}$

So range is $\dfrac{12}{7} < y < 3$

33 $x = \dfrac{1}{1+t} \Rightarrow t = \dfrac{1}{x} - 1 = \dfrac{1-x}{x}$

$$y = \frac{1}{1 - \frac{1-x}{x}} = \frac{x}{x-(1-x)} = \frac{x}{2x-1}$$

34 a $y = \cos 3t = \cos(2t + t) = \cos 2t \cos t - \sin 2t \sin t$

$= (\cos^2 t - 1)\cos t - 2\sin^2 t \cos t$

$= 2\cos^3 t - \cos t - 2(1 - \cos^2 t)\cos t$

$= 4\cos^3 t - 3\cos t$

$x = 2\cos t \Rightarrow \cos t = \dfrac{x}{2}$

$$y = 4\left(\frac{x}{2}\right)^3 - 3\left(\frac{x}{2}\right) = \frac{1}{2}x^3 - \frac{3}{2}x = \frac{x}{2}(x^2 - 3)$$

b $0 \le t \le \dfrac{\pi}{2}$

So $0 \le \cos t \le 1$ and $-1 \le \cos 3t \le 1$

So $0 \le x \le 2, -1 \le y \le 1$

35 a $y = \sin\left(t + \dfrac{\pi}{6}\right) = \sin t \cos\dfrac{\pi}{6} + \cos t \sin\dfrac{\pi}{6}$

$= \dfrac{\sqrt{3}}{2}\sin t + \dfrac{1}{2}\cos t$

$= \dfrac{\sqrt{3}}{2}\sin t + \dfrac{1}{2}\sqrt{1 - \sin^2 t}$

$= \dfrac{\sqrt{3}}{2}x + \dfrac{1}{2}\sqrt{1 - x^2}$

As $-\dfrac{\pi}{2} \le t \le \dfrac{\pi}{2}, -1 \le \sin t \le 1 \Rightarrow -1 \le x \le 1$

35 b At A, $\sin\left(t+\frac{\pi}{6}\right)=0 \Rightarrow t=-\frac{\pi}{6}$

$x=\sin\left(-\frac{\pi}{6}\right)=-\frac{1}{2}$

Coordinates of A are $\left(-\frac{1}{2},0\right)$

At B, $x=\sin t=0 \Rightarrow t=0$

$y=\sin\left(t+\frac{\pi}{6}\right)=\sin\frac{\pi}{6}=\frac{1}{2}$

Coordinates of B are $\left(0,\frac{1}{2}\right)$

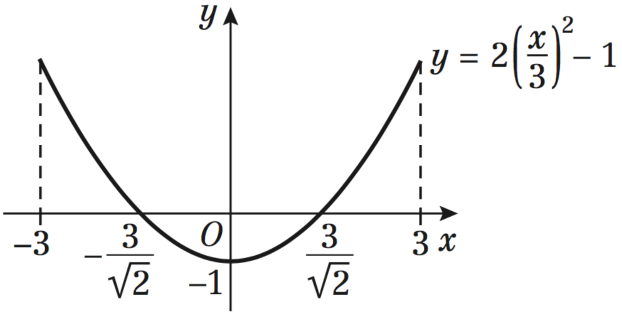
36 a $y=\cos 2t=2\cos^2 t-1$

$y=2\left(\frac{x}{3}\right)^2-1,\ -3\leq x\leq 3$

b Curve is a parabola, with a minima and y-intercept at (0,–1) and x-intercepts when

$2\left(\frac{x}{3}\right)^2=1 \Rightarrow \frac{x}{3}=\pm\frac{1}{\sqrt{2}} \Rightarrow x=\pm\frac{3}{\sqrt{2}}$

Coordinates $\left(-\frac{3}{\sqrt{2}},0\right),\left(\frac{3}{\sqrt{2}},0\right)$



37 $y=3x+c$ would intersect curve C if

$8t(2t-1)=3(4t)+c$

$16t^2-20t-c=0$

Using the quadratic formula, this equation has no real solutions if

$(-20)^2-4(16)(-c)<0$

$\Rightarrow\ 64c<-400 \Rightarrow c<-\frac{25}{4}$

38 a The curve intersects the x-axis when $2\cos t + 1 = 0 \Rightarrow \cos t = -\frac{1}{2}$

Solutions in the interval are $t = \frac{2\pi}{3}, \frac{4\pi}{3}$

$$\Rightarrow x = 3\sin\left(\frac{4\pi}{3}\right), 3\sin\left(\frac{8\pi}{3}\right)$$

So coordinates are $\left(-\frac{3\sqrt{3}}{2}, 0\right)$ and $\left(\frac{3\sqrt{3}}{2}, 0\right)$

b $3\sin 2t = 1.5 \Rightarrow \sin 2t = \frac{1}{2}$

In the interval $\pi \leq 2t \leq 3\pi$ solutions are

$$2t = \frac{13\pi}{6}, \frac{17\pi}{6}$$

$$\Rightarrow t = \frac{13\pi}{12}, \frac{17\pi}{12}$$

39 a Find the time the ball hits the ground by solving $-4.9t^2 + 25t + 50 = 0$

$$t = \frac{-25 \pm \sqrt{25^2 - 4(-4.9)(50)}}{2(-4.9)}$$

$t \geq 0$, so only valid solution is $t = 6.64$ s (2 d.p.)

$\Rightarrow k = 6.64$ (2 d.p.)

b $t = \frac{x}{25\sqrt{3}}$

$$y = 25\left(\frac{x}{25\sqrt{3}}\right) - 4.9\left(\frac{x}{25\sqrt{3}}\right)^2 + 50$$

$$= \frac{x}{\sqrt{3}} - \frac{49}{18750}x^2 + 50$$

Domain of the function is from where the ball is hit at $x = 0$ to where it hits the ground when $t = 6.64$ seconds.

When $t = 6.64$, $x = 25\sqrt{3}(6.64) = 287.5$ (1 d.p.)

So domain is $0 \leq x \leq 287.5$

Challenge

1 Angle of minor arc = $\frac{\pi}{2}$ because it is a quarter circle

Let the chord meet the circle at *R* and *T*. The area of *P* is the area of sector formed by *O*, *R* and *T* less the area of the triangle *ORT*.

So area of P = $\frac{1}{2}r^2\frac{\pi}{2}-\frac{1}{2}r^2\sin\frac{\pi}{2}=r^2\left(\frac{\pi}{4}-\frac{1}{2}\right)=\frac{r^2}{4}(\pi-2)$

Area of Q = $\pi r^2 -$ area of P

$$=r^2\left(\pi-\frac{\pi}{4}+\frac{1}{2}\right)=r^2\left(\frac{3\pi}{4}+\frac{1}{2}\right)=\frac{r^2}{4}(3\pi+2)$$

So ratio = $(\pi-2):(3\pi+2)=\frac{\pi-2}{3\pi+2}:1$

2 a $\sin x$

b $\cos x$

c $\angle COA=\frac{\pi}{2}-x\Rightarrow\angle CAO=x$

$OA=1\div\sin x=\operatorname{cosec} x$

d $AC=1\div\tan x=\cot x$

e $\tan x$

f $OB=1\div\cos x=\sec x$

3 a $\sin t = \frac{x-3}{4}, \cos t = \frac{y+1}{4}$

As $\sin^2 t + \cos^2 t = 1$

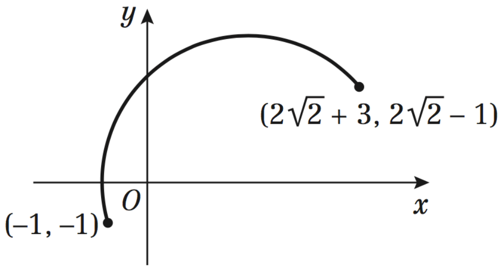
$$\left(\frac{x-3}{4}\right)^2 + \left(\frac{y+1}{4}\right)^2 = 1$$

$$\Rightarrow (x-3)^2 + (y+1)^2 = 16$$

The curve is a circle centre (3, –1) and radius 4.

Endpoints when $t = -\frac{\pi}{2}, x = -1, y = -1$

and when $t = \frac{\pi}{4}, x = 2\sqrt{2}+3, y = 2\sqrt{2}-1$



b C is $\frac{3}{8}$ths of a circle, radius 4

So length = $\frac{3}{8} \times 8\pi = 3\pi$

Differentiation 9A

1 a $\mathrm{f}(x) = \cos x$

$$\begin{aligned}\mathrm{f}'(x) &= \lim_{h\to 0}\frac{\mathrm{f}(x+h)-\mathrm{f}(x)}{h}\\ &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}\\ &= \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}\\ &= \lim_{h\to 0}\left(\left(\frac{\cos h-1}{h}\right)\cos x-\frac{\sin h}{h}\sin x\right)\end{aligned}$$

b Since $\frac{\cos h-1}{h}\to 0$ and $\frac{\sin h}{h}\to 1$

the expression inside the limit in part **a** tends to $0\times\cos x-1\times\sin x=-\sin x$

So $\mathrm{f}'(x)=-\sin x$

2 a $y=2\cos x$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=2\times(-\sin x)=-2\sin x$$

b $y=2\sin\frac{1}{2}x$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=2\times\frac{1}{2}\cos\frac{1}{2}x=\cos\frac{1}{2}x$$

c $y=\sin 8x$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=8\cos 8x$$

d $y=6\sin\frac{2}{3}x$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=6\times\frac{2}{3}\cos\frac{2}{3}x=4\cos\frac{2}{3}x$$

3 a $\mathrm{f}(x)=2\cos x$

$$\mathrm{f}'(x)=2\times(-\sin x)=-2\sin x$$

b $\mathrm{f}(x)=6\cos\frac{5}{6}x$

$$\mathrm{f}'(x)=6\times\left(-\frac{5}{6}\sin\frac{5}{6}x\right)=-5\sin\frac{5}{6}x$$

c $\mathrm{f}(x)=4\cos\frac{1}{2}x$

$$\mathrm{f}'(x)=4\times\left(-\frac{1}{2}\sin\frac{1}{2}x\right)=-2\sin\frac{1}{2}x$$

d $\mathrm{f}(x)=3\cos 2x$

$$\mathrm{f}'(x)=3(-2\sin 2x)=-6\sin 2x$$

4 a $y=\sin 2x+\cos 3x$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}&=2\cos 2x+(-3\sin 3x)\\ &=2\cos 2x-3\sin 3x\end{aligned}$$

b $y=2\cos 4x-4\cos x+2\cos 7x$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}&=2\times(-4\sin 4x)-4\times(-\sin x)\\ &\quad+2\times(-7\sin 7x)\\ &=-8\sin 4x+4\sin x-14\sin 7x\end{aligned}$$

c $y=x^2+4\cos 3x$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=2x+4(-3\sin 3x)=2x-12\sin 3x$$

d $y=\frac{1+2x\sin 5x}{x}=\frac{1}{x}+2\sin 5x$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}&=-\frac{1}{x^2}+2\times(5\cos 5x)\\ &=-\frac{1}{x^2}+10\cos 5x\end{aligned}$$

5 $y = x - \sin 3x$

$\frac{dy}{dx} = 1 - 3\cos 3x$

At stationary points $\frac{dy}{dx} = 0$

$1 - 3\cos 3x = 0$

$\cos 3x = \frac{1}{3}$

$3x = 1.23..., 5.05...$ or $7.51...$

$x = 0.410, 1.68$ or 2.50 (3 s.f.)

$x = 0.410 \Rightarrow y = 0.41 - \sin 1.23 = -0.532$
$x = 1.68 \Rightarrow y = 1.68 - \sin 5.04 = 2.63$
$x = 2.50 \Rightarrow y = 2.50 - \sin 7.50 = 1.56$

Stationary points in the interval $0 \le x \le \pi$ are (0.410, −0.532), (1.68, 2.63) and (2.50, 1.56).

6 $y = 2\sin 4x - 4\cos 2x$

$\frac{dy}{dx} = 2 \times 4\cos 4x - 4 \times (-2\sin 2x)$
$= 8\cos 4x + 8\sin 2x$

When $x = \frac{\pi}{2}$:

$\frac{dy}{dx} = 8\cos 2\pi + 8\sin \pi$
$= 8 \times 1 + 8 \times 0 = 8$

So the gradient of the curve at the point where $x = \frac{\pi}{2}$ is 8.

7 $y = 2\sin 2x + \cos 2x$

$\frac{dy}{dx} = 2 \times 2\cos 2x + (-2\sin 2x)$
$= 4\cos 2x - 2\sin 2x$

At stationary points $\frac{dy}{dx} = 0$
$4\cos 2x - 2\sin 2x = 0$
$4 - 2\tan 2x = 0$
$\tan 2x = 2$
$2x = 1.107...$ or $4.248...$
$x = 0.554$ or 2.12 (3 s.f.)

When $x = 0.554$:
$y = 2\sin(2 \times 0.554) + \cos(2 \times 0.554) = 2.24$
When $x = 2.12$:
$y = 2\sin(2 \times 2.12) + \cos(2 \times 2.12) = -2.24$

Stationary points in the interval $0 \leqslant x \leqslant \pi$ are (0.554, 2.24) and (2.12, −2.24).

8 $y = \sin 5x + \cos 3x$

$\frac{dy}{dx} = 5\cos 5x - 3\sin 3x$

At $(\pi, -1)$, $\frac{dy}{dx} = 5\cos 5\pi - 3\sin 3\pi$
$= 5 \times (-1) - 3 \times 0 = -5$

Equation of tangent is $y - (-1) = -5(x - \pi)$
or $y = -5x + 5\pi - 1$

9 $y = 2x^2 - \sin x$

$\frac{dy}{dx} = 4x - \cos x$

When $x = \pi$, $y = 2\pi^2$ and

$\frac{dy}{dx} = 4\pi - \cos \pi = 4\pi + 1$

Gradient of normal is $-\frac{1}{4\pi + 1}$

Equation of normal is

$y - 2\pi^2 = -\frac{1}{4\pi + 1}(x - \pi)$

Multiplying through by (4π + 1) and rearranging gives
$x + (4\pi + 1)y - \pi(8\pi^2 + 2\pi + 1) = 0$

10 Let $\mathrm{f}(x)=\sin x$

$$\mathrm{f}'(x)=\lim_{h\to 0}\frac{\mathrm{f}(x+h)-\mathrm{f}(x)}{h}=\frac{\sin(x+h)-\sin x}{h}$$

$$=\lim_{h\to 0}\left(\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}\right)$$

$$=\lim_{h\to 0}\left(\left(\frac{\cos h-1}{h}\right)\sin x+\left(\frac{\sin h}{h}\right)\cos x\right)$$

Since $\frac{\cos h-1}{h}\to 0$ and $\frac{\sin h}{h}\to 1$,

the expression inside the limit tends to

$(0\times\sin x+1\times\cos x)$

So $\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}=\cos x$

Hence the derivative of $\sin x$ is $\cos x$.

Challenge

Let $\mathrm{f}(x)=\sin(kx)$

$$\mathrm{f}'(x)=\lim_{h\to 0}\left(\frac{\mathrm{f}(x+h)-\mathrm{f}(x)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{\sin(kx+kh)-\sin(kx)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{\sin kx\cos kh+\cos kx\sin kh-\sin kx}{h}\right)$$

$$=\lim_{h\to 0}\left(\left(\frac{\cos kh-1}{h}\right)\sin kx+\left(\frac{\sin kh}{h}\right)\cos kx\right)$$

As $h\to 0$, $\left(\frac{\sin kh}{h}\right)\to k$ and $\left(\frac{\cos kh-1}{h}\right)\to 0$,

so the expression inside the limit tends to

$0\times\sin kx+k\times\cos kx=k\cos kx$

Hence the derivative of $\sin(kx)$ is $k\cos(kx)$.

Differentiation 9B

1 a $y = 4e^{7x}$

$$\frac{dy}{dx} = 4 \times 7e^{7x} = 28e^{7x}$$

b $y = 3^x$

$$y = e^{\ln(3^x)} = e^{x\ln 3} = e^{(\ln 3)x}$$

$$\frac{dy}{dx} = \ln 3\, e^{(\ln 3)x} = \ln 3\, e^{\ln(3^x)} = 3^x \ln 3$$

c $y = \left(\frac{1}{2}\right)^x$

Using the result $y = a^{kx} \Rightarrow \frac{dy}{dx} = a^{kx} k \ln a$

with $a = \frac{1}{2}$ and $k = 1$:

$$\frac{dy}{dx} = \left(\frac{1}{2}\right)^x \ln\frac{1}{2}$$

d $y = \ln 5x$

$$y = \ln 5 + \ln x$$

$$\frac{dy}{dx} = 0 + \frac{1}{x} = \frac{1}{x}$$

e $y = 4\left(\frac{1}{3}\right)^x$

Using the result $y = a^{kx} \Rightarrow \frac{dy}{dx} = a^{kx} k \ln a$

with $a = \frac{1}{3}$ and $k = 1$:

$$\frac{dy}{dx} = 4\left(\frac{1}{3}\right)^x \ln\frac{1}{3}$$

f $y = \ln(2x^3)$

$$y = \ln 2 + \ln(x^3) = \ln 2 + 3\ln x$$

$$\frac{dy}{dx} = 0 + 3 \times \frac{1}{x} = \frac{3}{x}$$

g $y = e^{3x} - e^{-3x}$

$$\frac{dy}{dx} = 3e^{3x} - (-3e^{-3x})$$

$$= 3e^{3x} + 3e^{-3x}$$

h $y = \frac{(1+e^x)^2}{e^x}$

$$y = \frac{1 + 2e^x + (e^x)^2}{e^x} = e^{-x} + 2 + e^x$$

$$\frac{dy}{dx} = -e^{-x} + 0 + e^x = -e^{-x} + e^x$$

2 a $f(x) = 3^{4x}$

$$f(x) = e^{\ln(3^{4x})} = e^{4x\ln 3} = e^{(4\ln 3)x}$$

$$f'(x) = (4\ln 3)e^{(4\ln 3)x} = 4\ln 3\, e^{4x\ln 3}$$

$$= 4\ln 3\, e^{\ln 3^{4x}} = 3^{4x} 4\ln 3$$

b $f(x) = \left(\frac{3}{2}\right)^{2x}$

Using the result $y = a^{kx} \Rightarrow \frac{dy}{dx} = a^{kx} k \ln a$

with $a = \frac{3}{2}$ and $k = 2$:

$$f'(x) = \left(\frac{3}{2}\right)^{2x} 2\ln\frac{3}{2}$$

c $f(x) = 2^{4x} + 4^{2x}$

Using the result $y = a^{kx} \Rightarrow \frac{dy}{dx} = a^{kx} k \ln a$

for each term:

$$f'(x) = 2^{4x} 4\ln 2 + 4^{2x} 2\ln 4$$

Alternatively,

$$f(x) = 2^{4x} + (2^2)^{2x} = 2^{4x} + 2^{4x} = 2 \times 2^{4x}$$

$$f'(x) = 2 \times 2^{4x} 4\ln 2$$

d $f(x) = \frac{2^{7x} + 8^x}{4^{2x}}$

$$f(x) = \frac{2^{7x} + 2^{3x}}{2^{4x}} = 2^{3x} + 2^{-x}$$

$$f'(x) = 2^{3x} 3\ln 2 + 2^{-x}(-1)\ln 2$$

$$= 2^{3x} 3\ln 2 - 2^{-x}\ln 2$$

3 $y = (e^{2x} - e^{-2x})^2 = e^{4x} - 2 + e^{-4x}$

$\frac{dy}{dx} = 4e^{4x} - 4e^{-4x} = 4(e^{4x} - e^{-4x})$

Where $x = \ln 3$:

$\frac{dy}{dx} = 4(e^{4\ln 3} - e^{-4\ln 3}) = 4\left(e^{\ln 3^4} - e^{\ln 3^{-4}}\right)$

$= 4\left(3^4 - 3^{-4}\right) = 4\left(81 - \frac{1}{81}\right)$

≈ 323.95

4 $y = 2^x + 2^{-x}$

$\frac{dy}{dx} = 2^x \ln 2 - 2^{-x} \ln 2$

When $x = 2$, $\frac{dy}{dx} = 4 \ln 2 - \frac{1}{4} \ln 2 = \frac{15}{4} \ln 2$

$\therefore$ the equation of the tangent at $\left(2, \frac{17}{4}\right)$ is

$y - \frac{17}{4} = \frac{15}{4} \ln 2(x - 2)$

or $4y = (15 \ln 2)x + (17 - 30 \ln 2)$

5 $y = e^{2x} - \ln x$

$\frac{dy}{dx} = 2e^{2x} - \frac{1}{x}$

When $x = 1$, $y = e^2$ and $\frac{dy}{dx} = 2e^2 - 1$

Equation of tangent at $(1, e^2)$ is

$y - e^2 = (2e^2 - 1)(x - 1)$

Rearranging gives

$y = (2e^2 - 1)x - 2e^2 + 1 + e^2$

or $y = (2e^2 - 1)x - e^2 + 1$

6 $R = 200 \times 0.9^t$

$\frac{dR}{dt} = 200 \times (0.9)^t \ln 0.9 = 100 \ln 0.9 \times (0.9)^t$

When $t = 8$:

$\frac{dR}{dt} = 200 \ln 0.9 \times 0.9^8 = -9.07$ (3 s.f.)

7 **a** When $t = 0$, $P = 37\,000$

So $37\,000 = P_0 k^0 = P_0$

$P_0 = 37\,000$

When $t = 100$, $P = 109\,000$

So $109\,000 = 37\,000 k^{100}$

$\frac{109}{37} = k^{100}$

and hence $k = \left(\frac{109}{37}\right)^{\frac{1}{100}}$

$= 1.01086287...$

$= 1.01$ (2 d.p.)

b $P = P_0 k^t \Rightarrow \frac{dP}{dt} = P_0 k^t \ln k$

With $P_0 = 37\,000$, $k = 1.01086...$, $t = 100$:

$\frac{dP}{dt} = 37\,000 \times 1.0108629^{100} \times \ln 1.0108629$

$= 1178$

c The rate of change of population in the year 2000.

8 The student has treated $\ln kx$ as if it were e^{kx} – they applied the incorrect differentiation formula.

The correct derivative is $\frac{1}{x}$

9 Let $y = a^{kx}$

Then $y = e^{\ln a^{kx}} = e^{kx \ln a} = e^{(k \ln a)x}$

$\frac{dy}{dx} = (k \ln a) e^{(k \ln a)x} = k \ln a\, e^{kx \ln a}$

$= k \ln a\, e^{\ln a^{kx}} = a^{kx} k \ln a$

10 a $\mathrm{f}(x)=\mathrm{e}^{2x}-\ln(x^2)+4=\mathrm{e}^{2x}-2\ln x+4$

$$\mathrm{f}'(x)=2\mathrm{e}^{2x}-\frac{2}{x}$$

b At P,

$\mathrm{f}'(x)=2$ and $x=a$

$$\text{so } 2\mathrm{e}^{2a}-\frac{2}{a}=2$$

$$\mathrm{e}^{2a}-\frac{1}{a}-1=0$$

$$a\mathrm{e}^{2a}-1-a=0$$

$$\therefore\ a(\mathrm{e}^{2a}-1)=1$$

11 a $y=5\sin 3x+2\cos 3x$

When $x=0$,

$y=5\sin 0+2\cos 0=0+2=2$

Hence P (0, 2) lies on the curve.

b $\frac{\mathrm{d}y}{\mathrm{d}x}=15\cos 3x-6\sin 3x$

When $x=0$, $\frac{\mathrm{d}y}{\mathrm{d}x}=15\cos 0-6\sin 0=15$

Equation of normal at P is

$$y-2=-\frac{1}{15}(x-0)$$

or $y=-\frac{1}{15}x+2$

12 $y=2\times 3^{4x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=2\times 3^{4x}\,4\ln 3=8\ln 3\times 3^{4x}$$

When $x=1$, $y=2\times 81=162$

and $\frac{\mathrm{d}y}{\mathrm{d}x}=8\ln 3\times 3^4=648\ln 3$

Equation of normal at P is

$$y-162=-\frac{1}{648\ln 3}\left(x-1\right)$$

or $y=-\frac{1}{648\ln 3}x+\frac{1}{648\ln 3}+162$

Challenge

$$y=\mathrm{e}^{4x}-5x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=4\mathrm{e}^{4x}-5$$

Lines parallel to $y=3x+4$ have gradient 3.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=3\Rightarrow 4\mathrm{e}^{4x}-5=3$$

$$\mathrm{e}^{4x}=2$$

$$4x=\ln 2$$

$$x=\frac{\ln 2}{4}$$

When $x=\frac{\ln 2}{4}$, $y=\mathrm{e}^{\ln 2}-5\frac{\ln 2}{4}=2-5\frac{\ln 2}{4}$

Equation of tangent at this point is

$$y-\left(2-5\frac{\ln 2}{4}\right)=3\left(x-\frac{\ln 2}{4}\right)$$

$$y=3x-3\frac{\ln 2}{4}+2-5\frac{\ln 2}{4}$$

$$y=3x-2\ln 2+2$$

Differentiation 9C

1 a $y=(1+2x)^4$

Let $u=1+2x$; then $y=u^4$

$$\frac{du}{dx}=2 \quad \text{and} \quad \frac{dy}{du}=4u^3$$

Using the chain rule,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=4u^3\times 2=8u^3 = 8(1+2x)^3$$

b $y=(3-2x^2)^{-5}$

Let $u=3-2x^2$; then $y=u^{-5}$

$$\frac{du}{dx}=-4x \quad \text{and} \quad \frac{dy}{du}=-5u^{-6}$$

Using the chain rule,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=(-5u^{-6})\times(-4x)=20xu^{-6}$$
$$=20x(3-2x^2)^{-6}$$

c $y=(3+4x)^{\frac{1}{2}}$

Let $u=3+4x$; then $y=u^{\frac{1}{2}}$

$$\frac{du}{dx}=4 \quad \text{and} \quad \frac{dy}{du}=\frac{1}{2}u^{-\frac{1}{2}}$$

Using the chain rule,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{1}{2}u^{-\frac{1}{2}}\times 4$$
$$=2u^{-\frac{1}{2}}=2(3+4x)^{-\frac{1}{2}}$$

d $y=(6x+x^2)^7$

Let $u=6x+x^2$; then $y=u^7$

$$\frac{du}{dx}=6+2x \quad \text{and} \quad \frac{dy}{du}=7u^6$$

Using the chain rule,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=7u^6\times(6+2x)$$
$$=7(6+2x)(6x+x^2)^6$$

e $y=\dfrac{1}{3+2x}=(3+2x)^{-1}$

Let $u=3+2x$; then $y=\dfrac{1}{u}=u^{-1}$

$$\frac{du}{dx}=2 \quad \text{and} \quad \frac{dy}{du}=-u^{-2}$$

Using the chain rule,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=-u^{-2}\times 2$$
$$=-2u^{-2}=\frac{-2}{(3+2x)^2}$$

f $y=\sqrt{7-x}=(7-x)^{\frac{1}{2}}$

Let $u=7-x$; then $y=u^{\frac{1}{2}}$

$$\frac{du}{dx}=-1 \quad \text{and} \quad \frac{dy}{du}=\frac{1}{2}u^{-\frac{1}{2}}$$

Using the chain rule,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{1}{2}u^{-\frac{1}{2}}\times(-1)$$
$$=-\frac{1}{2}(7-x)^{-\frac{1}{2}}=-\frac{1}{2\sqrt{7-x}}$$

1 g $y = 4(2+8x)^4$

Let $u = 2+8x$; then $y = 4u^4$

$$\frac{du}{dx} = 8 \quad \text{and} \quad \frac{dy}{du} = 16u^3$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 16u^3 \times 8 = 128(2+8x)^3$$

h $y = 3(8-x)^{-6}$

Let $u = 8-x$; then $y = 3u^{-6}$

$$\frac{du}{dx} = -1 \quad \text{and} \quad \frac{dy}{du} = -18u^{-7}$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -18u^{-7} \times (-1) = 18(8-x)^{-7}$$

2 a $y = e^{\cos x}$

Let $u = \cos x$; then $y = e^u$

$$\frac{du}{dx} = -\sin x \quad \text{and} \quad \frac{dy}{du} = e^u$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= e^u \times (-\sin x) = -\sin x \, e^{\cos x}$$

b $y = \cos(2x-1)$

Let $u = 2x-1$; then $y = \cos u$

$$\frac{du}{dx} = 2 \quad \text{and} \quad \frac{dy}{du} = -\sin u$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\sin u \times 2 = -2\sin(2x-1)$$

c $y = \sqrt{\ln x}$

Let $u = \ln x$; then $y = \sqrt{u} = u^{\frac{1}{2}}$

$$\frac{du}{dx} = -\frac{1}{x} \quad \text{and} \quad \frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \times \left(-\frac{1}{x}\right)$$

$$= \frac{1}{2xu^{\frac{1}{2}}} = \frac{1}{2x\sqrt{\ln x}}$$

d $y = (\sin x + \cos x)^5$

Let $u = \sin x + \cos x$; then $y = u^5$

$$\frac{du}{dx} = \cos x - \sin x \quad \text{and} \quad \frac{dy}{du} = 5u^4$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 5u^4 \times (\cos x - \sin x)$$

$$= 5(\cos x - \sin x)(\sin x + \cos x)^4$$

e $y = \sin(3x^2 - 2x + 1)$

Let $u = 3x^2 - 2x + 1$; then $y = \sin u$

$$\frac{du}{dx} = 6x - 2 \quad \text{and} \quad \frac{dy}{du} = \cos u$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \cos u \times (6x-2)$$

$$= (6x-2)\cos(3x^2 - 2x + 1)$$

f $y = \ln(\sin x)$

Let $u = \sin x$; then $y = \ln u$

$$\frac{du}{dx} = \cos x \quad \text{and} \quad \frac{dy}{du} = \frac{1}{u}$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times \cos x = \frac{\cos x}{\sin x} = \cot x$$

g $y = 2e^{\cos 4x}$

Let $u = \cos 4x$; then $y = 2e^u$

$$\frac{du}{dx} = -4\sin 4x \quad \text{and} \quad \frac{dy}{du} = 2e^u$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 2e^u \times (-4\sin 4x)$$

$$= -8\sin 4x \, e^{\cos x}$$

2 h $y = \cos(e^{2x} + 3)$

Let $u = e^{2x} + 3$; then $y = \cos u$

$\frac{du}{dx} = 2e^{2x}$ and $\frac{dy}{du} = -\sin u$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\sin u \times 2e^{2x}$$
$$= -2e^{2x} \sin(e^{2x} + 3)$$

3 $y = \frac{1}{(4x+1)^2}$

Let $u = 4x+1$; then $y = u^{-2}$

$\frac{du}{dx} = 4$ and $\frac{dy}{du} = -2u^{-3}$

$$\therefore \frac{dy}{dx} = -8u^{-3} = \frac{-8}{(4x+1)^3}$$

When $x = \frac{1}{4}$,

$$\frac{dy}{dx} = \frac{-8}{\left(4\times\frac{1}{4}+1\right)^3} = \frac{-8}{2^3} = -1$$

4 $y = (5-2x)^3$

Let $u = 5-2x$; then $y = u^3$

$\frac{du}{dx} = -2$ and $\frac{dy}{du} = 3u^2$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3u^2 \times (-2) = -6(5-2x)^2$$

When $x = 1$,

$y = 3^3 = 27$ and $\frac{dy}{dx} = -6 \times 3^2 = -54$

Equation of tangent at point P (1, 27) is

$y - 27 = -54(x-1)$

or $y = -54x + 81$

5 $y = (1 + \ln 4x)^{\frac{3}{2}}$

Let $u = 1 + \ln 4x$; then $y = u^{\frac{3}{2}}$

$\frac{du}{dx} = \frac{1}{x}$ and $\frac{dy}{du} = \frac{3}{2}u^{\frac{1}{2}}$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{3}{2}u^{\frac{1}{2}} \times \frac{1}{x} = \frac{3}{2x}\sqrt{1+\ln 4x}$$

When $x = \frac{1}{4}e^3$,

$$\frac{dy}{dx} = \frac{3}{\frac{1}{2}e^3}\sqrt{1+\ln e^3} = \frac{6}{e^3}\sqrt{1+3} = 12e^{-3}$$

6 a $x = y^2 + y$

$$\frac{dx}{dy} = 2y + 1$$
$$\therefore \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{2y+1}$$

b $x = e^y + 4y$

$$\frac{dx}{dy} = e^y + 4$$
$$\therefore \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{e^y + 4}$$

c $x = \sin 2y$

$$\frac{dx}{dy} = 2\cos 2y$$
$$\therefore \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{2\cos 2y} = \frac{1}{2}\sec 2y$$

d $4x = \ln y + y^3$

$$x = \frac{1}{4}\ln y + \frac{1}{4}y^3$$
$$\frac{dx}{dy} = \frac{1}{4y} + \frac{3}{4}y^2 = \frac{1+3y^3}{4y}$$
$$\therefore \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{4y}{1+3y^3}$$

7 $x=3y^2-2y$

$$\frac{dx}{dy}=6y-2$$

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}=\frac{1}{6y-2}$$

At (8, 2) the value of y is 2.

$$\therefore\ \frac{dy}{dx}=\frac{1}{12-2}=\frac{1}{10}$$

8 $x=y^{\frac{1}{2}}+y^{-\frac{1}{2}}$

$$\frac{dx}{dy}=\frac{1}{2}y^{-\frac{1}{2}}-\frac{1}{2}y^{-\frac{3}{2}}$$

$$\frac{dy}{dx}=\frac{1}{\frac{1}{2}y^{-\frac{1}{2}}-\frac{1}{2}y^{-\frac{3}{2}}}$$

At the point $\left(\frac{5}{2},4\right)$ the value of y is 4.

$$\therefore\ \frac{dy}{dx}=\frac{1}{\frac{1}{2}(4)^{-\frac{1}{2}}-\frac{1}{2}(4)^{-\frac{3}{2}}}=\frac{1}{\frac{1}{2}\times\frac{1}{2}-\frac{1}{2}\times\frac{1}{8}}$$
$$=\frac{1}{\frac{1}{4}-\frac{1}{16}}=\frac{1}{\frac{3}{16}}=\frac{16}{3}$$

9 **a** $x=e^y$

$$\Rightarrow\frac{dx}{dy}=e^y$$

b $y=\ln x\ \Rightarrow\ e^y=x$

From part **a**, $\frac{dx}{dy}=e^y$

$$\therefore\ \frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}=\frac{1}{e^y}=\frac{1}{x}$$

10 a $x=4\cos 2y$

When $x=2,\ \cos 2y=\frac{1}{2}$

So $2y=\frac{\pi}{3},\frac{5\pi}{3},\ldots$

$y=\frac{\pi}{6},\frac{5\pi}{6},\ldots$

Hence $Q\left(2,\frac{\pi}{6}\right)$ lies on C.

b $\frac{dx}{dy}=-8\sin 2y\ \Rightarrow\ \frac{dy}{dx}=-\frac{1}{8\sin 2y}$

At Q, $y=\frac{\pi}{6}$

so $\frac{dy}{dx}=-\frac{1}{8\sin\frac{\pi}{3}}=-\frac{1}{8\times\frac{\sqrt{3}}{2}}=-\frac{1}{4\sqrt{3}}$

c Equation of normal to C at Q is

$$y-\frac{\pi}{6}=4\sqrt{3}(x-2)$$

or $4\sqrt{3}x-y-8\sqrt{3}+\frac{\pi}{6}=0$

11 a $y=\sin^2 3x=(\sin 3x)^2$

Let $u=\sin 3x$; then $y=u^2$

$\frac{du}{dx}=3\cos 3x$ and $\frac{dy}{du}=2u$

Using the chain rule,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=2u\times 3\cos 3x$$
$$=6\sin 3x\cos 3x$$

b $y=e^{(x+1)^2}$

Let $u=(x+1)^2$; then $y=e^u$

$\frac{du}{dx}=2(x+1)$ and $\frac{dy}{du}=e^u$

Using the chain rule,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=e^u\times 2(x+1)=2(x+1)e^{(x+1)^2}$$

11 c $y = \ln(\cos x)^2$

Let $u = \cos x$; then $y = \ln u^2 = 2\ln u$

$$\frac{du}{dx} = -\sin x \quad \text{and} \quad \frac{dy}{du} = \frac{2}{u}$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{2}{u} \times (-\sin x)$$

$$= -2\frac{\sin x}{\cos x} = -2\tan x$$

d $y = \dfrac{1}{3+\cos 2x}$

Let $u = 3 + \cos 2x$; then $y = \dfrac{1}{u} = u^{-1}$

$$\frac{du}{dx} = -2\sin 2x \quad \text{and} \quad \frac{dy}{du} = -u^{-2} = -\frac{1}{u^2}$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{1}{u^2} \times (-2\sin 2x)$$

$$= \frac{2\sin 2x}{(3+\cos 2x)^2}$$

e $y = \sin\left(\dfrac{1}{x}\right)$

Let $u = \dfrac{1}{x}$; then $y = \sin u$

$$\frac{du}{dx} = -\frac{1}{x^2} \quad \text{and} \quad \frac{dy}{du} = \cos u$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \cos u \times \left(-\frac{1}{x^2}\right)$$

$$= -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$$

12 $y = \dfrac{4}{(2-4x)^2}$

Let $u = 2 - 4x$; then $y = \dfrac{4}{u^2} = 4u^{-2}$

$$\frac{du}{dx} = -4 \quad \text{and} \quad \frac{dy}{du} = -8u^{-3} = -\frac{8}{u^3}$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{8}{u^3} \times (-4) = \frac{32}{(2-4x)^3}$$

When $x = 3$, $y = \dfrac{4}{(-10)^2} = 0.04$

and $\dfrac{dy}{dx} = \dfrac{32}{(-10)^3} = -0.032$

Equation of normal at A is

$$y - 0.04 = \frac{1}{0.032}(x-3)$$

Multiplying through by 100 and rearranging gives

$100y - 4 = 3125x - 9375$

$3125x - 100y - 9371 = 0$

13 $y = 3^{x^3}$

Let $u = x^3$; then $y = 3^u$

$$\frac{du}{dx} = 3x^2 \quad \text{and} \quad \frac{dy}{du} = 3^u \ln 3$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3^u \ln 3 \times 3x^2 = 3x^2 3^{x^3} \ln 3$$

When $x = 1$, $\dfrac{dy}{dx} = 3 \times 1^2 \times 3^{1^3} \times \ln 3 = 9\ln 3$

Challenge

a $y=\sqrt{\sin\sqrt{x}}$

Let $u=\sqrt{x}=x^{\frac{1}{2}}$; then $y=\sqrt{\sin u}$

$$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2}x^{-\frac{1}{2}}$$

Let $v=\sin u$; then $y=\sqrt{v}=v^{\frac{1}{2}}$

$$\frac{\mathrm{d}v}{\mathrm{d}u}=\cos u \quad\text{and}\quad \frac{\mathrm{d}y}{\mathrm{d}v}=\frac{1}{2}v^{-\frac{1}{2}}$$

Using the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}u}=\frac{\mathrm{d}y}{\mathrm{d}v}\times\frac{\mathrm{d}v}{\mathrm{d}u}=\frac{1}{2}v^{-\frac{1}{2}}\times\cos u$$

$$=\frac{\cos u}{2y}=\frac{\cos\sqrt{x}}{2\sqrt{\sin\sqrt{x}}}$$

Using the chain rule again,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{\cos\sqrt{x}}{2\sqrt{\sin\sqrt{x}}}\times\frac{1}{2}x^{-\frac{1}{2}}$$

$$=\frac{\cos\sqrt{x}}{4\sqrt{x\sin\sqrt{x}}}$$

b $\ln y=\sin^3(3x+4)$

Hence $y=\mathrm{e}^{\sin^3(3x+4)}$

Let $u=\sin(3x+4)$; then $y=\mathrm{e}^{u^3}$

$$\frac{\mathrm{d}u}{\mathrm{d}x}=3\cos(3x+4)$$

Let $v=u^3$; then $y=\mathrm{e}^{v}$

$$\frac{\mathrm{d}v}{\mathrm{d}u}=3u^2 \quad\text{and}\quad \frac{\mathrm{d}y}{\mathrm{d}v}=\mathrm{e}^{v}$$

Using the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}u}=\frac{\mathrm{d}y}{\mathrm{d}v}\times\frac{\mathrm{d}v}{\mathrm{d}u}=\mathrm{e}^{v}\times 3u^2=3u^2\mathrm{e}^{u^3}$$

Using the chain rule again,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}=3u^2\mathrm{e}^{u^3}\times 3\cos(3x+4)$$

$$=3\sin^2(3x+4)\mathrm{e}^{\sin^3(3x+4)}\times 3\cos(3x+4)$$

$$=9\mathrm{e}^{\sin^3(3x+4)}\cos(3x+4)\sin^2(3x+4)$$

Differentiation 9D

1 a Let $y = x(1+3x)^5$

Let $u = x$ and $v = (1+3x)^5$

Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = 3 \times 5(1+3x)^4$
(using the chain rule)

Using $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= x \times 15(1+3x)^4 + (1+3x)^5 \times 1 \\ &= (1+3x)^4(15x+1+3x) \\ &= (1+3x)^4(1+18x)\end{aligned}$$

b Let $y = 2x(1+3x^2)^3$

Let $u = 2x$ and $v = (1+3x^2)^3$

Then $\frac{du}{dx} = 2$ and $\frac{dv}{dx} = 18x(1+3x^2)^2$

Using $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= 2x \times 18x(1+3x^2)^2 + (1+3x^2)^3 \times 2 \\ &= (1+3x^2)^2\left(36x^2 + 2(1+3x^2)\right) \\ &= (1+3x^2)^2(42x^2+2) \\ &= 2(1+3x^2)^2(21x^2+1)\end{aligned}$$

c Let $y = x^3(2x+6)^4$

Let $u = x^3$ and $v = (2x+6)^4$

Then $\frac{du}{dx} = 3x^2$ and $\frac{dv}{dx} = 8(2x+6)^3$

Using $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= x^3 \times 8(2x+6)^3 + (2x+6)^4 \times 3x^2 \\ &= x^2(2x+6)^3\left(8x+3(2x+6)\right) \\ &= x^2(2x+6)^3(14x+18) \\ &= x^2 \times 2^3(x+3)^3 \times 2(7x+9) \\ &= 16x^2(x+3)^3(7x+9)\end{aligned}$$

d Let $y = 3x^2(5x-1)^{-1}$

Let $u = 3x^2$ and $v = (5x-1)^{-1}$

Then $\frac{du}{dx} = 6x$ and $\frac{dv}{dx} = -5(5x-1)^{-2}$

Using $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= 3x^2 \times \left(-5(5x-1)^{-2}\right) + (5x-1)^{-1} \times 6x \\ &= -15x^2(5x-1)^{-2} + 6x(5x-1)^{-1} \\ &= 3x(5x-1)^{-2}\left(-5x+2(5x-1)\right) \\ &= 3x(5x-2)(5x-1)^{-2}\end{aligned}$$

2 a Let $y = e^{-2x}(2x-1)^5$

Let $u = e^{-2x}$ and $v = (2x-1)^5$

Then $\frac{du}{dx} = -2e^{-2x}$ and $\frac{dv}{dx} = 10(2x-1)^4$

Using $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= e^{-2x} \times 10(2x-1)^4 + (2x-1)^5(-2e^{-2x}) \\ &= e^{-2x}(2x-1)^4\left(10-2(2x-1)\right) \\ &= e^{-2x}(2x-1)^4(12-4x) \\ &= -4(x-3)(2x-1)^4e^{-2x}\end{aligned}$$

2 b Let $y = \sin 2x \cos 3x$

Let $u = \sin 2x$ and $v = \cos 3x$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 2\cos 2x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = -3\sin 3x$

Using $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sin 2x(-3\sin 3x) + \cos 3x(2\cos 2x)$$
$$= 2\cos 2x\cos 3x - 3\sin 2x\sin 3x$$

c Let $y = \mathrm{e}^x \sin x$

Let $u = \mathrm{e}^x$ and $v = \sin x$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$

Using $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x \cos x + \sin x\,\mathrm{e}^x = \mathrm{e}^x(\sin x + \cos x)$$

d Let $y = \sin(5x)\ln(\cos x)$

Let $u = \sin 5x$ and $v = \ln(\cos x)$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 5\cos 5x$

and $\frac{\mathrm{d}v}{\mathrm{d}x} = (-\sin x)\times\frac{1}{\cos x} = -\tan x$

Using $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sin(5x)(-\tan x) + 5\cos(5x)\ln(\cos x)$$
$$= 5\cos 5x\ln(\cos x) - \tan x\sin 5x$$

3 a $y = x^2(3x-1)^3$

Let $u = x^2$ and $v = (3x-1)^3$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 9(3x-1)^2$

Using the product rule $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 \times 9(3x-1)^2 + (3x-1)^3 \times 2x$$
$$= x(3x-1)^2\big(9x + 2(3x-1)\big)$$
$$= x(3x-1)^2(15x-2) \qquad (*)$$

At the point (1, 8), $x = 1$.

Substituting $x = 1$ into expression (*):

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \times 2^2 \times 13 = 52$$

b $y = 3x(2x+1)^{\frac{1}{2}}$

Let $u = 3x$ and $v = (2x+1)^{\frac{1}{2}}$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 3$

and $\frac{\mathrm{d}v}{\mathrm{d}x} = 2\times\frac{1}{2}(2x+1)^{-\frac{1}{2}} = (2x+1)^{-\frac{1}{2}}$

Using the product rule $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x(2x+1)^{-\frac{1}{2}} + 3(2x+1)^{\frac{1}{2}}$$
$$= 3(2x+1)^{-\frac{1}{2}}\big(x + (2x+1)\big)$$
$$= 3(3x+1)(2x+1)^{-\frac{1}{2}} \qquad (*)$$

At the point (4, 36), $x = 4$.

Substituting $x = 4$ into (*):

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\times 13\times 9^{-\frac{1}{2}} = 3\times 13\times\frac{1}{3} = 13$$

3 c $y=(x-1)(2x+1)^{-1}$

Let $u=x-1$ and $v=(2x+1)^{-1}$

Then $\frac{du}{dx}=1$ and $\frac{dv}{dx}=-2(2x+1)^{-2}$

Using the product rule $\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

$$\frac{dy}{dx}=(x-1)\left(-2(2x+1)^{-2}\right)+(2x+1)^{-1}\times 1$$
$$=(2x+1)^{-2}\left(-2(x-1)+(2x+1)\right)$$
$$=3(2x+1)^{-2} \qquad (*)$$

At the point $\left(2,\frac{1}{5}\right)$, $x=2$.

Substituting $x=2$ into (*):

$$\frac{dy}{dx}=3\times 5^{-2}=\frac{3}{25}$$

4 $y=(x-2)^2(2x+3)$

Let $u=(x-2)^2$ and $v=(2x+3)$

Then $\frac{du}{dx}=2(x-2)$ and $\frac{dv}{dx}=2$

Using $\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

$$\frac{dy}{dx}=(x-2)^2\times 2+(2x+3)\times 2(x-2)$$
$$=2(x-2)(x-2+2x+3)$$
$$=2(x-2)(3x+1)$$

At stationary points $\frac{dy}{dx}=0$

$2(x-2)(3x+1)=0$

$(x-2)=0$ or $(3x+1)=0$

$\therefore\ x=2$ or $-\frac{1}{3}$

$x=2\Rightarrow y=0$

$x=-\frac{1}{3}\Rightarrow y=\left(-\frac{7}{3}\right)^2\left(\frac{7}{3}\right)=\frac{343}{27}$

So the stationary points are

$(2,0)$ and $\left(-\frac{1}{3},\frac{343}{27}\right)$

5 $y=\left(x-\frac{\pi}{2}\right)^5\sin 2x$

Let $u=\left(x-\frac{\pi}{2}\right)^5$ and $v=\sin 2x$

$\frac{du}{dx}=5\left(x-\frac{\pi}{2}\right)^4$ and $\frac{dv}{dx}=2\cos 2x$

Using $\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

$$\frac{dy}{dx}=\left(x-\frac{\pi}{2}\right)^5 2\cos 2x+\sin 2x\times 5\left(x-\frac{\pi}{2}\right)^4$$
$$=\left(x-\frac{\pi}{2}\right)^4\left(2\left(x-\frac{\pi}{2}\right)\cos 2x+5\sin 2x\right)$$

When $x=\frac{\pi}{4}$,

$$\frac{dy}{dx}=\left(-\frac{\pi}{4}\right)^4\left(2\left(-\frac{\pi}{4}\right)\cos\frac{\pi}{2}+5\sin\frac{\pi}{2}\right)$$
$$=\frac{\pi^4}{256}(0+5)=\frac{5\pi^4}{256}$$

6 $y = x^2\cos(x^2)$

Let $u = x^2$ and $v = \cos(x^2)$

$\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = -2x\sin(x^2)$

Using $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = x^2\left(-2x\sin(x^2)\right) + \cos(x^2)\times 2x$$
$$= 2x\left(\cos(x^2) - x^2\sin(x^2)\right)$$

When $x = \frac{\sqrt{\pi}}{2}$,

$$\frac{dy}{dx} = \sqrt{\pi}\left(\cos\frac{\pi}{4} - \frac{\pi}{4}\sin\frac{\pi}{4}\right)$$
$$= \sqrt{\pi}\left(\frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}}\right) = \sqrt{\frac{\pi}{2}}\left(1 - \frac{\pi}{4}\right)$$

Equation of tangent at $P\left(\frac{\sqrt{\pi}}{2}, \frac{\pi\sqrt{2}}{8}\right)$ is

$$y - \frac{\pi\sqrt{2}}{8} = \sqrt{\frac{\pi}{2}}\left(1 - \frac{\pi}{4}\right)\left(x - \frac{\sqrt{\pi}}{2}\right)$$
$$8y - \pi\sqrt{2} = 4\sqrt{2\pi}\left(1 - \frac{\pi}{4}\right)\left(x - \frac{\sqrt{\pi}}{2}\right)$$
$$8y - \pi\sqrt{2} = \sqrt{2\pi}(4 - \pi)\left(x - \frac{\sqrt{\pi}}{2}\right)$$
$$8y - \pi\sqrt{2} = \sqrt{2\pi}(4 - \pi)x - \frac{\pi\sqrt{2}}{2}(4 - \pi)$$
$$\sqrt{2\pi}(\pi - 4)x + 8y - \pi\sqrt{2} + \frac{\pi\sqrt{2}}{2}(4 - \pi) = 0$$
$$\sqrt{2\pi}(\pi - 4)x + 8y - \pi\sqrt{2}\left(\frac{\pi - 2}{2}\right) = 0$$

This is in the form $ax + by + c = 0$ with

$a = \sqrt{2\pi}(\pi - 4)$, $b = 8$ and $c = -\pi\sqrt{2}\left(\frac{\pi - 2}{2}\right)$

7 $y = 3x^2(5x - 3)^3$

Let $u = 3x^2$ and $v = (5x - 3)^3$

$\frac{du}{dx} = 6x$ and $\frac{dv}{dx} = 15(5x - 3)^2$

Using $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = 3x^2 \times 15(5x - 3)^2 + 6x(5x - 3)^3$$
$$= 3x(5x - 3)^2\left(15x + 2(5x - 3)\right)$$
$$= 3x(5x - 3)^2(25x - 6)$$

Hence $A = 3$, $n = 2$, $B = 25$ and $C = -6$.

8 a $y = (x + 3)^2 e^{3x}$

Let $u = (x + 3)^2$ and $v = e^{3x}$

$\frac{du}{dx} = 2(x + 3)$ and $\frac{dv}{dx} = 3e^{3x}$

Using $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = (x + 3)^2 \times 3e^{3x} + e^{3x} \times 2(x + 3)$$
$$= e^{3x}(x + 3)\left(3(x + 3) + 2\right)$$
$$= e^{3x}(x + 3)(3x + 11)$$

b When $x = 2$, $\frac{dy}{dx} = e^6 \times 5 \times 17 = 85e^6$

Hence the gradient at point C is $85e^6$.

9 a Let $y = (2\sin x - 3\cos x)\ln 3x$

Let $u = 2\sin x - 3\cos x$ and $v = \ln 3x$

Then $\frac{du}{dx} = 2\cos x + 3\sin x$ and $\frac{dv}{dx} = \frac{1}{x}$

Using $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = \frac{2\sin x - 3\cos x}{x}$$
$$+ (2\cos x + 3\sin x)\ln 3x$$

9 b Let $y = x^4 e^{7x-3}$

Let $u = x^4$ and $v = e^{7x-3}$

Then $\frac{du}{dx} = 4x^3$ and $\frac{dv}{dx} = 7e^{7x-3}$

Using $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = x^4 \times 7e^{7x-3} + 4x^3 e^{7x-3}$$

$$= x^3 e^{7x-3}(7x+4)$$

10 Let $y = x^5\sqrt{10x+6}$

Let $u = x^5$ and $v = \sqrt{10x+6} = (10x+6)^{\frac{1}{2}}$

Then $\frac{du}{dx} = 5x^4$

and $\frac{dv}{dx} = 10 \times \frac{1}{2}(10x+6)^{-\frac{1}{2}} = \frac{5}{\sqrt{10x+6}}$

Using $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = \frac{5x^5}{\sqrt{10x+6}} + 5x^4\sqrt{10x+6}$$

When $x = 1$, $\frac{dy}{dx} = \frac{5}{\sqrt{16}} + 5\sqrt{16} = 21.25$

Challenge

a Let $y = e^x \sin^2 x \cos x$

Let $p = e^x$ and $q = \sin^2 x \cos x$

Then $y = pq$, $\frac{dp}{dx} = e^x$

and $\frac{dy}{dx} = p\frac{dq}{dx} + q\frac{dp}{dx}$

Let $u = \sin^2 x = (\sin x)^2$ and $v = \cos x$

Then $q = uv$,

$\frac{du}{dx} = 2\sin x \cos x$ and $\frac{dv}{dx} = -\sin x$

Using $\frac{dq}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dq}{dx} = \sin^2 x\,(-\sin x) + \cos x\,(2\sin x \cos x)$$

$$= -\sin^3 x + 2\sin x \cos^2 x$$

Using $\frac{dy}{dx} = p\frac{dq}{dx} + q\frac{dp}{dx}$

$$\frac{dy}{dx} = e^x(-\sin^3 x + 2\sin x \cos^2 x)$$

$$+ \sin^2 x \cos x \times e^x$$

$$= -e^x \sin x\,(\sin^2 x - 2\cos^2 x - \sin x \cos x)$$

Challenge

b Let $y = x(4x-3)^6(1-4x)^9$

Let $p = x(4x-3)^6$ and $q = (1-4x)^9$

Then $y = pq$, $\dfrac{dq}{dx} = -36(1-4x)^8$

and $\dfrac{dy}{dx} = p\dfrac{dq}{dx} + q\dfrac{dp}{dx}$

Let $u = x$ and $v = (4x-3)^6$

Then $p = uv$,

$\dfrac{du}{dx} = 1$ and $\dfrac{dv}{dx} = 24(4x-3)^5$

Using $\dfrac{dp}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

$$\begin{aligned}\frac{dp}{dx} &= 24x(4x-3)^5 + (4x-3)^6 \\ &= (4x-3)^5(24x+4x-3) \\ &= (4x-3)^5(28x-3)\end{aligned}$$

Using $\dfrac{dy}{dx} = p\dfrac{dq}{dx} + q\dfrac{dp}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= x(4x-3)^6\left(-36(1-4x)^8\right) \\ &\quad + (1-4x)^9(4x-3)^5(28x-3) \\ &= (4x-3)^5(1-4x)^8 \\ &\quad \times\left(-36x(4x-3) + (1-4x)(28x-3)\right) \\ &= (4x-3)^5(1-4x)^8 \\ &\quad \times\left(-144x^2 + 108x + 40x - 112x^2 - 3\right) \\ &= -(4x-3)^5(1-4x)^8(256x^2 - 148x + 3)\end{aligned}$$

Differentiation 9E

1 a Let $y = \frac{5x}{x+1}$

Let $u = 5x$ and $v = x+1$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 5$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 1$

Using $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x+1)\times 5 - 5x\times 1}{(x+1)^2} = \frac{5}{(x+1)^2}$$

b Let $y = \frac{2x}{3x-2}$

Let $u = 2x$ and $v = 3x-2$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 2$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 3$

Using $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(3x-2)\times 2 - 2x\times 3}{(3x-2)^2}$$
$$= \frac{6x-4-6x}{(3x-2)^2} = -\frac{4}{(3x-2)^2}$$

c Let $y = \frac{x+3}{2x+1}$

Let $u = x+3$ and $v = 2x+1$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 2$

Using $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(2x+1)\times 1 - (x+3)\times 2}{(2x+1)^2}$$
$$= \frac{2x+1-2x-6}{(2x+1)^2} = -\frac{5}{(2x+1)^2}$$

d Let $y = \frac{3x^2}{(2x-1)^2}$

Let $u = 3x^2$ and $v = (2x-1)^2$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 6x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 4(2x-1)$

Using $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(2x-1)^2\times 6x - 3x^2\times 4(2x-1)}{(2x-1)^4}$$
$$= \frac{6x(2x-1)\big((2x-1)-2x\big)}{(2x-1)^4}$$
$$= \frac{-6x(2x-1)}{(2x-1)^4} = -\frac{6x}{(2x-1)^3}$$

e Let $y = \frac{6x}{(5x+3)^{\frac{1}{2}}}$

Let $u = 6x$ and $v = (5x+3)^{\frac{1}{2}}$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 6$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{5}{2}(5x+3)^{-\frac{1}{2}}$

Using $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(5x+3)^{\frac{1}{2}}\times 6 - 6x\times \frac{5}{2}(5x+3)^{-\frac{1}{2}}}{\left((5x+3)^{\frac{1}{2}}\right)^2}$$
$$= \frac{3(5x+3)^{-\frac{1}{2}}\big(2(5x+3)-5x\big)}{(5x+3)}$$
$$= \frac{3(5x+3)^{-\frac{1}{2}}(10x+6-5x)}{(5x+3)} = \frac{3(5x+6)}{(5x+3)^{\frac{3}{2}}}$$

2 a Let $y = \frac{e^{4x}}{\cos x}$

Let $u = e^{4x}$ and $v = \cos x$

$\frac{du}{dx} = 4e^{4x}$ and $\frac{dv}{dx} = -\sin x$

Using $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{4e^{4x}\cos x - e^{4x}(-\sin x)}{\cos^2 x}$$

$$= \frac{e^{4x}(4\cos x + \sin x)}{\cos^2 x}$$

b Let $y = \frac{\ln x}{x+1}$

Let $u = \ln x$ and $v = x+1$

$\frac{du}{dx} = \frac{1}{x}$ and $\frac{dv}{dx} = 1$

Using $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{\frac{(x+1)}{x} - \ln x}{(x+1)^2} = \frac{1}{x(x+1)} - \frac{\ln x}{(x+1)^2}$$

c Let $y = \frac{e^{-2x} + e^{2x}}{\ln x}$

Let $u = e^{-2x} + e^{2x}$ and $v = \ln x$

$\frac{du}{dx} = -2e^{-2x} + 2e^{2x}$ and $\frac{dv}{dx} = \frac{1}{x}$

Using $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{\ln x\,(2e^{2x} - e^{-2x}) - \frac{e^{-2x} + e^{2x}}{x}}{(\ln x)^2}$$

$$= \frac{2x\ln x\,(e^{2x} - e^{-2x}) - (e^{-2x} + e^{2x})}{x(\ln x)^2}$$

$$= \frac{e^{-2x}\left(2x(e^{4x} - 1)\ln x - e^{4x} - 1\right)}{x(\ln x)^2}$$

d Let $y = \frac{(e^x + 3)^3}{\cos x}$

Let $u = (e^x + 3)^3$ and $v = \cos x$

$\frac{du}{dx} = 3e^x(e^x + 3)^2$ and $\frac{dv}{dx} = -\sin x$

Using $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{3e^x(e^x + 3)^2\cos x - (-\sin x)(e^x + 3)^3}{\cos^2 x}$$

$$= \frac{(e^x + 3)^2\left(3e^x\cos x + (e^x + 3)\sin x\right)}{\cos^2 x}$$

e Let $y = \frac{\sin^2 x}{\ln x}$

Let $u = \sin^2 x$ and $v = \ln x$

$\frac{du}{dx} = 2\sin x\cos x$ and $\frac{dv}{dx} = \frac{1}{x}$

Using $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{\ln x(2\sin x\cos x) - \frac{1}{x}\sin^2 x}{(\ln x)^2}$$

$$= \frac{2\sin x\cos x}{\ln x} - \frac{\sin^2 x}{x(\ln x)^2}$$

3 $y = \dfrac{x}{3x+1}$

Let $u = x$ and $v = 3x+1$

$\dfrac{du}{dx} = 1$ and $\dfrac{dv}{dx} = 3$

Using $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{(3x+1) - 3x}{(3x+1)^2} = \frac{1}{(3x+1)^2}$$

At the point $\left(1, \frac{1}{4}\right)$, $x = 1$

so $\dfrac{dy}{dx} = \dfrac{1}{4^2} = \dfrac{1}{16}$

4 $y = \dfrac{x+3}{(2x+1)^{\frac{1}{2}}}$

Let $u = x+3$ and $v = (2x+1)^{\frac{1}{2}}$

$\dfrac{du}{dx} = 1$ and $\dfrac{dv}{dx} = (2x+1)^{-\frac{1}{2}}$

Using $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{(2x+1)^{\frac{1}{2}} - (x+3)(2x+1)^{-\frac{1}{2}}}{2x+1}$$

At the point (12, 3), $x = 12$

$$\text{so } \frac{dy}{dx} = \frac{25^{\frac{1}{2}} - (15 \times 25^{-\frac{1}{2}})}{25}$$

$$= \frac{5 - 15 \times \frac{1}{5}}{25} = \frac{2}{25}$$

5 $y = \dfrac{e^{2x+3}}{x}$

Let $u = e^{2x+3}$ and $v = x$

$\dfrac{du}{dx} = 2e^{2x+3}$ and $\dfrac{dv}{dx} = 1$

Using $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{2xe^{2x+3} - e^{2x+3}}{x^2} = \frac{e^{2x+3}(2x-1)}{x^2}$$

At stationary points $\dfrac{dy}{dx} = 0$

so $2x - 1 = 0$

$x = 0.5$ and $y = 2e^4$

There is one stationary point at $(0.5, 2e^4)$.

6 $y = \dfrac{e^{\frac{1}{3}x}}{x}$

Let $u = e^{\frac{1}{3}x}$ and $v = x$

$\dfrac{du}{dx} = \dfrac{1}{3}e^{\frac{1}{3}x}$ and $\dfrac{dv}{dx} = 1$

Using $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{\frac{x}{3}e^{\frac{1}{3}x} - e^{\frac{1}{3}x}}{x^2} = \frac{e^{\frac{1}{3}x}\left(\frac{x}{3} - 1\right)}{x^2}$$

At the point $\left(3, \frac{1}{3}e\right)$, $x = 3$ so $\dfrac{dy}{dx} = 0$

Equation of tangent is

$y - \dfrac{1}{3}e = 0(x-3)$

i.e. $y = \dfrac{1}{3}e$

7 $y = \dfrac{\ln x}{\sin 3x}$

Let $u = \ln x$ and $v = \sin 3x$

$\dfrac{du}{dx} = \dfrac{1}{x}$ and $\dfrac{dv}{dx} = 3\cos 3x$

Using $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{\frac{\sin 3x}{x} - 3\ln x \cos 3x}{\sin^2 3x}$$

$$= \frac{\sin 3x - 3x\ln x \cos 3x}{x\sin^2 3x}$$

When $x = \dfrac{\pi}{9}$,

$$\frac{dy}{dx} = \frac{\sin\frac{\pi}{3} - \frac{\pi}{3}\ln\left(\frac{\pi}{9}\right)\cos\frac{\pi}{3}}{\frac{\pi}{9}\sin^2\frac{\pi}{3}}$$

$$= \frac{\frac{\sqrt{3}}{2} - \frac{\pi}{6}\ln\left(\frac{\pi}{9}\right)}{\frac{3\pi}{36}} = \frac{18\sqrt{3} - 6\pi\ln\left(\frac{\pi}{9}\right)}{3\pi}$$

$$= \frac{6\sqrt{3} - 2\pi\ln\left(\frac{\pi}{9}\right)}{\pi}$$

8 a $x = \dfrac{e^y}{3+2y}$

When $y = 0$, $x = \dfrac{e^0}{3} = \dfrac{1}{3}$

Coordinates of P are $\left(\dfrac{1}{3}, 0\right)$.

b Let $u = e^y$ and $v = 3 + 2y$

$\dfrac{du}{dy} = e^y$ and $\dfrac{dv}{dy} = 2$

$\dfrac{dx}{dy} = \dfrac{e^y(3+2y) - 2e^y}{(3+2y)^2} = \dfrac{e^y(2y+1)}{(3+2y)^2}$

Gradient of normal to the curve is

$-\dfrac{1}{\frac{dy}{dx}} = -\dfrac{dx}{dy} = -\dfrac{e^y(2y+1)}{(3+2y)^2}$

Gradient of normal at $P\left(\dfrac{1}{3}, 0\right)$ is

$-\dfrac{e^0(2\times 0+1)}{3^2} = -\dfrac{1}{9}$

Equation of normal at P is

$y - 0 = -\dfrac{1}{9}\left(x - \dfrac{1}{3}\right)$

$y = -\dfrac{1}{9}x + \dfrac{1}{27}$

This is in the form $y = mx + c$ with

$m = -\dfrac{1}{9}$ and $c = \dfrac{1}{27}$

9 Let $y = \dfrac{x^4}{\cos 3x}$

Let $u = x^4$ and $v = \cos 3x$

$\dfrac{du}{dx} = 4x^3$ and $\dfrac{dv}{dx} = -3\sin 3x$

Using $\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx} = \dfrac{4x^3\cos 3x - x^4(-3\sin 3x)}{\cos^2 3x}$

$= \dfrac{x^3(4\cos 3x + 3x\sin 3x)}{\cos^2 3x}$

10 a $y = \dfrac{e^{2x}}{(x-2)^2}$

Let $u = e^{2x}$ and $v = (x-2)^2$

$\dfrac{du}{dx} = 2e^{2x}$ and $\dfrac{dv}{dx} = 2(x-2)$

Using $\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx} = \dfrac{2(x-2)^2e^{2x} - 2e^{2x}(x-2)}{(x-2)^4}$

$= \dfrac{2e^{2x}(x-2)((x-2)-1)}{(x-2)^4}$

$= \dfrac{2e^{2x}(x-3)}{(x-2)^3}$

So $A = 2$, $B = 1$ and $C = 3$.

b When $x = 1$, $y = e^2$

and $\dfrac{dy}{dx} = \dfrac{2e^2(-2)}{-1} = 4e^2$

Equation of tangent is

$y - e^2 = 4e^2(x-1)$

$y = 4e^2x - 3e^2$

11 a $f(x) = \dfrac{2x}{x+5} + \dfrac{6x}{x^2+7x+10}$

$f(x) = \dfrac{2x}{x+5} + \dfrac{6x}{(x+2)(x+5)}$

$= \dfrac{2x(x+2)}{(x+2)(x+5)} + \dfrac{6x}{(x+2)(x+5)}$

$= \dfrac{2x^2+4x+6x}{(x+2)(x+5)} = \dfrac{2x^2+10x}{(x+2)(x+5)}$

$= \dfrac{2x(x+5)}{(x+2)(x+5)} = \dfrac{2x}{x+2}$

In the last line, dividing through by $(x + 5)$ is allowed because $x > 0$ so $x + 5 \neq 0$.

b Let $u = 2x$ and $v = x + 2$

$\dfrac{du}{dx} = 2$ and $\dfrac{dv}{dx} = 1$

$f'(x) = \dfrac{2(x+2) - 2x}{(x+2)^2} = \dfrac{4}{(x+2)^2}$

Hence $f'(3) = \dfrac{4}{5^2} = \dfrac{4}{25}$

12 a $\mathrm{f}(x)=\dfrac{2\cos 2x}{\mathrm{e}^{2-x}}$

Let $u=2\cos 2x$ and $v=\mathrm{e}^{2-x}$

$\dfrac{\mathrm{d}u}{\mathrm{d}x}=-4\sin 2x$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x}=-\mathrm{e}^{2-x}$

$$\mathrm{f}'(x)=\frac{-4\mathrm{e}^{2-x}\sin 2x-2\cos 2x\,(-\mathrm{e}^{2-x})}{(\mathrm{e}^{2-x})^2}$$

$$=\frac{2\mathrm{e}^{2-x}(\cos 2x-2\sin 2x)}{(\mathrm{e}^{2-x})^2}$$

At stationary points, $\mathrm{f}'(x)=0$

$\cos 2x-2\sin 2x=0$

$2\sin 2x=\cos 2x$

$\therefore\ \tan 2x=\dfrac{1}{2}$

b The range of f(x) is between the y-coordinate of B and the y-coordinate of the right endpoint of the interval.

$\tan 2x=\dfrac{1}{2}\Rightarrow 2x=0.4636$ or 3.6052

$x=0.2318$ or 1.8026

So the x-coordinate of B is 1.8026.

Range of f(x) is

$\mathrm{f}(1.8026)\leqslant y<\mathrm{f}(\pi)$

$-1.47\leqslant y<6.26$ (3 s.f.)

Differentiation 9F

1 a $y = \tan 3x$

Using the result

$$y = \tan kx \Rightarrow \frac{dy}{dx} = k \sec^2 kx$$

$$\frac{dy}{dx} = 3\sec^2 3x$$

b $y = 4\tan^3 x$

Let $u = \tan x$; then $y = 4u^3$

$$\frac{du}{dx} = \sec^2 x \quad \text{and} \quad \frac{dy}{du} = 12u^2$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 12u^2 \sec^2 x$$
$$= 12\tan^2 x \sec^2 x$$

c $y = \tan(x-1)$

$$\frac{dy}{dx} = \sec^2(x-1)$$

d $y = x^2 \tan\frac{1}{2}x + \tan\left(x - \frac{1}{2}\right)$

The first term is a product with $u = x^2$ and $v = \tan\frac{1}{2}x$

$$\frac{du}{dx} = 2x \quad \text{and} \quad \frac{dv}{dx} = \frac{1}{2}\sec^2\frac{1}{2}x$$

Using the product rule for the first term:

$$\frac{dy}{dx} = x^2\left(\frac{1}{2}\sec^2\frac{1}{2}x\right) + \tan\frac{1}{2}x \times 2x$$
$$+ \sec^2\left(x - \frac{1}{2}\right)$$
$$= \frac{1}{2}x^2\sec^2\frac{1}{2}x + 2x\tan\frac{1}{2}x$$
$$+ \sec^2\left(x - \frac{1}{2}\right)$$

2 a $y = \cot 4x$

Let $u = 4x$; then $y = \cot u$

$$\frac{du}{dx} = 4 \quad \text{and} \quad \frac{dy}{du} = -\text{cosec}^2\, u$$

Using the chain rule,

$$\frac{dy}{dx} = -\text{cosec}^2 u \times 4 = -4\,\text{cosec}^2\, 4x$$

b $y = \sec 5x$

Let $u = 5x$; then $y = \sec u$

$$\frac{du}{dx} = 5 \quad \text{and} \quad \frac{dy}{du} = \sec u \tan u$$

Using the chain rule,

$$\frac{dy}{dx} = 5\sec u \tan u = 5\sec 5x \tan 5x$$

c $y = \operatorname{cosec} 4x$

Let $u = 4x$; then $y = \operatorname{cosec} u$

$$\frac{du}{dx} = 4 \text{ and } \frac{dy}{du} = -\operatorname{cosec} u \cot u$$

Using the chain rule,

$$\frac{dy}{dx} = -4\operatorname{cosec} u \cot u$$
$$= -4\operatorname{cosec} 4x \cot 4x$$

d $y = \sec^2 3x = (\sec 3x)^2$

Let $u = \sec 3x$; then $y = u^2$

$$\frac{du}{dx} = 3\sec 3x \tan 3x \text{ and } \frac{dy}{du} = 2u$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 2u \times 3\sec 3x \tan 3x$$
$$= 2\sec 3x \times 3\sec 3x \tan 3x$$
$$= 6\sec^2 3x \tan 3x$$

e $y = x\cot 3x$

This is a product, so let

$u = x$ and $v = \cot 3x$

and use the product rule.

$$\frac{du}{dx} = 1 \text{ and } \frac{dv}{dx} = -3\operatorname{cosec}^2 3x$$

$$\frac{dy}{dx} = x(-3\operatorname{cosec}^2 3x) + \cot 3x \times 1$$
$$= \cot 3x - 3x\operatorname{cosec}^2 3x$$

f $y = \dfrac{\sec^2 x}{x}$

This is a quotient, so let

$u = \sec^2 x$ and $v = x$

and use the quotient rule.

$$\frac{du}{dx} = 2\sec x(\sec x \tan x) \text{ and } \frac{dv}{dx} = 1$$

$$\frac{dy}{dx} = \frac{x(2\sec^2 x \tan x) - \sec^2 x \times 1}{x^2}$$
$$= \frac{\sec^2 x(2x\tan x - 1)}{x^2}$$

g $y = \operatorname{cosec}^3 2x$

Let $u = \operatorname{cosec} 2x$; then $y = u^3$

$$\frac{du}{dx} = -2\operatorname{cosec} 2x \cot 2x \text{ and } \frac{dy}{du} = 3u^2$$

Using the chain rule,

$$\frac{dy}{dx} = 3u^2(-2\operatorname{cosec} 2x \cot 2x)$$
$$= -6\operatorname{cosec}^2 2x \operatorname{cosec} 2x \cot 2x$$
$$= -6\operatorname{cosec}^3 2x \cot 2x$$

h $y = \cot^2(2x-1)$

Let $u = \cot(2x-1)$; then $y = u^2$

$$\frac{du}{dx} = -2\operatorname{cosec}^2(2x-1) \text{ and } \frac{dy}{du} = 2u$$

Using the chain rule,

$$\frac{dy}{dx} = 2u\left(-2\operatorname{cosec}^2(2x-1)\right)$$
$$= -4\cot(2x-1)\operatorname{cosec}^2(2x-1)$$

3 a $f(x) = (\sec x)^{\frac{1}{2}}$

Using the chain rule,

$$f'(x) = \frac{1}{2}(\sec x)^{-\frac{1}{2}} \times \sec x \tan x$$
$$= \frac{1}{2}(\sec x)^{\frac{1}{2}} \tan x$$

3 **b** $\mathrm{f}(x)=\sqrt{\cot x}=(\cot x)^{\frac{1}{2}}$

Using the chain rule,

$$\mathrm{f}'(x)=\frac{1}{2}(\cot x)^{-\frac{1}{2}}\times(-\operatorname{cosec}^2 x)$$

$$=-\frac{1}{2}(\cot x)^{-\frac{1}{2}}\operatorname{cosec}^2 x$$

c $\mathrm{f}(x)=\operatorname{cosec}^2 x=(\operatorname{cosec} x)^2$

Using the chain rule,

$$\mathrm{f}'(x)=2(\operatorname{cosec} x)^1(-\operatorname{cosec} x\cot x)$$

$$=-2\operatorname{cosec}^2 x\cot x$$

d $\mathrm{f}(x)=\tan^2 x=(\tan x)^2$

Using the chain rule,

$$\mathrm{f}'(x)=2\tan x\times\sec^2 x=2\tan x\sec^2 x$$

e $\mathrm{f}(x)=\sec^3 x=(\sec x)^3$

Using the chain rule,

$$\mathrm{f}'(x)=3(\sec x)^2\sec x\tan x=3\sec^3 x\tan x$$

f $\mathrm{f}(x)=\cot^3 x=(\cot x)^3$

Using the chain rule,

$$\mathrm{f}'(x)=3(\cot x)^2(-\operatorname{cosec}^2 x)$$

$$=-3\cot^2 x\operatorname{cosec}^2 x$$

4 **a** $\mathrm{f}(x)=x^2\sec 3x$

Let $u=x^2$ and $v=\sec 3x$

$$\frac{\mathrm{d}u}{\mathrm{d}x}=2x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x}=3\sec 3x\tan 3x$$

Using the product rule,

$$\mathrm{f}'(x)=3x^2\sec 3x\tan 3x+2x\sec 3x$$

b $\mathrm{f}(x)=\dfrac{\tan 2x}{x}$

Let $u=\tan 2x$ and $v=x$

$$\frac{\mathrm{d}u}{\mathrm{d}x}=2\sec^2 2x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x}=1$$

Using the quotient rule,

$$\mathrm{f}'(x)=\frac{2x\sec^2 2x-\tan 2x}{x^2}$$

c $\mathrm{f}(x)=\dfrac{x^2}{\tan x}$

Let $u=x^2$ and $v=\tan x$

$$\frac{\mathrm{d}u}{\mathrm{d}x}=2x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x}=\sec^2 x$$

Using the quotient rule,

$$\mathrm{f}'(x)=\frac{2x\tan x-x^2\sec^2 x}{\tan^2 x}$$

d $\mathrm{f}(x)=\mathrm{e}^x\sec 3x$

Let $u=\mathrm{e}^x$ and $v=\sec 3x$

$$\frac{\mathrm{d}u}{\mathrm{d}x}=\mathrm{e}^x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x}=3\sec 3x\tan 3x$$

Using the product rule,

$$\mathrm{f}'(x)=3\mathrm{e}^x\sec 3x\tan 3x+\mathrm{e}^x\sec 3x$$

$$=\mathrm{e}^x\sec 3x(3\tan 3x+1)$$

e $\mathrm{f}(x)=\dfrac{\ln x}{\tan x}$

Let $u=\ln x$ and $v=\tan x$

$$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x}=\sec^2 x$$

Using the quotient rule,

$$\mathrm{f}'(x)=\frac{\left(\frac{1}{x}\right)\tan x-\ln x\sec^2 x}{\tan^2 x}$$

$$=\frac{\tan x-x\ln x\sec^2 x}{x\tan^2 x}$$

f $\mathrm{f}(x)=\dfrac{\mathrm{e}^{\tan x}}{\cos x}$

Let $u=\mathrm{e}^{\tan x}$ and $v=\cos x$

$$\frac{\mathrm{d}u}{\mathrm{d}x}=\mathrm{e}^{\tan x}\sec^2 x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x}=-\sin x$$

Using the quotient rule,

$$\mathrm{f}'(x)=\frac{\mathrm{e}^{\tan x}\sec^2 x\cos x-\mathrm{e}^{\tan x}(-\sin x)}{\cos^2 x}$$

$$=\frac{\mathrm{e}^{\tan x}\sec x+\mathrm{e}^{\tan x}\sin x}{\cos^2 x}$$

$$=\frac{\mathrm{e}^{\tan x}(\sec x+\sin x)}{\cos^2 x}$$

$$=\mathrm{e}^{\tan x}(\sec^3 x+\sec x\tan x)$$

$$=\mathrm{e}^{\tan x}\sec x(\sec^2 x+\tan x)$$

5 **a** $y = \dfrac{1}{\cos x \sin x} = \sec x \operatorname{cosec} x$

Let $u = \sec x$ and $v = \operatorname{cosec} x$

$\dfrac{du}{dx} = \sec x \tan x$ and $\dfrac{dv}{dx} = -\operatorname{cosec} x \cot x$

Using the product rule,

$$\frac{dy}{dx} = \sec x(-\operatorname{cosec} x \cot x) + \operatorname{cosec} x(\sec x \tan x)$$

$$= -\frac{\cos x}{\cos x \sin x \sin x} + \frac{\sin x}{\sin x \cos x \cos x}$$

$$= -\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x}$$

Alternative solution:

$$y = \frac{1}{\cos x \sin x} = \frac{2}{\sin 2x} = 2\operatorname{cosec} 2x$$

(because $\sin 2x = 2\sin x \cos x$)

$$\frac{dy}{dx} = -4\operatorname{cosec} 2x \cot 2x$$

b At stationary points $\dfrac{dy}{dx} = 0$

$$\frac{1}{\cos^2 x} - \frac{1}{\sin^2 x} = 0$$

$$\frac{1}{\cos^2 x} = \frac{1}{\sin^2 x}$$

$$\tan^2 x = 1$$

$$\tan x = \pm 1$$

In the interval $0 < x \leqslant \pi$

there are two solutions, $\dfrac{\pi}{4}$ and $\dfrac{3\pi}{4}$.

So the number of stationary points is 2.

Alternative solution:

$-4\operatorname{cosec} 2x \cot 2x = 0$

$\operatorname{cosec} 2x \neq 0$

but $\cot 2x = 0$ has two solutions

in the interval $0 < x \leqslant \pi$.

So there are 2 stationary points.

c When $x = \dfrac{\pi}{3}$,

$$y = \frac{1}{\cos\frac{\pi}{3}\sin\frac{\pi}{3}} = \frac{1}{\frac{1}{2} \times \frac{\sqrt{3}}{2}} = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$$

$$\frac{dy}{dx} = -\frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{\left(\frac{1}{2}\right)^2} = -\frac{4}{3} + 4 = \frac{8}{3}$$

or, using the alternative expression,

$$\frac{dy}{dx} = -4 \times \frac{2}{\sqrt{3}} \times \left(-\frac{1}{\sqrt{3}}\right) = \frac{8}{3}$$

Equation of tangent is

$$y - \frac{4\sqrt{3}}{3} = \frac{8}{3}\left(x - \frac{\pi}{3}\right)$$

$$3y - 4\sqrt{3} = 8x - \frac{8\pi}{3}$$

$$24x - 9y + 12\sqrt{3} - 8\pi = 0$$

This is in the required form $ax + by + c = 0$

With $a = 24$, $b = -9$ and $c = 12\sqrt{3} - 8\pi$.

6 $y = \sec x = \dfrac{1}{\cos x}$

Let $u = 1$ and $v = \cos x$

$\dfrac{du}{dx} = 0$ and $\dfrac{dv}{dx} = -\sin x$

Using the quotient rule,

$$\frac{dy}{dx} = \frac{\cos x \times 0 - 1 \times (-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} = \sec x \tan x$$

7 $y = \cot x = \dfrac{1}{\tan x}$

Let $u = 1$ and $v = \tan x$

$\dfrac{du}{dx} = 0$ and $\dfrac{dv}{dx} = \sec^2 x$

Using the quotient rule,

$$\frac{dy}{dx} = \frac{\tan x \times 0 - 1 \times \sec^2 x}{\tan^2 x} = -\frac{\sec^2 x}{\tan^2 x}$$

$$= -\frac{\frac{1}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x}} = -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x$$

8 $x = \cos 2y$

$$\frac{dx}{dy} = -2\sin 2y$$

$$\frac{dy}{dx} = \frac{-1}{2\sin 2y}$$

$$\sin^2 2y + \cos^2 2y = 1$$

$$\sin 2y = \sqrt{1 - \cos^2 2y}$$

$$= \sqrt{1 - x^2}$$

$$\therefore \frac{dy}{dx} = \frac{-1}{2\sqrt{1 - x^2}}$$

9 **a** $x = \text{cosec } 5y$

$$\frac{dx}{dy} = \frac{d}{dy}(5y) \bullet (-\cot 5y \text{ cosec} 5y)$$

$$\frac{dx}{dy} = -5\cot 5y \text{ cosec} 5y$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{-1}{5\cot 5y \text{ cosec} 5y}$$

b $\frac{dy}{dx} = \frac{-1}{5\cot 5y \text{ cosec} 5y}$

$x = \text{cosec } 5y$

$$\cot^2 5y + 1 = \text{cosec}^2 5y$$

$$\Rightarrow \cot 5y = \sqrt{\text{cosec}^2 5y - 1} = \sqrt{x^2 - 1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{5x\sqrt{x^2 - 1}}$$

Challenge

a Let $y = \arccos x$

So $x = \cos y$

$$\frac{dx}{dy} = -\sin y$$

$$\frac{dy}{dx} = -\frac{1}{\sin y}$$

$$\sin^2 y + \cos^2 y = 1$$

$$\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$$

b Let $y = \arctan x$

So $x = \tan y$

$$\frac{dx}{dy} = \sec^2 y$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$= \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

Differentiation 9G

1 **a** $x = 2t,\ y = t^2 - 3t + 2$

$$\frac{dx}{dt} = 2,\ \frac{dy}{dt} = 2t - 3$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t-3}{2} = t - \frac{3}{2}$$

b $x = 3t^2,\ y = 2t^3$

$$\frac{dx}{dt} = 6t,\ \frac{dy}{dt} = 6t^2$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t^2}{6t} = t$$

c $x = t + 3t^2,\ y = 4t$

$$\frac{dx}{dt} = 1 + 6t,\ \frac{dy}{dt} = 4$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4}{1+6t}$$

d $x = t^2 - 2,\ y = 3t^5$

$$\frac{dx}{dt} = 2t,\ \frac{dy}{dt} = 15t^4$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{15t^4}{2t} = \frac{15t^3}{2}$$

e $x = \frac{2}{t},\ y = 3t^2 - 2$

$$\frac{dx}{dt} = -\frac{2}{t^2},\ \frac{dy}{dt} = 6t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{6t}{\frac{2}{t^2}} = -3t^3$$

f $x = \frac{1}{2t-1},\ y = \frac{t^2}{2t-1}$

$$\frac{dx}{dt} = -\frac{2}{(2t-1)^2},\ \frac{dy}{dt} = \frac{2t(2t-1) - 2t^2}{(2t-1)^2}$$

$$= \frac{2t^2 - 2t}{(2t-1)^2}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{2t^2 - 2t}{2} = t(1-t)$$

g $x = \frac{2t}{1+t^2},\ y = \frac{1-t^2}{1+t^2}$

$$\frac{dx}{dt} = \frac{2(1+t^2) - 4t^2}{(1+t^2)^2} = \frac{2(1-t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{-2t(1+t^2) - 2t(1-t^2)}{(1+t^2)^2} = -\frac{4t}{(1+t^2)^2}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-4t}{2(1-t^2)} = \frac{2t}{t^2-1}$$

h $x = t^2e^t,\ y = 2t$

$$\frac{dx}{dt} = t^2e^t + 2te^t,\ \frac{dy}{dt} = 2$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2}{t^2e^t + 2te^t} = \frac{2}{(t^2+2t)e^t}$$

i $x = 4\sin 3t,\ y = 3\cos 3t$

$$\frac{dx}{dt} = 12\cos 3t,\ \frac{dy}{dt} = -9\sin 3t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-9\sin 3t}{12\cos 3t} = -\frac{3}{4}\tan 3t$$

j $x = 2 + \sin t,\ y = 3 - 4\cos t$

$$\frac{dx}{dt} = \cos t,\ \frac{dy}{dt} = 4\sin t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4\sin t}{\cos t} = 4\tan t$$

1 k $x=\sec t,\ y=\tan t$

$$\frac{dx}{dt}=\sec t\tan t,\ \frac{dy}{dt}=\sec^2 t$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\sec^2 t}{\sec t\tan t}=\frac{\sec t}{\tan t}=\text{cosec } t$$

l $x=2t-\sin 2t,\ y=1-\cos 2t$

$$\frac{dx}{dt}=2-2\cos 2t,\ \frac{dy}{dt}=2\sin 2t$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2\sin 2t}{2-2\cos 2t}=\frac{\sin 2t}{1-\cos 2t}$$

$$=\frac{2\sin t\cos t}{1-(1-2\sin^2 t)}=\cot t$$

m $x=e^t-5,\ y=\ln t$

$$\frac{dx}{dt}=e^t,\ \frac{dy}{dt}=\frac{1}{t}$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{1}{te^t}$$

n $x=\ln t,\ y=t^2-64$

$$\frac{dx}{dt}=\frac{1}{t},\ \frac{dy}{dt}=2t$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2t}{\frac{1}{t}}=2t^2$$

o $x=e^{2t}+1,\ y=2e^t-1$

$$\frac{dx}{dt}=2e^{2t},\ \frac{dy}{dt}=2e^t$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2e^t}{2e^{2t}}=\frac{1}{e^t}=e^{-t}$$

2 a $x=3-2\sin t,\ y=t\cos t$

$$\frac{dx}{dt}=-2\cos t,\ \frac{dy}{dt}=-t\sin t+\cos t$$

$$\therefore \frac{dy}{dx}=\frac{-t\sin t+\cos t}{-2\cos t}=\frac{t}{2}\tan t-\frac{1}{2}$$

At point P, where $t=\pi$,

$$x=3,\ y=-\pi \text{ and } \frac{dy}{dx}=-\frac{1}{2}$$

Equation of tangent is

$$y-(-\pi)=-\frac{1}{2}(x-3)$$

$$y=-\frac{1}{2}x+\frac{3}{2}-\pi$$

b $x=9-t^2,\ y=t^2+6t$

$$\frac{dx}{dt}=-2t,\ \frac{dy}{dt}=2t+6$$

$$\therefore \frac{dy}{dx}=-\frac{2t+6}{2t}=-\frac{t+3}{t}$$

At point P, where $t=2$,

$$x=5,\ y=16 \text{ and } \frac{dy}{dx}=-\frac{5}{2}$$

Equation of tangent is

$$y-16=-\frac{5}{2}(x-5)$$

$$2y-32=25-5x$$

$$2y+5x=57$$

3 a $x=e^t,\ y=e^t+e^{-t}$

$$\frac{dx}{dt}=e^t,\ \frac{dy}{dt}=e^t-e^{-t}$$

$$\therefore \frac{dy}{dx}=\frac{e^t-e^{-t}}{e^t}=1-e^{-2t}$$

At point P, where $t=0$,

$$\frac{dy}{dx}=1-1=0$$

Gradient of curve is 0

$\therefore$ normal is parallel to the y-axis.

When $t=0,\ x=1$ and $y=2$

Equation of the normal is $x=1$.

3 **b** $x = 1 - \cos 2t,\ y = \sin 2t$

$$\frac{dx}{dt} = 2\sin 2t,\ \frac{dy}{dt} = 2\cos 2t$$

$$\therefore \frac{dy}{dx} = \frac{2\cos 2t}{2\sin 2t} = \cot 2t$$

At point P, where $t = \frac{\pi}{6}$,

$$\frac{dy}{dx} = \frac{1}{\tan \frac{\pi}{3}} = \frac{1}{\sqrt{3}}$$

$\therefore$ gradient of the normal is $-\sqrt{3}$

When $t = \frac{\pi}{6}$, $x = 1 - \cos\frac{\pi}{3} = \frac{1}{2}$

and $y = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

Equation of the normal is

$$y - \frac{\sqrt{3}}{2} = -\sqrt{3}\left(x - \frac{1}{2}\right)$$

$$y - \frac{\sqrt{3}}{2} = -\sqrt{3}x + \frac{\sqrt{3}}{2}$$

$$y + \sqrt{3}x = \sqrt{3}$$

4 $x = \frac{t}{1-t},\ y = \frac{t^2}{1-t}$

Using the quotient rule,

$$\frac{dx}{dt} = \frac{(1-t)\times 1 - t(-1)}{(1-t)^2} = \frac{1}{(1-t)^2}$$

$$\frac{dy}{dt} = \frac{(1-t)2t - t^2(-1)}{(1-t)^2} = \frac{2t - t^2}{(1-t)^2}$$

$$\therefore \frac{dy}{dx} = \frac{2t - t^2}{(1-t)^2} \div \frac{1}{(1-t)^2}$$

$$= 2t - t^2 = t(2-t)$$

$\frac{dy}{dx} = 0$ when $t = 0$ or 2

When $t = 0,\ x = 0$ and $y = 0$

When $t = 2,\ x = -2$ and $y = -4$

$\therefore$ (0, 0) and (−2, −4) are the points of zero gradient on the curve.

5 **a** $x = e^{2t},\ y = e^t - 1$

$$\frac{dx}{dt} = 2e^{2t},\ \frac{dy}{dt} = e^t$$

$$\therefore \frac{dy}{dx} = \frac{e^t}{2e^{2t}} = \frac{1}{2e^t}$$

When $t = \ln 2$,

$x = 4,\ y = 1$ and $\frac{dy}{dx} = \frac{1}{4}$

Equation of tangent is

$$y - 1 = \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}x$$

b At stationary points $\frac{dy}{dx} = 0$

$$\frac{1}{2e^t} = 0 \Rightarrow e^{-t} = 0$$

This has no solutions, so the curve has no stationary points.

6 $x = \dfrac{t^2 - 3t - 4}{t}$, $y = 2t$

$x = \dfrac{t^2}{t} - 3 - \dfrac{4}{t} = t - 3 - 4t^{-1}$

$\dfrac{dx}{dt} = 1 + \dfrac{4}{t^2}$, $\dfrac{dy}{dt} = 2$

$\therefore \dfrac{dy}{dx} = \dfrac{2}{1 + \dfrac{4}{t^2}} = \dfrac{2t^2}{t^2 + 4}$

l_1 is parallel to $y = x + 5$

so gradient of l_1 is 1

$\dfrac{dy}{dx} = 1 \Rightarrow \dfrac{2t^2}{t^2 + 4} = 1$

$\Rightarrow t^2 = 4$

so $t = 2$ (because $t > 0$)

When $t = 2$, $x = -3$ and $y = 4$

Equation of l_1 is

$y - 4 = 1 \times (x + 3)$

$y = x + 7$

7 **a** $x = 2\sin^2 t$, $y = 2\cot t$

$\dfrac{dx}{dt} = 4\sin t\cos t$, $\dfrac{dy}{dt} = -2\operatorname{cosec}^2 t$

$\therefore \dfrac{dy}{dx} = -\dfrac{2\operatorname{cosec}^2 t}{4\sin t\cos t} = -\dfrac{1}{2}\sec t\operatorname{cosec}^3 t$

b When $t = \dfrac{\pi}{6}$, $x = \dfrac{1}{2}, y = 2\sqrt{3}$

and $\dfrac{dy}{dx} = -\dfrac{1}{2} \times \dfrac{2}{\sqrt{3}} \times 2^3 = -\dfrac{8}{\sqrt{3}} = -\dfrac{8\sqrt{3}}{3}$

Equation of tangent is

$y - 2\sqrt{3} = -\dfrac{8\sqrt{3}}{3}\left(x - \dfrac{1}{2}\right)$

$\sqrt{3}y - 6 = -8x + 4$

$8x + \sqrt{3}y - 10 = 0$

8 **a** $x = 4\sin t$, $y = 2\operatorname{cosec} 2t$

$x = 2\sqrt{3} \Rightarrow 4\sin t = 2\sqrt{3}$

$\sin t = \dfrac{\sqrt{3}}{2} \quad \therefore t = \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

$t = \dfrac{\pi}{3} \Rightarrow y = 2\operatorname{cosec}\dfrac{2\pi}{3} = \dfrac{4}{\sqrt{3}} = \dfrac{4\sqrt{3}}{3}$,

which is the y-coordinate of point A.

So $t = \dfrac{\pi}{3}$ at point A.

b $\dfrac{dx}{dt} = 4\cos t$, $\dfrac{dy}{dt} = -4\operatorname{cosec} 2t\cot 2t$

$\therefore \dfrac{dy}{dx} = -\dfrac{-4\operatorname{cosec} 2t\cot 2t}{4\cos t}$

$= -\dfrac{\cot 2t\operatorname{cosec} 2t}{\cos t}$

When $t = \dfrac{\pi}{3}$, $\dfrac{dy}{dx} = -\dfrac{\cot\dfrac{2\pi}{3}\operatorname{cosec}\dfrac{2\pi}{3}}{\cos\dfrac{\pi}{3}}$

$= -\dfrac{\left(-\dfrac{1}{\sqrt{3}}\right) \times \dfrac{2}{\sqrt{3}}}{\dfrac{1}{2}} = \dfrac{4}{3}$

$\therefore$ gradient of normal is $-\dfrac{3}{4}$

Equation of normal, l, is

$y - \dfrac{4\sqrt{3}}{3} = -\dfrac{3}{4}(x - 2\sqrt{3})$

$12y - 16\sqrt{3} = -9(x - 2\sqrt{3})$

$9x + 12y - 34\sqrt{3} = 0$

9 **a** $x = t^2 + t$, $y = t^2 - 10t + 5$

$\dfrac{dx}{dt} = 2t + 1$, $\dfrac{dy}{dt} = 2t - 10$

$\therefore \dfrac{dy}{dx} = \dfrac{2t - 10}{2t + 1}$

When gradient is 2, $\dfrac{2t - 10}{2t + 1} = 2$

$2t - 10 = 4t + 2 \Rightarrow t = -6$

At P, $x = (-6)^2 - 6 = 30$

and $y = (-6)^2 - 10(-6) + 5 = 101$

Coordinates of P are (30, 101).

9 b Equation of tangent at *P* is

$$y-101=2(x-30)$$
$$y=2x+41$$

c Substituting for *y* and *x* in the tangent equation:

$$t^2-10t+5=2(t^2+t)+41$$
$$t^2+12t+36=0$$

Discriminant $=12^2-4\times 36=0$

Therefore the curve and the line only intersect once, so the tangent at *P* does not intersect the curve again.

10 a $x=2\sin t,\ y=\sqrt{2}\cos 2t$

$$\frac{dx}{dt}=2\cos t,\ \frac{dy}{dt}=-2\sqrt{2}\sin 2t$$
$$\therefore\ \frac{dy}{dx}=\frac{-2\sqrt{2}\sin 2t}{2\cos t}=\frac{-\sqrt{2}\times 2\sin t\cos t}{\cos t}$$
$$=-2\sqrt{2}\sin t$$

b When $t=\frac{\pi}{3}$:

$$x=2\left(\frac{\sqrt{3}}{2}\right)=\sqrt{3},\ y=\sqrt{2}\left(-\frac{1}{2}\right)=-\frac{\sqrt{2}}{2}$$

and $\frac{dy}{dx}=-2\sqrt{2}\left(\frac{\sqrt{3}}{2}\right)=-\sqrt{6}$

Equation of normal at *A* is

$$y-\left(-\frac{\sqrt{2}}{2}\right)=\frac{1}{\sqrt{6}}(x-\sqrt{3})$$
$$\sqrt{6}y+\sqrt{3}=x-\sqrt{3}$$
$$x-\sqrt{6}y-2\sqrt{3}=0$$

c Substituting for *y* and *x* in the normal equation:

$$2\sin t-\sqrt{6}\times\sqrt{2}\cos 2t-2\sqrt{3}=0$$
$$\sin t-\sqrt{3}\cos 2t-\sqrt{3}=0$$
$$\sin t-\sqrt{3}(1-2\sin^2 t)-\sqrt{3}=0$$
$$2\sqrt{3}\sin^2 t+\sin t-2\sqrt{3}=0$$
$$(2\sin t-\sqrt{3})(\sqrt{3}\sin t+2)=0$$
$$\sin t=\frac{\sqrt{3}}{2}\ \text{ or }\ \sin t=-\frac{2}{\sqrt{3}}$$

(2nd option not possible since $|\sin t|\le 1$)

$$\sin t=\frac{\sqrt{3}}{2}\Rightarrow t=\frac{\pi}{3}\ \text{ or }\ \frac{2\pi}{3}$$

When $t=\frac{2\pi}{3}$:

$$x=2\sin\frac{2\pi}{3}=\sqrt{3},\ y=\sqrt{2}\cos\frac{4\pi}{3}=-\frac{\sqrt{2}}{2},$$

which is the same as point *A*, so *l* does not intersect *C* other than at point *A*.

11 a $x=\cos t,\ y=\frac{1}{2}\sin 2t$

$$\frac{dx}{dt}=-\sin t,\ \frac{dy}{dt}=\cos 2t$$
$$\therefore\ \frac{dy}{dx}=-\frac{\cos 2t}{\sin t}$$

b When $t=\frac{\pi}{6}$: $x=\frac{\sqrt{3}}{2},\ y=\frac{\sqrt{3}}{4}$

and $\frac{dy}{dx}=-\frac{\frac{1}{2}}{\frac{1}{2}}=-1$

Equation of tangent at *A* is

$$y-\frac{\sqrt{3}}{4}=-\left(x-\frac{\sqrt{3}}{2}\right)$$

i.e. $y=-x+\frac{3\sqrt{3}}{4}$

11 c l_1 and l_2 both have gradient −1

$\therefore$ values of t at points where the tangents cut the curve will be solutions to

$$-\frac{\cos 2t}{\sin t} = -1$$

$$1 - 2\sin^2 t = \sin t$$

$$2\sin^2 t + \sin t - 1 = 0$$

$$(2\sin t - 1)(\sin t + 1) = 0$$

$$\sin t = \frac{1}{2} \text{ or } -1$$

$$t = \frac{\pi}{6}, \frac{5\pi}{6} \text{ or } \frac{3\pi}{2}$$

So lines l_1 and l_2 touch the curve when

$t = \frac{5\pi}{6}$ and $t = \frac{3\pi}{2}$.

$$t = \frac{5\pi}{6} \Rightarrow x = -\frac{\sqrt{3}}{2}, y = -\frac{\sqrt{3}}{4}$$

Equation of l_1 is

$$y - \left(-\frac{\sqrt{3}}{4}\right) = -1\left(x - \left(-\frac{\sqrt{3}}{2}\right)\right)$$

i.e. $y = -x - \frac{3\sqrt{3}}{4}$

$$t = \frac{3\pi}{2} \Rightarrow x = 0, y = 0$$

Equation of l_2 is

$$y - 0 = -(x - 0)$$

i.e. $y = -x$

Differentiation 9H

1 $u = y^n$

$$\frac{\mathrm{d}u}{\mathrm{d}y} = ny^{n-1}$$

$$\frac{\mathrm{d}(y^n)}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y} \times \frac{\mathrm{d}y}{\mathrm{d}x} = ny^{n-1}\frac{\mathrm{d}y}{\mathrm{d}x}$$

2 $$\frac{\mathrm{d}(xy)}{\mathrm{d}x} = x\frac{\mathrm{d}(y)}{\mathrm{d}x} + \frac{\mathrm{d}(x)}{\mathrm{d}x}y = x\frac{\mathrm{d}(y)}{\mathrm{d}x} + 1 \times y$$

$$= x\frac{\mathrm{d}y}{\mathrm{d}x} + y$$

3 a $x^2 + y^3 = 2$

Differentiate with respect to *x*:

$$2x + 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$3y^2\frac{\mathrm{d}y}{\mathrm{d}x} = -2x$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x}{3y^2}$$

b $x^2 + 5y^2 = 14$

$$2x + 10y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2x}{10y} = -\frac{x}{5y}$$

c $x^2 + 6x - 8y + 5y^2 = 13$

$$2x + 6 - 8\frac{\mathrm{d}y}{\mathrm{d}x} + 10y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$2x + 6 = (8 - 10y)\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x+6}{8-10y} = \frac{x+3}{4-5y}$$

d $y^3 + 3x^2y - 4x = 0$

$$3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + \left(3x^2\frac{\mathrm{d}y}{\mathrm{d}x} + y \times 6x\right) - 4 = 0$$

$$(3y^2 + 3x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 4 - 6xy$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4 - 6xy}{3(x^2 + y^2)}$$

e $3y^2 - 2y + 2xy = x^3$

$$6y\frac{\mathrm{d}y}{\mathrm{d}x} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + \left(2x\frac{\mathrm{d}y}{\mathrm{d}x} + y \times 2\right) = 3x^2$$

$$(6y - 2 + 2x)\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 2y$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2 - 2y}{2x + 6y - 2}$$

f $x = \dfrac{2y}{x^2 - y}$

$$x^3 - xy = 2y$$

$$x^3 - xy - 2y = 0$$

Differentiate with respect to *x*:

$$3x^2 - \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right) - 2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$3x^2 - y = (x + 2)\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2 - y}{x + 2}$$

3 **g** $(x-y)^4 = x+y+5$

Differentiate with respect to x
(using the chain rule on the first term):

$$4(x-y)^3\left(1-\frac{\mathrm{d}y}{\mathrm{d}x}\right)=1+\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$4(x-y)^3-1=\left(1+4(x-y)^3\right)\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\therefore\ \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{4(x-y)^3-1}{1+4(x-y)^3}$$

h $\mathrm{e}^x y = x\mathrm{e}^y$

$$\mathrm{e}^x\frac{\mathrm{d}y}{\mathrm{d}x}+y\mathrm{e}^x = x\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{e}^y\times 1$$

$$\mathrm{e}^x\frac{\mathrm{d}y}{\mathrm{d}x}-x\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^y-y\mathrm{e}^x$$

$$(\mathrm{e}^x-x\mathrm{e}^y)\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^y-y\mathrm{e}^x$$

$$\therefore\ \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{e}^y-y\mathrm{e}^x}{\mathrm{e}^x-x\mathrm{e}^y}$$

i $\sqrt{xy}+x+y^2=0$

$$(xy)^{\frac{1}{2}}+x+y^2=0$$

$$\frac{1}{2}(xy)^{-\frac{1}{2}}\left(x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)+1+2y\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

Multiply both sides by $2\sqrt{xy}$:

$$\left(x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)+2\sqrt{xy}+4y\sqrt{xy}\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

$$\left(x+4y\sqrt{xy}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=-2\sqrt{xy}-y$$

$$\therefore\ \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-2\sqrt{xy}-y}{x+4y\sqrt{xy}}$$

4 $x^2+3xy^2-y^3=9$

Differentiate with respect to x:

$$2x+\left(3x\times 2y\frac{\mathrm{d}y}{\mathrm{d}x}+y^2\times 3\right)-3y^2\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

Substitute $x=2$ and $y=1$ to give

$$4+\left(12\frac{\mathrm{d}y}{\mathrm{d}x}+3\right)-3\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

$$9\frac{\mathrm{d}y}{\mathrm{d}x}=-7$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{7}{9}$$

$\therefore$ gradient of the tangent at (2, 1) is $-\frac{7}{9}$

Equation of the tangent is

$$(y-1)=-\frac{7}{9}(x-2)$$

$$y=-\frac{7}{9}x+\frac{23}{9}$$

or $7x+9y-23=0$

5 $(x+y)^3=x^2+y$

Differentiate with respect to x:

$$3(x+y)^2\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)=2x+\frac{\mathrm{d}y}{\mathrm{d}x}$$

Substitute $x = 1$ and $y = 0$ to give

$$3\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)=2+\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$2\frac{\mathrm{d}y}{\mathrm{d}x}=-1\quad\therefore\ \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-1}{2}$$

$\therefore$ gradient of the normal at (1, 0) is 2.

Equation of the normal is

$$y-0=2(x-1)$$

or $y=2x-2$

6 $x^2+4y^2-6x-16y+21=0$

Differentiate with respect to *x*:

$$\begin{aligned}2x+8y\frac{dy}{dx}-6-16\frac{dy}{dx}&=0\\8y\frac{dy}{dx}-16\frac{dy}{dx}&=6-2x\\(8y-16)\frac{dy}{dx}&=6-2x\\\frac{dy}{dx}&=\frac{6-2x}{8y-16}\end{aligned}$$

For zero gradient:

$$\frac{6-2x}{8y-16}=0$$
$$6-2x=0$$
$$x=3$$

Substitute $x=3$ into the equation of the curve to give

$$9+4y^2-18-16y+21=0$$
$$4y^2-16y+12=0$$
$$y^2-4y+3=0$$
$$(y-1)(y-3)=0$$
$$y=1 \text{ or } 3$$

$\therefore$ the coordinates of the points of zero gradient are (3, 1) and (3, 3).

7 $2x^2+3y^2-x+6xy+5=0$

$$4x+6y\frac{dy}{dx}-1+6\left(x\frac{dy}{dx}+y\right)=0$$
$$(6y+6x)\frac{dy}{dx}=1-6y-4x$$
$$\frac{dy}{dx}=\frac{1-6y-4x}{6(x+y)}$$

When $x=1$ and $y=-2$,

$$\frac{dy}{dx}=\frac{1-6(-2)-4}{6(1-2)}=-\frac{3}{2}$$

Equation of tangent at (1, −2) is

$$y-(-2)=-\frac{3}{2}(x-1)$$
$$2y+4=-3x+3$$
$$3x+2y+1=0$$

8 $3^x=y-2xy$

$$3^x\ln 3=\frac{dy}{dx}-2\left(x\frac{dy}{dx}+y\right)$$
$$3^x\ln 3+2y=(1-2x)\frac{dy}{dx}$$
$$\frac{dy}{dx}=\frac{3^x\ln 3+2y}{1-2x}$$

Substitute $x = 2$ and $y = -3$ to give

$$\frac{dy}{dx}=\frac{3^2\ln 3-6}{1-4}=2-3\ln 3$$

9 $\ln(y^2)=\frac{1}{2}x\ln(x-1)$

$$2\ln y=\frac{1}{2}x\ln(x-1)$$
$$\frac{2}{y}\frac{dy}{dx}=\frac{1}{2}\left(\left(x\times\frac{1}{x-1}\right)+\ln(x-1)\right)$$
$$\frac{dy}{dx}=\frac{y}{4}\left(\frac{x}{x-1}+\ln(x-1)\right)$$

When $x = 4$,
the equation of the curve gives

$$\ln(y^2)=2\ln 3=\ln 9\Rightarrow y^2=9$$
$$\therefore\ y=3\ \ (\text{because } y>0)$$

Hence $\frac{dy}{dx}=\frac{3}{4}\left(\frac{4}{3}+\ln 3\right)=1+\frac{3}{4}\ln 3$

10 a $\sin x + \cos y = 0.5$

$$\cos x - \sin y \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x}{\sin y}$$

b At stationary points $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$$\frac{\cos x}{\sin y} = 0 \text{ when } \cos x = 0$$

$$\therefore x = \pm\frac{\pi}{2} \text{ (in the interval } -\pi < x < \pi\text{)}$$

When $x = \frac{\pi}{2}$, $1 + \cos y = 0.5$

$$\cos y = -0.5 \Rightarrow y = \pm\frac{2\pi}{3}$$

When $x = -\frac{\pi}{2}$, $-1 + \cos y = 0.5$

$\cos y = 1.5 \Rightarrow$ no solutions

Therefore the stationary points are $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and $\left(\frac{\pi}{2}, -\frac{2\pi}{3}\right)$.

11 a $y\mathrm{e}^{-3x} - 3x = y^2$

$$y(-3\mathrm{e}^{-3x}) + \mathrm{e}^{-3x}\frac{\mathrm{d}y}{\mathrm{d}x} - 3 = 2y\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$(\mathrm{e}^{-3x} - 2y)\frac{\mathrm{d}y}{\mathrm{d}x} = 3(y\mathrm{e}^{-3x} + 1)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3(y\mathrm{e}^{-3x} + 1)}{\mathrm{e}^{-3x} - 2y}$$

b Substitute $x = 0$ and $y = 0$ to give

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3(0 \times \mathrm{e}^0 + 1)}{\mathrm{e}^0 - 2 \times 0} = 3$$

Equation of tangent at (0, 0) is

$$y - 0 = 3(x - 0)$$

$$y = 3x$$

Challenge

a $6x + y^2 + 2xy = x^2$

$$6 + 2y\frac{\mathrm{d}y}{\mathrm{d}x} + 2\left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right) = 2x$$

$$(2y + 2x)\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 2y - 6$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x - 2y - 6}{2y + 2x} = \frac{x - y - 3}{y + x}$$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ only when $x - y - 3 = 0$

or $y = x - 3$

Substitute $y = x - 3$ into the equation of the curve to give

$$6x + (x - 3)^2 + 2x(x - 3) = x^2$$

$$2x^2 - 6x + 9 = 0$$

The discriminant of this quadratic is

$$(-6)^2 - 4 \times 2 \times 9 = -36 < 0$$

so there are no real solutions.

Hence there are no points on C such that $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

b $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{y + x}{x - y - 3}$

$\frac{\mathrm{d}x}{\mathrm{d}y} = 0$ when $y + x = 0$

or $y = -x$

Substitute $y = -x$ into the equation of the curve to give

$$6x + (-x)^2 + 2x(-x) = x^2$$

$$2x^2 - 6x = 0$$

$$x(x - 3) = 0$$

$$x = 0 \text{ or } x = 3$$

So $y = 0$ or $y = -3$

Therefore the coordinates of the points on C such that $\frac{\mathrm{d}x}{\mathrm{d}y} = 0$ are (0, 0) and (3, −3).

Differentiation 9I

1 a $f(x) = x^3 - 3x^2 + x - 2$

$f'(x) = 3x^2 - 6x + 1$

$f''(x) = 6x - 6$

i $f(x)$ is convex when $f''(x) \geqslant 0$

$6x - 6 \geqslant 0$ for $x \geqslant 1$

So $f(x)$ is convex for all $x \geqslant 1$

or on the interval $[1, \infty)$.

ii $f(x)$ is concave when $f''(x) \leqslant 0$

$6x - 6 \leqslant 0$ for $x \leqslant 1$

So $f(x)$ is concave for all $x \leqslant 1$

or on the interval $(-\infty, 1]$.

b $f(x) = x^4 - 3x^3 + 2x - 1$

$f'(x) = 4x^3 - 9x^2 + 2$

$f''(x) = 12x^2 - 18x = 6x(2x - 3)$

i $f(x)$ is convex when $f''(x) \geqslant 0$

$6x(2x - 3) \geqslant 0$ for $x \leqslant 0$ or $x \geqslant \frac{3}{2}$

So $f(x)$ is convex for $x \leqslant 0$ or $x \geqslant \frac{3}{2}$,

or on $(-\infty, 0] \cup \left[\frac{3}{2}, \infty\right)$.

ii $f(x)$ is concave when $f''(x) \leqslant 0$

$6x(2x - 3) \leqslant 0$ for $0 \leqslant x \leqslant \frac{3}{2}$

So $f(x)$ is concave for all $0 \leqslant x \leqslant \frac{3}{2}$

or on the interval $\left[0, \frac{3}{2}\right]$.

c $f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

i $f(x)$ is convex when $f''(x) \geqslant 0$

$-\sin x \geqslant 0$ for $\pi \leqslant x \leqslant 2\pi$

So $f(x)$ is convex on the interval $[\pi, 2\pi]$.

ii $f(x)$ is concave when $f''(x) \leqslant 0$

$-\sin x \leqslant 0$ for $\leqslant 0\ x \leqslant \pi$

So $f(x)$ is concave on the interval $[0, \pi]$.

d $f(x) = -x^2 + 3x - 7$

$f'(x) = -2x + 3$

$f''(x) = -2$

i $f(x)$ is convex when $f''(x) \geqslant 0$

So $f(x)$ is not convex anywhere.

ii $f(x)$ is concave when $f''(x) \leqslant 0$

So $f(x)$ is concave for all $x \in \mathbb{R}$

or on the interval $(-\infty, \infty)$.

e $f(x) = e^x - x^2$

$f'(x) = e^x - 2x$

$f''(x) = e^x - 2$

i $f(x)$ is convex when $f''(x) \geqslant 0$

$e^x - 2 \geqslant 0$ for $x \geqslant \mathrm{In}\ 2$

So $f(x)$ is convex on $[\ln 2, \infty)$.

ii $f(x)$ is concave when $f''(x) \leqslant 0$

$e^x - 2 \leqslant 0$ for $x \leqslant \mathrm{In}\ 2$

So $f(x)$ is concave on $(-\infty, \ln 2]$.

f $f(x) = \ln x, \quad x > 0$

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2}$

i $f(x)$ is convex when $f''(x) \geqslant 0$

But $-\frac{1}{x^2} < 0$ for all $x > 0$

So $f(x)$ is not convex anywhere.

ii $f(x)$ is concave when $f''(x) \leqslant 0$

$-\frac{1}{x^2} < 0$ for all $x > 0$

So $f(x)$ is concave on $(0, \infty)$.

2 a Let $y = \mathrm{f}(x)$. Then $x = \sin y$.

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \cos y \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}}$$

so $\mathrm{f}'(x) = \dfrac{1}{\sqrt{1-x^2}}$

b $\mathrm{f}(x) = \arcsin x$

$$\mathrm{f}'(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$$

$$\mathrm{f}''(x) = (-2x)\left(-\frac{1}{2}\right)(1-x^2)^{-\frac{3}{2}} = \frac{x}{(1-x^2)^{\frac{3}{2}}}$$

On the interval $(-1, 0)$, $x < 0$

$\therefore\ \mathrm{f}''(x) \leqslant 0$

So $\mathrm{f}(x)$ is concave on the interval $(-1, 0)$.

c On the interval $(0, 1)$, $x > 0$

$\therefore\ \mathrm{f}''(x) \geqslant 0$

So $\mathrm{f}(x)$ is convex on the interval $(0, 1)$.

d $\mathrm{f}(x)$ changes from concave to convex at $x = 0$

When $x = 0$, $y = 0$.

$\therefore$ point of inflection is $(0, 0)$.

3 a $\mathrm{f}(x) = \cos^2 x - 2\sin x$

$$\mathrm{f}'(x) = -2\cos x \sin x - 2\cos x$$

$$\begin{aligned}\mathrm{f}''(x) &= -2(\cos^2 x - \sin^2 x) + 2\sin x \\ &= -2(1 - 2\sin^2 x) + 2\sin x \\ &= -2 + 4\sin^2 x + 2\sin x \\ &= 2(2\sin^2 x + \sin x - 1)\end{aligned}$$

At points of inflection $\mathrm{f}''(x) = 0$

$$2\sin^2 x + \sin x - 1 = 0$$

$$(2\sin x - 1)(\sin x + 1) = 0$$

$$\sin x = \frac{1}{2} \text{ or } -1$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6} \text{ or } \frac{3\pi}{2}$$

Check the sign of $\mathrm{f}''(x)$ on either side of each point:

$$\mathrm{f}''(0) = 2(0 + 0 - 1) < 0$$

$$\mathrm{f}''\left(\frac{\pi}{2}\right) = 2(2 + 1 - 1) > 0$$

$\Rightarrow x = \dfrac{\pi}{6}$ is an inflection point

$$\mathrm{f}''(\pi) = 2(0 + 0 - 1) < 0$$

$\Rightarrow x = \dfrac{5\pi}{6}$ is an inflection point

$$\mathrm{f}''(2\pi) = 2(0 + 0 - 1) < 0$$

$\Rightarrow x = \dfrac{3\pi}{2}$ is not an inflection point

$$x = \frac{\pi}{6} \Rightarrow y = \left(\frac{\sqrt{3}}{2}\right)^2 - 2\left(\frac{1}{2}\right) = -\frac{1}{4}$$

$$x = \frac{5\pi}{6} \Rightarrow y = \left(-\frac{\sqrt{3}}{2}\right)^2 - 2\left(\frac{1}{2}\right) = -\frac{1}{4}$$

So the points of inflection are $\left(\dfrac{\pi}{6}, -\dfrac{1}{4}\right)$ and $\left(\dfrac{5\pi}{6}, -\dfrac{1}{4}\right)$.

3 b $\mathrm{f}(x) = -\dfrac{x^3 - 2x^2 + x - 1}{x - 2}$

$$= -\left(x^2 + 1 + \frac{1}{x-2}\right)$$

$$\mathrm{f}'(x) = \frac{1}{(x-2)^2} - 2x$$

$$\mathrm{f}''(x) = -\frac{2}{(x-2)^3} - 2 = -2\left(\frac{1}{(x-2)^3} + 1\right)$$

At points of inflection $\mathrm{f}''(x) = 0$

$$\frac{1}{(x-2)^3} + 1 = 0$$

$$(x-2)^3 = -1$$

$$x - 2 = -1 \quad \therefore \; x = 1$$

Check the sign of $\mathrm{f}''(x)$ on either side of $x = 1$:

$\mathrm{f}''(0.5) = -1.407... < 0$

$\mathrm{f}''(1.5) = 14 > 0$

$\therefore \; x = 1$ is a point of inflection

When $x = 1$, $y = -(1 + 1 - 1) = -1$

So the point of inflection is $(1, -1)$.

c $\mathrm{f}(x) = \dfrac{x^3}{x^2 - 4} = x + \dfrac{2}{x-2} + \dfrac{2}{x+2}$

$$\mathrm{f}'(x) = 1 - \frac{2}{(x-2)^2} - \frac{2}{(x+2)^2}$$

$$\mathrm{f}''(x) = \frac{4}{(x-2)^3} + \frac{4}{(x+2)^3}$$

$$= 4\left(\frac{1}{(x-2)^3} + \frac{1}{(x+2)^3}\right)$$

At points of inflection $\mathrm{f}''(x) = 0$

$$\frac{1}{(x-2)^3} + \frac{1}{(x+2)^3} = 0$$

$$(x-2)^3 = -(x+2)^3$$

$$x - 2 = -(x+2)$$

$$\therefore \; x = 0$$

Check the sign of $\mathrm{f}''(x)$ on either side of $x = 0$:

$$\mathrm{f}''(-1) = \frac{104}{27} > 0$$

$$\mathrm{f}''(1) = -\frac{104}{27} < 0$$

$\therefore \; x = 0$ is a point of inflection

When $x = 0$, $y = 0$

So the point of inflection is $(0, 0)$.

4 $f(x) = 2x^2 \ln x$

$$f'(x) = 2x^2\left(\frac{1}{x}\right) + 4x\ln x = 2x(1 + 2\ln x)$$

$$f''(x) = 2x\left(\frac{2}{x}\right) + 2(1 + 2\ln x) = 6 + 4\ln x$$

At a point of inflection $f''(x) = 0$

$$6 + 4\ln x = 0 \Rightarrow \ln x = -\frac{3}{2} \Rightarrow x = e^{-\frac{3}{2}}$$

So there is one point of inflection, where $x = e^{-\frac{3}{2}}$

5 a $y = e^x(x^2 - 2x + 2)$

$$\frac{dy}{dx} = e^x(2x - 2) + e^x(x^2 - 2x + 2) = e^x x^2$$

At stationary points $\frac{dy}{dx} = 0$

$e^x x^2 = 0$ when $x = 0$ and $y = 2$

$\therefore$ stationary point at (0, 2)

$$\frac{d^2y}{dx^2} = 2xe^x + e^x x^2 = e^x x(x + 2)$$

When $x = 0$, $\frac{d^2y}{dx^2} = 0$

so $x = 0$ is neither a maximum nor a minimum point

When $x > 0$, $\frac{d^2y}{dx^2} > 0$

When $-2 < x < 0$, $\frac{d^2y}{dx^2} < 0$

$\therefore$ (0, 2) is a stationary point of inflection.

b At points of inflection $\frac{d^2y}{dx^2} = 0$

$e^x x(2 + x) = 0$

$x = 0$ or -2

From part **a** it is known that $x = 0$ is a stationary point of inflection.

When $x < -2$, $\frac{d^2y}{dx^2} > 0$

When $-2 < x < 0$, $\frac{d^2y}{dx^2} < 0$

so $x = -2$ is a point of inflection

$x = -2 \Rightarrow y = 10e^{-2}$

$\therefore$ $(-2, 10e^{-2})$ is a non-stationary point of inflection.

6 a $y = xe^x$

$$\frac{dy}{dx} = xe^x + e^x = e^x(x + 1)$$

At stationary points $\frac{dy}{dx} = 0$

$e^x(x + 1) = 0$ when $x = -1$ and $y = -e^{-1}$

$\therefore$ stationary point at $\left(-1, -\frac{1}{e}\right)$

$$\frac{d^2y}{dx^2} = e^x + e^x(x + 1) = e^x(x + 2)$$

When $x = -1$, $\frac{d^2y}{dx^2} = e^{-1} > 0$

Therefore $\left(-1, -\frac{1}{e}\right)$ is a minimum.

b At points of inflection $\frac{d^2y}{dx^2} = 0$

$e^x(x + 2) = 0$

$\Rightarrow x = -2$, $y = -2e^{-2} = -\frac{2}{e^2}$

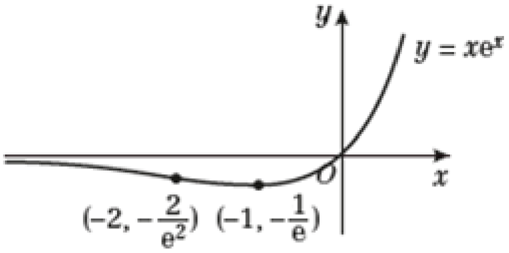
When $x < -2$, $\frac{d^2y}{dx^2} < 0$

When $x > -2$, $\frac{d^2y}{dx^2} > 0$

so $x = -2$ is a point of inflection

$\therefore$ non-stationary point of inflection at $\left(-2, -\frac{2}{e^2}\right)$

c



7 i $f'(x)$ is the gradient, so it is
negative for A
zero for B
positive for C
zero for D

7 ii $f''(x)$ determines whether the curve is convex, is concave or has a point of inflection. Hence $f''(x)$ is
positive for A
positive for B
negative for C
zero for D

8 $f(x) = \tan x$

$f'(x) = \sec^2 x$

$f''(x) = 2\sec^2 x \tan x = 2\dfrac{\sin x}{\cos^3 x}$

At points of inflection $f''(x) = 0$

$2\dfrac{\sin x}{\cos^3 x} = 0$ only when $\sin x = 0$, which has only one solution, $x = 0$, in the interval $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

When $x = 0$, $f(x) = 0$

When $x < 0$, $f''(x) < 0$

When $x > 0$, $f''(x) > 0$

$\therefore$ there is one point of inflection at (0, 0).

9 a $y = x(3x-1)^5$

$$\frac{dy}{dx} = 15x(3x-1)^4 + (3x-1)^5$$

$$\frac{d^2y}{dx^2} = 15(3x-1)^4 + 15(3x-1)^4 + 180x(3x-1)^3$$

$$= 30(3x-1)^4 + 180x(3x-1)^3$$

$$= 30(3x-1)^3(9x-1)$$

b At points of inflection $\dfrac{d^2y}{dx^2} = 0$

$30(3x-1)^3(9x-1) = 0$

$x = \dfrac{1}{3}$ or $\dfrac{1}{9}$

$x = \dfrac{1}{9} \Rightarrow y = \dfrac{1}{9} \times \left(\dfrac{1}{3} - 1\right)^5 = -\dfrac{32}{2187}$

$x = \dfrac{1}{3} \Rightarrow y = \dfrac{1}{3} \times (1-1)^5 = 0$

Points of inflection are

$\left(\dfrac{1}{9}, -\dfrac{32}{2187}\right)$ and $\left(\dfrac{1}{3}, 0\right)$

10 a $\dfrac{d^2y}{dx^2} = 12(x-5)^2 \geqslant 0$ for all x, so even though $\dfrac{d^2y}{dx^2} = 0$ at $x = 5$, the sign of $\dfrac{d^2y}{dx^2}$ does not change on either side of $x = 5$ and hence it is not a point of inflection.

b $\dfrac{dy}{dx} = 4(x-5)^3 = 0$ when

$x = 5$ and $y = 0$

Stationary point is at (5, 0).

When $x < 5$, $\dfrac{dy}{dx} < 0$

When $x > 5$, $\dfrac{dy}{dx} > 0$

$\therefore$ (5, 0) is a minimum point.

11 $y = \dfrac{1}{3}x^2 \ln x - 2x + 5$

$$\frac{dy}{dx} = \frac{1}{3}x^2\left(\frac{1}{x}\right) + \frac{2}{3}x\ln x - 2 = \frac{x}{3} + \frac{2}{3}x\ln x - 2$$

$$\frac{d^2y}{dx^2} = \frac{1}{3} + \frac{2}{3}(1 + \ln x) = 1 + \frac{2}{3}\ln x$$

C is convex when $\dfrac{d^2y}{dx^2} \geqslant 0$

$1 + \dfrac{2}{3}\ln x \geqslant 0$

$\text{In } x \geqslant -\dfrac{3}{2}$

$x \geqslant e^{-\frac{3}{2}}$

Challenge

1 A general cubic can be written as

$$\mathrm{f}(x) = ax^3 + bx^2 + cx + d$$
$$\mathrm{f}'(x) = 3ax^2 + 2bx + c$$
$$\mathrm{f}''(x) = 6ax + 2b$$
$$\mathrm{f}''(x) = 0 \text{ when } x = -\frac{b}{3a}$$

Let $\varepsilon > 0$; then

$$\mathrm{f}''\left(-\frac{b}{3a} + \varepsilon\right) = 6a\left(-\frac{b}{3a} + \varepsilon\right) + 2b$$
$$= -2b + 6a\varepsilon + 2b = 6a\varepsilon > 0$$
$$\mathrm{f}''\left(-\frac{b}{3a} - \varepsilon\right) = 6a\left(-\frac{b}{3a} - \varepsilon\right) + 2b$$
$$= -2b - 6a\varepsilon + 2b = -6a\varepsilon < 0$$

The sign of $\mathrm{f}''(x)$ changes either side of $x = -\frac{b}{3a}$, so this is the single point of inflection.

2 a $y = ax^4 + bx^3 + cx^2 + dx + e$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4ax^3 + 3bx^2 + 2cx + d$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12ax^2 + 6bx + 2c$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \Leftrightarrow 12ax^2 + 6bx + 2c = 0$$

As this is a quadratic equation, there are at most two values of x for which $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$.

So there are at most two points of inflection.

b If the discriminant of a quadratic is less than zero, there are no real solutions.

$$\text{Discriminant} = (6b)^2 - 4 \times 12a \times 2c$$
$$= 36b^2 - 96ac$$
$$= 12(3b^2 - 8ac)$$

If $3b^2 < 8ac$ then discriminant < 0 and so there are no solutions to $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$.

Therefore if $3b^2 < 8ac$, then C has no points of inflection.

Differentiation 9J

1 $A=\frac{1}{4}\pi r^2$

$$\frac{dA}{dr}=\frac{1}{2}\pi r$$

$$\frac{dA}{dt}=\frac{dA}{dr}\times\frac{dr}{dt}$$

$$=\frac{1}{2}\pi r\times 6=3\pi r$$

When $r=2$,

$$\frac{dA}{dt}=3\pi\times 2=6\pi$$

2 $y=xe^x$

$$\frac{dy}{dx}=xe^x+e^x=(x+1)e^x$$

$$\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$$

$$=(x+1)e^x\times 5=5(x+1)e^x$$

When $x=2$,

$$\frac{dy}{dt}=5(2+1)e^2=15e^2$$

3 $r=1+3\cos\theta$

$$\frac{dr}{d\theta}=-3\sin\theta$$

$$\frac{dr}{dt}=\frac{dr}{d\theta}\times\frac{d\theta}{dt}$$

$$=-3\sin\theta\times 3=-9\sin\theta$$

When $\theta=\frac{\pi}{6}$,

$$\frac{dr}{dt}=-9\sin\frac{\pi}{6}=-\frac{9}{2}$$

4 $V=\frac{1}{3}\pi r^3$

$$\frac{dV}{dr}=\pi r^2 \Rightarrow \frac{dr}{dV}=\frac{1}{\pi r^2}$$

$$\frac{dr}{dt}=\frac{dr}{dV}\times\frac{dV}{dt}$$

$$=\frac{1}{\pi r^2}\times 8=\frac{8}{\pi r^2}$$

When $r=3$,

$$\frac{dr}{dt}=\frac{8}{\pi\times 3^2}=\frac{8}{9\pi}$$

5 Let P be the size of the population and let t be time. Then the rate of growth of the population is $\frac{dP}{dt}$

$$\frac{dP}{dt}\propto P$$

i.e. $\frac{dP}{dt}=kP$ where k is a positive constant.

6 The gradient of the curve is $\frac{dy}{dx}$

$\therefore \frac{dy}{dx}\propto xy$ (product of x- and y-coordinates)

i.e. $\frac{dy}{dx}=kxy$,

where k is the constant of proportion.

When $x=4$, $y=2$ and $\frac{dy}{dx}=\frac{1}{2}$.

Substituting into $\frac{dy}{dx}=kxy$ gives

$$\frac{1}{2}=k\times 4\times 2$$

$$k=\frac{1}{16}$$

$$\therefore \frac{dy}{dx}=\frac{xy}{16}$$

7 The rate of increase of the volume of liquid in the container is $\frac{dV}{dt}$

$$\frac{dV}{dt}=\text{rate in}-\text{rate out}$$

$$=30-\frac{2}{15}V$$

Multiply both sides by –15 to give

$$-15\frac{dV}{dt}=2V-450$$

8 The rate of change of the charge is $\frac{\mathrm{d}Q}{\mathrm{d}t}$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} \propto Q$$

i.e. $\frac{\mathrm{d}Q}{\mathrm{d}t} = -kQ$

where k is a positive constant.
(The negative sign is required as the body is *losing* charge.)

9 The rate of increase of x is $\frac{\mathrm{d}x}{\mathrm{d}t}$

$\therefore \frac{\mathrm{d}x}{\mathrm{d}t} \propto \frac{1}{x^2}$ (inverse proportion)

i.e. $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{k}{x^2}$
where k is the constant of proportion.

10 a Let r be the radius of the circle and let t be time. Then $\frac{\mathrm{d}r}{\mathrm{d}t} = 0.4 \text{ cm s}^{-1}$.

$$C = 2\pi r$$

$$\frac{\mathrm{d}C}{\mathrm{d}r} = 2\pi$$

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \frac{\mathrm{d}C}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$$

$$= 2\pi \times 0.4 = 0.8\pi \text{ cm s}^{-1}$$

This means that the circumference is increasing at a constant rate of 0.8π cm per second.

b Let A be the radius of the circle; then

$$A = \pi r^2$$

$$\frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$$

$$= 2\pi r \times 0.4 = 0.8\pi r$$

When $r = 10$,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 0.8\pi \times 10 = 8\pi \text{ cm}^2 \text{ s}^{-1}$$

c $\frac{\mathrm{d}A}{\mathrm{d}t} = 0.8\pi r$

When $\frac{\mathrm{d}A}{\mathrm{d}t} = 20$,

$$0.8\pi r = 20$$

$$r = \frac{20}{0.8\pi} = \frac{25}{\pi} \text{ cm}$$

11 a Let l be the side length of the cube and let V be its volume.

Then $V = l^3$ and $\frac{\mathrm{d}V}{\mathrm{d}t} = -4.5$

$$\frac{\mathrm{d}V}{\mathrm{d}l} = 3l^2 \quad \text{so} \quad \frac{\mathrm{d}l}{\mathrm{d}V} = \frac{1}{3l^2}$$

$$\frac{\mathrm{d}l}{\mathrm{d}t} = \frac{\mathrm{d}l}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$$

$$= \frac{1}{3l^2} \times (-4.5) = -\frac{3}{2l^2}$$

When $V = 100$, $l = \sqrt[3]{100}$

$$\frac{\mathrm{d}l}{\mathrm{d}t} = -\frac{3}{2\left(\sqrt[3]{100}\right)^2} = 0.070 \text{ cm s}^{-1} \text{ (2 s.f.)}$$

b $\frac{\mathrm{d}l}{\mathrm{d}t} = -\frac{3}{2l^2}$

2 mm is 0.2 cm

When $\frac{\mathrm{d}l}{\mathrm{d}t} = -0.2$,

$$-\frac{3}{2l^2} = -0.2$$

$$2l^2 = 15$$

$$l = \sqrt{7.5}$$

$$\therefore V = l^3 = \left(\sqrt{7.5}\right)^3 = 20.5 \text{ cm}^3 \text{ (3 s.f.)}$$

12 The rate of change of the volume of fluid in the tank is $\frac{\mathrm{d}V}{\mathrm{d}t}$

$$\frac{\mathrm{d}V}{\mathrm{d}t} \propto \sqrt{V}$$

i.e. $\frac{\mathrm{d}V}{\mathrm{d}t} = -K\sqrt{V}$

where K is a positive constant.
(The negative sign is present because fluid is flowing *out* of the tank, so the volume left in the tank is *decreasing*.)

Let A be the constant cross-section;
then $V = Ah$ (where h is the depth)

$$\therefore \frac{\mathrm{d}V}{\mathrm{d}h} = A$$

Use the chain rule to find $\frac{\mathrm{d}h}{\mathrm{d}t}$:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}h} \times \frac{\mathrm{d}h}{\mathrm{d}t}$$

$$\therefore \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t} \div \frac{\mathrm{d}V}{\mathrm{d}h}$$

$$= \frac{-K\sqrt{V}}{A} = \frac{-K\sqrt{Ah}}{A}$$

$$= \left(\frac{-K}{\sqrt{A}}\right)\sqrt{h} = -k\sqrt{h}$$

where $k = \frac{K}{\sqrt{A}}$ is a positive constant.

13 a Let l be the length of one side of the cube.
Surface area of cube $A = 6l^2$.

So $l = \sqrt{\frac{A}{6}}$

Volume of cube $V = l^3 = \left(\sqrt{\frac{A}{6}}\right)^3 = \left(\frac{A}{6}\right)^{\frac{3}{2}}$

b $\frac{\mathrm{d}V}{\mathrm{d}A} = \frac{1}{6} \times \frac{3}{2}\left(\frac{A}{6}\right)^{\frac{1}{2}} = \frac{1}{4}\left(\frac{A}{6}\right)^{\frac{1}{2}}$

c Rate of expansion of surface area is $\frac{\mathrm{d}A}{\mathrm{d}t}$

Given $\frac{\mathrm{d}A}{\mathrm{d}t} = 2$

Need $\frac{\mathrm{d}V}{\mathrm{d}t}$ so use the chain rule:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}A} \times \frac{\mathrm{d}A}{\mathrm{d}t}$$

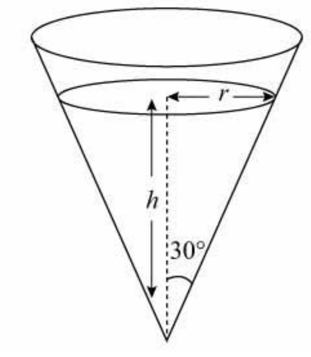
$$= \frac{1}{4}\left(\frac{A}{6}\right)^{\frac{1}{2}} \times 2 = \frac{1}{2}\left(\frac{A}{6}\right)^{\frac{1}{2}}$$

From $A = 6l^2$ and $V = l^3$,

$$A = 6\left(\sqrt[3]{V}\right)^2 = 6V^{\frac{2}{3}}$$

$$\therefore \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{2}\left(\frac{6V^{\frac{2}{3}}}{6}\right)^{\frac{1}{2}} = \frac{1}{2}V^{\frac{1}{3}}$$

14



Let V be the volume of salt in the funnel at time t.

$$V = \frac{1}{3}\pi r^2 h$$

$$\tan 30° = \frac{r}{h} \Rightarrow r = h \tan 30° = \frac{h}{\sqrt{3}}$$

$$\therefore V = \frac{1}{3}\pi\left(\frac{h^2}{3}\right)h = \frac{1}{9}\pi h^3$$

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{1}{3}\pi h^2 \text{ and hence } \frac{\mathrm{d}h}{\mathrm{d}V} = \frac{3}{\pi h^2}$$

Given that $\frac{\mathrm{d}V}{\mathrm{d}t} = -6$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{3}{\pi h^2} \times (-6) = -\frac{18}{\pi h^2}$$

So the rate of change of the height, h, is inversely proportional to h^2 and is given by the differential equation $\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{18}{\pi h^2}$

Differentiation, Mixed Exercise 9

1 a $y = \ln x^2 = 2\ln x$

(using properties of logs)

$$\therefore \frac{dy}{dx} = 2 \times \frac{1}{x} = \frac{2}{x}$$

Alternative method:

When $y = \ln f(x)$, $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

(by the chain rule)

$$\therefore y = \ln x^2 \Rightarrow \frac{dy}{dx} = \frac{2x}{x^2} = \frac{2}{x}$$

b $y = x^2 \sin 3x$

Using the product rule,

$$\frac{dy}{dx} = x^2(3\cos 3x) + (\sin 3x) \times 2x$$

$$= 3x^2 \cos 3x + 2x \sin 3x$$

2 a $2y = x - \sin x \cos x$

$$\therefore y = \frac{x}{2} - \frac{1}{2}\sin x \cos x$$

Using the product rule,

$$\frac{dy}{dx} = \frac{1}{2} - \frac{1}{2}\left(\sin x(-\sin x) + \cos x \cos x\right)$$

$$= \frac{1}{2} + \frac{1}{2}\sin^2 x - \frac{1}{2}\cos^2 x$$

$$= \frac{1}{2}(1 - \cos^2 x) + \frac{1}{2}\sin^2 x$$

$$= \frac{1}{2}\sin^2 x + \frac{1}{2}\sin^2 x$$

$$= \sin^2 x$$

b $y = \frac{x}{2} - \frac{1}{2}\sin x \cos x$

$$\frac{dy}{dx} = \sin^2 x$$

$$\frac{d^2y}{dx^2} = 2\sin x \cos x = \sin 2x$$

At points of inflection $\frac{d^2y}{dx^2} = 0$

i.e. $\sin 2x = 0$

$2x = \pi, 2\pi$ or 3π

$x = \frac{\pi}{2}, \pi$ or $\frac{3\pi}{2}$

When $x = \frac{\pi}{2}$, $y = \frac{\pi}{4}$

At $x = \frac{\pi}{3}$, $\frac{d^2y}{dx^2} > 0$; at $x = \frac{3\pi}{4}$, $\frac{d^2y}{dx^2} < 0$

So $\frac{d^2y}{dx^2}$ changes sign either side of $x = \frac{\pi}{2}$

When $x = \pi$, $y = \frac{\pi}{2}$

At $x = \frac{3\pi}{4}$, $\frac{d^2y}{dx^2} < 0$; at $x = \frac{5\pi}{4}$, $\frac{d^2y}{dx^2} > 0$

So $\frac{d^2y}{dx^2}$ changes sign either side of $x = \pi$

When $x = \frac{3\pi}{2}$, $y = \frac{3\pi}{4}$

At $x = \frac{5\pi}{4}$, $\frac{d^2y}{dx^2} > 0$; at $x = \frac{7\pi}{4}$, $\frac{d^2y}{dx^2} < 0$

So $\frac{d^2y}{dx^2}$ changes sign either side of $x = \frac{3\pi}{2}$

Hence the points of inflection are

$\left(\frac{\pi}{2}, \frac{\pi}{4}\right)$, $\left(\pi, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, \frac{3\pi}{4}\right)$

3 **a** $y = \dfrac{\sin x}{x}$

Using the quotient rule:

$$\frac{dy}{dx} = \frac{x\cos x - \sin x \times 1}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

b $y = \ln\dfrac{1}{x^2+9} = \ln 1 - \ln(x^2+9)$

$$= -\ln(x^2+9)$$

(by the laws of logarithms)

Using the chain rule:

$$\frac{dy}{dx} = -\frac{1}{x^2+9} \times 2x = -\frac{2x}{x^2+9}$$

4 **a** $f(x) = \dfrac{x}{x^2+2}$

$$f'(x) = \frac{(x^2+2)\times 1 - x \times 2x}{(x^2+2)^2} = \frac{2-x^2}{(x^2+2)^2}$$

The function is increasing when $f'(x) \geqslant 0$

i.e. $\dfrac{2-x^2}{(x^2+2)^2} \geqslant 0$

$$x^2 \leqslant 2$$

$$-\sqrt{2} \leqslant x \leqslant \sqrt{2}$$

Hence $f(x)$ is increasing on the interval $[-k, k]$ where $k = \sqrt{2}$.

b $f''(x) = \dfrac{-2x(x^2+2)^2 - 4x(x^2+2)(2-x^2)}{(x^2+2)^4}$

$$= \frac{2x(x^2+2)\left(-(x^2+2) - 2(2-x^2)\right)}{(x^2+2)^4}$$

$$= \frac{2x(x^2+2)(x^2-6)}{(x^2+2)^4}$$

$f''(x)$ changes sign when the numerator $2x(x^2+2)(x^2-6)$ is zero

i.e. at $x = 0$ and $x = \pm\sqrt{6}$

where $y = 0$ and $y = \dfrac{\pm\sqrt{6}}{6+2}$

Points of inflection are

$(0, 0)$ and $\left(\pm\sqrt{6}, \pm\dfrac{\sqrt{6}}{8}\right)$

5 **a** $f(x) = 12\ln x + x^{\frac{3}{2}}, \quad x > 0$

$$f'(x) = \frac{12}{x} + \frac{3}{2}x^{\frac{1}{2}} = \frac{12}{x} + \frac{3}{2}\sqrt{x}$$

f(x) is an increasing function when $f'(x) \geq 0$

As $x > 0$, $\dfrac{12}{x} + \dfrac{3}{2}\sqrt{x}$ is always positive.

∴ f(x) is increasing for all $x > 0$.

b $f''(x) = -\dfrac{12}{x^2} + \dfrac{3}{4}x^{-\frac{1}{2}} = -\dfrac{12}{x^2} + \dfrac{3}{4\sqrt{x}}$

At a point of inflection $f''(x) = 0$

$$-\frac{12}{x^2} + \frac{3}{4\sqrt{x}} = 0$$

$$\frac{12}{x^2} = \frac{3}{4\sqrt{x}}$$

$$x^2 = 16\sqrt{x}$$

$$x^{\frac{3}{2}} = 16$$

$$x = \sqrt[3]{256}$$

$$f\left(\sqrt[3]{256}\right) = 12\ln(256)^{\frac{1}{3}} + 256^{\frac{1}{2}}$$

$$= 4\ln 256 + 16$$

$$= 4\ln 2^8 + 16 = 32\ln 2 + 16$$

Coordinates of the point of inflection are $\left(\sqrt[3]{256}, 32\ln 2 + 16\right)$

6 $y = \cos^2 x + \sin x$

$$\frac{dy}{dx} = -2\cos x \sin x + \cos x$$
$$= \cos x(1 - 2\sin x)$$

At stationary points $\frac{dy}{dx} = 0$

$\cos x(1 - 2\sin x) = 0$

$\cos x = 0$ or $\sin x = \frac{1}{2}$

Solutions in the interval $(0, 2\pi]$ are

$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ and $\frac{3\pi}{2}$

$x = \frac{\pi}{6} \Rightarrow y = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$

$x = \frac{\pi}{2} \Rightarrow y = 1$

$x = \frac{5\pi}{6} \Rightarrow y = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$

$x = \frac{3\pi}{2} \Rightarrow y = -1$

So the stationary points are

$\left(\frac{\pi}{6}, \frac{5}{4}\right), \left(\frac{\pi}{2}, 1\right), \left(\frac{5\pi}{6}, \frac{5}{4}\right)$ and $\left(\frac{3\pi}{2}, -1\right)$

7 $y = x\sqrt{\sin x} = x(\sin x)^{\frac{1}{2}}$

$$\frac{dy}{dx} = x \times \frac{1}{2}(\sin x)^{-\frac{1}{2}} \cos x + (\sin x)^{\frac{1}{2}} \times 1$$
$$= \frac{1}{2}(\sin x)^{-\frac{1}{2}}\left(x\cos x + 2\sin x\right)$$

At the maximum point $\frac{dy}{dx} = 0$

$\frac{1}{2}(\sin x)^{-\frac{1}{2}}\left(x\cos x + 2\sin x\right) = 0$

$\therefore\ x\cos x + 2\sin x = 0$

(as $(\sin x)^{-\frac{1}{2}} = \frac{1}{\sqrt{\sin x}} \neq 0$)

Dividing through by $\cos x$ gives

$x + 2\tan x = 0$

So the x-coordinate of the maximum point satisfies $2\tan x + x = 0$.

8 a $f(x) = e^{0.5x} - x^2$

$f'(x) = 0.5e^{0.5x} - 2x$

b $f'(6) = -1.957... < 0$

$f'(7) = 2.557... > 0$

As the sign changes between $x = 6$ and $x = 7$ and $f'(x)$ is continuous, $f'(x) = 0$ has a root p between 6 and 7.

Therefore $y = f(x)$ has a stationary point at $x = p$ where $6 < p < 7$.

9 a $f(x) = e^{2x}\sin 2x$

$$f'(x) = e^{2x}(2\cos 2x) + \sin 2x(2e^{2x})$$
$$= 2e^{2x}(\cos 2x + \sin 2x)$$

At turning points $f'(x) = 0$

$2e^{2x}(\cos 2x + \sin 2x) = 0$

$\cos 2x + \sin 2x = 0$

$\sin 2x = -\cos 2x$

Divide both sides by $\cos 2x$:

$\tan 2x = -1$

$2x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$

$\therefore x = \frac{3\pi}{8}$ or $\frac{7\pi}{8}$ (in the interval $0 < x < \pi$)

When $x = \frac{3\pi}{8}$, $y = \frac{1}{\sqrt{2}}e^{\frac{3\pi}{4}}$

When $x = \frac{7\pi}{8}$, $y = -\frac{1}{\sqrt{2}}e^{\frac{7\pi}{4}}$

So the coordinates of the turning points are $\left(\frac{3\pi}{8}, \frac{1}{\sqrt{2}}e^{\frac{3\pi}{4}}\right)$ and $\left(\frac{7\pi}{8}, -\frac{1}{\sqrt{2}}e^{\frac{7\pi}{4}}\right)$.

9 **b** $\mathrm{f}'(x) = 2\mathrm{e}^{2x}(\cos 2x + \sin 2x)$

$$\begin{aligned}\mathrm{f}''(x) &= 2\mathrm{e}^{2x}(-2\sin 2x + 2\cos 2x)\\ &\quad + 4\mathrm{e}^{2x}(\cos 2x + \sin 2x)\\ &= \mathrm{e}^{2x}(-4\sin 2x + 4\cos 2x\\ &\quad + 4\cos 2x + 4\sin 2x)\\ &= 8\mathrm{e}^{2x}\cos 2x\end{aligned}$$

c $\mathrm{f}''\left(\frac{3\pi}{8}\right) = 8\mathrm{e}^{\frac{3\pi}{4}}\cos\frac{3\pi}{4}$

$$= 8\mathrm{e}^{\frac{3\pi}{4}}\left(-\frac{\sqrt{2}}{2}\right) = -4\sqrt{2}\,\mathrm{e}^{\frac{3\pi}{4}} < 0$$

$\therefore \left(\frac{3\pi}{8}, \frac{1}{\sqrt{2}}\mathrm{e}^{\frac{3\pi}{4}}\right)$ is a maximum.

$\mathrm{f}''\left(\frac{7\pi}{8}\right) = 8\mathrm{e}^{\frac{7\pi}{4}}\cos\frac{7\pi}{4}$

$$= 8\mathrm{e}^{\frac{7\pi}{4}}\left(\frac{\sqrt{2}}{2}\right) = 4\sqrt{2}\,\mathrm{e}^{\frac{7\pi}{4}} > 0$$

$\therefore \left(\frac{7\pi}{8}, -\frac{1}{\sqrt{2}}\mathrm{e}^{\frac{7\pi}{4}}\right)$ is a minimum.

d At points of inflection $\mathrm{f}''(x) = 0$

$8\mathrm{e}^{2x}\cos 2x = 0$

$\cos 2x = 0$

$2x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

$\therefore x = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

When $x = \frac{\pi}{4}$, $y = \mathrm{e}^{\frac{\pi}{2}}\sin\frac{\pi}{2} = \mathrm{e}^{\frac{\pi}{2}}$

When $x = \frac{3\pi}{4}$, $y = \mathrm{e}^{\frac{3\pi}{2}}\sin\frac{3\pi}{2} = -\mathrm{e}^{\frac{3\pi}{2}}$

Points of inflection are

$\left(\frac{\pi}{4}, \mathrm{e}^{\frac{\pi}{2}}\right)$ and $\left(\frac{3\pi}{4}, -\mathrm{e}^{\frac{3\pi}{2}}\right)$.

10 $y = 2\mathrm{e}^x + 3x^2 + 2$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\mathrm{e}^x + 6x$

When $x = 0$, $y = 4$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$

Equation of normal at (0, 4) is

$y - 4 = -\frac{1}{2}(x - 0)$

$2y - 8 = -x$

or $x + 2y - 8 = 0$

11 **a** $\mathrm{f}(x) = 3\ln x + \frac{1}{x}$

$\mathrm{f}'(x) = \frac{3}{x} - \frac{1}{x^2}$

At a stationary point $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$\frac{3}{x} - \frac{1}{x^2} = 0$

$3x - 1 = 0$

$x = \frac{1}{3}$

So the x-coordinate of the stationary point P is $\frac{1}{3}$

b At the point Q, $x = 1$ so $y = \mathrm{f}(1) = 1$

The gradient of the curve at point Q is $\mathrm{f}'(1) = 3 - 1 = 2$

So the gradient of the normal to the curve at Q is $-\frac{1}{2}$

Equation of the normal at Q is

$y - 1 = -\frac{1}{2}(x - 1)$

i.e. $y = -\frac{1}{2}x + \frac{3}{2}$

12 a Let $\mathrm{f}(x)=\mathrm{e}^{2x}\cos x$

Then $\mathrm{f}'(x)=\mathrm{e}^{2x}(-\sin x)+\cos x(2\mathrm{e}^{2x})$

$$=\mathrm{e}^{2x}(2\cos x-\sin x)$$

Turning points occur when $\mathrm{f}'(x)=0$

$$\mathrm{e}^{2x}(2\cos x-\sin x)=0$$
$$\sin x=2\cos x$$

Dividing both sides by $\cos x$ gives

$$\tan x=2$$

b When $x=0, y=\mathrm{f}(0)=\mathrm{e}^0\cos 0=1$

The gradient of the curve at (0, 1) is $\mathrm{f}'(0)=\mathrm{e}^0(2-0)=2$

This is also the gradient of the tangent at (0, 1).

So the equation of the tangent at (0, 1) is

$$y-1=2(x-0)$$

$$y=2x+1$$

13 a $x=y^2\ln y$

Using the product rule:

$$\frac{\mathrm{d}x}{\mathrm{d}y}=y^2\left(\frac{1}{y}\right)+\ln y\times 2y=y+2y\ln y$$

b $$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}=\frac{1}{y+2y\ln y}$$

When $y=\mathrm{e}$,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\mathrm{e}+2\mathrm{e}\ln\mathrm{e}}=\frac{1}{3\mathrm{e}}$$

14 a $\mathrm{f}(x)=(x^3-2x)\mathrm{e}^{-x}$

$$\mathrm{f}'(x)=(x^3-2x)(-\mathrm{e}^{-x})+(3x^2-2)\mathrm{e}^{-x}$$
$$=\mathrm{e}^{-x}(-x^3+3x^2+2x-2)$$

b When $x=0,\ \mathrm{f}'(x)=-2$

Gradient of normal is $\frac{1}{2}$

$\therefore$ equation of normal to the curve at the origin is

$$y=\frac{1}{2}x$$

This line will intersect the curve again when

$$\frac{1}{2}x=(x^3-2x)\mathrm{e}^{-x}$$
$$1=2(x^2-2)\mathrm{e}^{-x}$$
$$\mathrm{e}^x=2x^2-4$$
$$2x^2=\mathrm{e}^x+4$$

15 a $\mathrm{f}(x)=x(1+x)\ln x=(x+x^2)\ln x$

$$\mathrm{f}'(x)=(x+x^2)\times\frac{1}{x}+\ln x\times(1+2x)$$
$$=1+x+(1+2x)\ln x$$

b At minimum point $A, \mathrm{f}'(x)=0$

$$1+x+(1+2x)\ln x=0$$
$$(1+2x)\ln x=-(1+x)$$
$$\ln x=-\frac{1+x}{1+2x}$$

So x-coordinate of A is the solution to the equation $x=\mathrm{e}^{-\frac{1+x}{1+2x}}$

16 a $x=4t-3,\ \ y=\frac{8}{t^2}=8t^{-2}$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=4,\ \frac{\mathrm{d}y}{\mathrm{d}t}=-16t^{-3}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\frac{-16t^{-3}}{4}=\frac{-4}{t^3}$$

16 b When $t = 2$, the curve has gradient

$$\frac{dy}{dx} = \frac{-4}{2^3} = -\frac{1}{2}$$

$\therefore$ the normal has gradient 2.

Also, when $t = 2$, $x = 5$ and $y = 2$,
so the point A has coordinates (5, 2).

$\therefore$ the equation of the normal at A is

$$y - 2 = 2(x - 5)$$

i.e. $y = 2x - 8$

17 $x = 2t,\ y = t^2$

$$\frac{dx}{dt} = 2,\ \frac{dy}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{2} = t$$

At the point P where $t = 3$, the gradient of the curve is $\frac{dy}{dx} = 3$

$\therefore$ gradient of the normal is $-\frac{1}{3}$

Also, when $t = 3$, the coordinates are (6, 9).

$\therefore$ the equation of the normal at P is

$$y - 9 = -\frac{1}{3}(x - 6)$$

i.e. $3y + x = 33$

18 $x = t^3,\ y = t^2$

$$\frac{dx}{dt} = 3t^2,\ \frac{dy}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{3t^2} = \frac{2}{3t}$$

At the point (1, 1) the value of t is 1.

$\therefore$ the gradient of the curve is $\frac{2}{3}$, which is also the gradient of the tangent.

$\therefore$ the equation of the tangent is

$$y - 1 = \frac{2}{3}(x - 1)$$

i.e. $y = \frac{2}{3}x + \frac{1}{3}$

19 a $x = 2\cos t + \sin 2t,\ y = \cos t - 2\sin 2t$

$$\frac{dx}{dt} = -2\sin t + 2\cos 2t$$

$$\frac{dy}{dt} = -\sin t - 4\cos 2t$$

b $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\sin t - 4\cos 2t}{-2\sin t + 2\cos 2t}$$

When $t = \frac{\pi}{4}$, $\frac{dy}{dx} = \frac{-\frac{1}{\sqrt{2}} - 0}{-\frac{2}{\sqrt{2}} + 0} = \frac{1}{2}$

19 c The gradient of the normal at the point P where $t = \frac{\pi}{4}$ is –2.

The coordinates of P are found by substituting $t = \frac{\pi}{4}$ into the parametric equations:

$$x = \frac{2}{\sqrt{2}} + 1,\ y = \frac{1}{\sqrt{2}} - 2$$

$\therefore$ the equation of the normal at P is

$$y - \left(\frac{1}{\sqrt{2}} - 2\right) = -2\left(x - \left(\frac{2}{\sqrt{2}} + 1\right)\right)$$

$$y - \frac{1}{\sqrt{2}} + 2 = -2x + 2\sqrt{2} + 2$$

i.e. $y + 2x = \frac{5\sqrt{2}}{2}$

20 a $x = 2t + 3,\ y = t^3 - 4t$

At point A, where $t = -1$, $x = 1$ and $y = 3$.

$\therefore$ the coordinates of A are (1, 3).

$$\frac{dx}{dt} = 2,\ \frac{dy}{dt} = 3t^2 - 4$$

$$\therefore \frac{dy}{dx} = \frac{3t^2 - 4}{2}$$

At the point A, $\frac{dy}{dx} = -\frac{1}{2}$

$\therefore$ gradient of the tangent at A is $-\frac{1}{2}$

Equation of the tangent at A is

$$y - 3 = -\frac{1}{2}(x - 1)$$

$$2y - 6 = -x + 1$$

i.e. $2y + x = 7$

b The tangent line l meets the curve C at points A and B.

Substitute $x = 2t + 3$ and $y = t^3 - 4t$ into the equation of l:

$$2(t^3 - 4t) + (2t + 3) = 7$$
$$2t^3 - 6t = 4$$
$$t^3 - 3t - 2 = 0$$

At point A, $t = -1$, so $t = -1$ is a root of this equation, and hence $(t + 1)$ is a factor of the left-hand side expression.

$$\begin{aligned} t^3 - 3t - 2 &= (t+1)(t^2 - t - 2) \\ &= (t+1)(t+1)(t-2) \\ &= 2(t+1)^2(t-2) \end{aligned}$$

So line l meets the curve C at $t = -1$ (repeated root because the line is tangent to the curve there) and at $t = 2$.

Therefore, at point B, $t = 2$.

21 The rate of change of V is $\frac{dV}{dt}$

$$\therefore \frac{dV}{dt} \propto V$$

i.e. $\frac{dV}{dt} = -kV$

where k is a positive constant.
(The negative sign is needed as the value of the car is *decreasing*.)

22 The rate of change of mass is $\frac{dM}{dt}$

$$\therefore \frac{dM}{dt} \propto M$$

i.e. $\frac{dM}{dt} = -kM$

where k is a positive constant.
(The negative sign represents *loss* of mass.)

23 The rate of change of pondweed is $\frac{dP}{dt}$

The growth rate is proportional to P:

growth rate $\propto P$

i.e. growth rate $= kP$
where k is a positive constant.

But pondweed is also being removed at a constant rate Q.

$$\therefore \frac{dP}{dt} = \text{growth rate} - \text{removal rate}$$
$$\frac{dP}{dt} = kP - Q$$

24 The rate of increase of the radius is $\frac{dr}{dt}$

$\therefore \frac{dr}{dt} \propto \frac{1}{r}$, as the rate is *inversely* proportional to the radius.

Hence $\frac{dr}{dt} = \frac{k}{r}$
where k is the constant of proportion.

25 The rate of change of temperature is $\frac{d\theta}{dt}$

$$\therefore \frac{d\theta}{dt} \propto (\theta - \theta_0)$$

i.e. $\frac{d\theta}{dt} = -k(\theta - \theta_0)$,
where k is a positive constant.
(The negative sign indicates that the temperature is decreasing, i.e. *loss* of temperature.)

26 a $x = 4\cos 2t,\ y = 3\sin t$

The point $A\left(2, \frac{3}{2}\right)$ is on the curve, so

$$4\cos 2t = 2 \text{ and } 3\sin t = \frac{3}{2}$$

$$\cos 2t = \frac{1}{2} \text{ and } \sin t = \frac{1}{2}$$

The only value of t in the interval $-\frac{\pi}{2} < t < \frac{\pi}{2}$ that satisfies both equations is $\frac{\pi}{6}$. Therefore $t = \frac{\pi}{6}$ at the point A.

b $\frac{dx}{dt} = -8\sin 2t,\ \frac{dy}{dt} = 3\cos t$

$$\therefore \frac{dy}{dx} = \frac{3\cos t}{-8\sin 2t}$$
$$= -\frac{3\cos t}{16\sin t\cos t}$$
(using a double angle formula)
$$= -\frac{3}{16\sin t}$$
$$= -\frac{3}{16}\operatorname{cosec} t$$

c At point A, where $t = \frac{\pi}{6}$, $\frac{dy}{dx} = -\frac{3}{8}$

$\therefore$ gradient of the normal at A is $\frac{8}{3}$

Equation of the normal is

$$y - \frac{3}{2} = \frac{8}{3}(x - 2)$$

Multiply through by 6 and rearrange to give:

$$6y - 9 = 16x - 32$$
$$6y - 16x + 23 = 0$$

26 d To find where the normal cuts the curve, substitute $x = 4\cos 2t$ and $y = 3\sin t$ into the equation of the normal:

$6(3\sin t) - 16(4\cos 2t) + 23 = 0$

$18\sin t - 64\cos 2t + 23 = 0$

$18\sin t - 64(1 - 2\sin^2 t) + 23 = 0$

(using a double angle formula)

$128\sin^2 t + 18\sin t - 41 = 0$

But $\sin t = \frac{1}{2}$ is one solution of this equation, as point A lies on the line and on the curve. So

$128\sin^2 t + 18\sin t - 41$

$= (2\sin t - 1)(64\sin t + 41)$

$\therefore (2\sin t - 1)(64\sin t + 41) = 0$

Therefore, at point B, $\sin t = -\frac{41}{64}$

$\therefore$ the y-coordinate of point B is $3 \times \left(-\frac{41}{64}\right) = -\frac{123}{64}$

27 a $x = a\sin^2 t$, $y = a\cos t$

$\frac{\mathrm{d}x}{\mathrm{d}t} = 2a\sin t\cos t$, $\frac{\mathrm{d}y}{\mathrm{d}t} = -a\sin t$

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-a\sin t}{2a\sin t\cos t} = \frac{-1}{2\cos t} = -\frac{1}{2}\sec t$

b As $P\left(\frac{3}{4}a, \frac{1}{2}a\right)$ lies on the curve,

$a\sin^2 t = \frac{3}{4}a$ and $a\cos t = \frac{1}{2}a$

$\sin t = \pm\frac{\sqrt{3}}{2}$ and $\cos t = \frac{1}{2}$

The only value of t in the interval $0 \leqslant t \leqslant \frac{\pi}{2}$ that satisfies both equations is $\frac{\pi}{3}$. Therefore $t = \frac{\pi}{3}$ at the point P.

Gradient of the curve at point P is $-\frac{1}{2}\sec\frac{\pi}{3} = -1$.

$\therefore$ equation of the tangent at P is

$y - \frac{1}{2}a = -1\left(x - \frac{3}{4}a\right)$

$y - \frac{1}{2}a = -x + \frac{3}{4}a$

Multiply equation by 4 and rearrange to give

$4y + 4x = 5a$

c Equation of the tangent at C is $4y + 4x = 5a$

At A, $x = 0 \Rightarrow y = \frac{5a}{4}$

At B, $y = 0 \Rightarrow x = \frac{5a}{4}$

Area of $AOB = \frac{1}{2}\left(\frac{5a}{4}\right)^2 = \frac{25}{32}a^2$,

which is of the form ka^2 with $k = \frac{25}{32}$

28 $x = (t+1)^2,\ y = \frac{1}{2}t^3 + 3$

$$\frac{dx}{dt} = 2(t+1),\ \frac{dy}{dt} = \frac{3}{2}t^2$$

$$\therefore \frac{dy}{dx} = \frac{\frac{3}{2}t^2}{2(t+1)} = \frac{3t^2}{4(t+1)}$$

When $t = 2,\ \frac{dy}{dx} = \frac{3\times4}{4\times3} = 1$

$\therefore$ gradient of the normal at the point P, where $t = 2$, is -1.

The coordinates of P are (9, 7).

$\therefore$ equation of the normal is

$$y - 7 = -1(x-9)$$
$$y - 7 = -x + 9$$
i.e. $y + x = 16$

29 $5x^2 + 5y^2 - 6xy = 13$

Differentiate with respect to x:

$$10x + 10y\frac{dy}{dx} - 6\left(x\frac{dy}{dx} + y\right) = 0$$

$$(10y - 6x)\frac{dy}{dx} = 6y - 10x$$
$$\therefore \frac{dy}{dx} = \frac{6y - 10x}{10y - 6x}$$

At the point (1, 2)

$$\frac{dy}{dx} = \frac{12-10}{20-6} = \frac{2}{14} = \frac{1}{7}$$

So the gradient of the curve at (1, 2) is $\frac{1}{7}$

30 $e^{2x} + e^{2y} = xy$

Differentiate with respect to x:

$$2e^{2x} + 2e^{2y}\frac{dy}{dx} = x\frac{dy}{dx} + y\times1$$

$$2e^{2y}\frac{dy}{dx} - x\frac{dy}{dx} = y - 2e^{2x}$$

$$(2e^{2y} - x)\frac{dy}{dx} = y - 2e^{2x}$$

$$\therefore \frac{dy}{dx} = \frac{y - 2e^{2x}}{2e^{2y} - x}$$

31 $y^3 + 3xy^2 - x^3 = 3$

Differentiate with respect to x:

$$3y^2\frac{dy}{dx} + \left(3x\times2y\frac{dy}{dx} + y^2\times3\right) - 3x^2 = 0$$

$$(3y^2 + 6xy)\frac{dy}{dx} = 3x^2 - 3y^2$$

$$\therefore \frac{dy}{dx} = \frac{3(x^2-y^2)}{3y(y+2x)} = \frac{x^2-y^2}{y(y+2x)}$$

Turning points occur when $\frac{dy}{dx} = 0$

$$\frac{x^2-y^2}{y(y+2x)} = 0$$
$$x^2 = y^2$$
$$x = \pm y$$

When $x = y,\ y^3 + 3y^3 - y^3 = 3$
so $3y^3 = 3$
$y = 1$ and hence $x = 1$

When $x = -y,\ y^3 - 3y^3 + y^3 = 3$
so $-y^3 = 3$
$y = \sqrt[3]{-3}$ and hence $x = -\sqrt[3]{-3}$

$\therefore$ the coordinates of the turning points are (1, 1) and $(-\sqrt[3]{-3}, \sqrt[3]{-3})$.

32 a $(1+x)(2+y)=x^2+y^2$

Differentiate with respect to x:

$$(1+x)\left(\frac{dy}{dx}\right)+(2+y)(1)=2x+2y\frac{dy}{dx}$$

$$(1+x-2y)\frac{dy}{dx}=2x-y-2$$

$$\therefore \frac{dy}{dx}=\frac{2x-y-2}{1+x-2y}$$

b When the curve meets the y-axis, $x = 0$.

Substitute $x = 0$ into the equation of the curve:

$$2+y=y^2$$

i.e. $y^2-y-2=0$

$(y-2)(y+1)=0$

$y=2$ or $y=-1$

At $(0, 2)$, $\frac{dy}{dx}=\frac{0-2-2}{1+0-4}=\frac{4}{3}$

At $(0, -1)$, $\frac{dy}{dx}=\frac{0+1-2}{1+0+2}=-\frac{1}{3}$

c A tangent that is parallel to the y-axis has infinite gradient.

For $\frac{dy}{dx}=\frac{2x-y-2}{1+x-2y}$ to be infinite, the denominator $1+x-2y=0$,

i.e. $x=2y-1$

Substitute $x=2y-1$ into the equation of the curve:

$$(1+2y-1)(2+y)=(2y-1)^2+y^2$$

$$2y^2+4y=4y^2-4y+1+y^2$$

$$3y^2-8y+1=0$$

$$y=\frac{8\pm\sqrt{64-12}}{6}=\frac{4\pm\sqrt{13}}{3}$$

When $y-\frac{4+\sqrt{13}}{3}$, $x=\frac{5+2\sqrt{13}}{3}$

When $y=\frac{4-\sqrt{13}}{3}$, $x=\frac{5-2\sqrt{13}}{3}$

$\therefore$ there are two points at which the tangents are parallel to the y-axis.

They are $\left(\frac{5+2\sqrt{13}}{3},\frac{4+\sqrt{13}}{3}\right)$ and $\left(\frac{5-2\sqrt{13}}{3},\frac{4-\sqrt{13}}{3}\right)$.

33 $7x^2+48xy-7y^2+75=0$

Implicit differentiation with respect to x gives

$$14x+48\left(x\frac{dy}{dx}+y\right)-14y\frac{dy}{dx}=0$$
$$(48x-14y)\frac{dy}{dx}=-14x-48y$$
$$\therefore\ \frac{dy}{dx}=\frac{-14x-48y}{48x-14y}=\frac{7x+24y}{7y-24x}$$

When $\frac{dy}{dx}=\frac{2}{11}$,

$$\frac{7x+24y}{7y-24x}=\frac{2}{11}$$
$$14y-48x=77x+264y$$
$$125x+250y=0$$
$$\therefore\ x+2y=0$$

So the coordinates of the points at which the gradient is $\frac{2}{11}$ satisfy $x+2y=0$, which means that the points lie on the line $x+2y=0$.

34 $y=x^x$

Take natural logs of both sides:

$$\ln y=\ln x^x$$

$$\ln y=x\ln x\ \text{(using properties of logarithms)}$$

Differentiate with respect to x:

$$\frac{1}{y}\frac{dy}{dx}=x\times\frac{1}{x}+\ln x\times1$$
$$=1+\ln x$$

$$\therefore\ \frac{dy}{dx}=y(1+\ln x)$$

But $y=x^x$

$$\therefore\ \frac{dy}{dx}=x^x(1+\ln x)$$

35 a $a^x=e^{kx}$

Take natural 1ogs of both sides:

$$\ln a^x=\ln e^{kx}$$
$$x\ln a=kx$$

As this is true for all values of x, $k=\ln a$.

b Taking $a=2$,

$$y=2^x=e^{kx}\ \text{where}\ k=\ln 2$$

$$\frac{dy}{dx}=ke^{kx}=(\ln 2)e^{(\ln 2)x}=2^x\ln 2$$

c At the point (2, 4), $x=2$.

$\therefore$ gradient of the curve at (2, 4) is

$$\frac{dy}{dx}=2^2\ln 2=4\ln 2=\ln 2^4=\ln 16$$

36 a $P=P_0(1.09)^t$

Take natural logs of both sides:

$$\ln P=\ln\left(P_0(1.09)^t\right)$$
$$=\ln P_0+\ln(1.09)^t$$
$$=\ln P_0+t\ln 1.09$$

$$\therefore\ t\ln 1.09=\ln P-\ln P_0$$

$$t=\frac{\ln P-\ln P_0}{\ln 1.09}\ \text{or}\ \frac{\ln\left(\frac{P}{P_0}\right)}{\ln 1.09}$$

b When $t=T$, $P=2P_0$.

Substituting these into the expression in part **a** gives

$$T=\frac{\ln 2}{\ln 1.09}=8.04\ (3\text{ s.f.})$$

36 c $\dfrac{dP}{dt} = P_0(1.09)^t(\ln 1.09)$

When $t = T$, $P = 2P_0$ so $(1.09)^T = 2$

Hence $\dfrac{dP}{dt} = P_0(1.09)^T(\ln 1.09)$

$= P_0 \times 2 \times \ln 1.09$

$= 0.172P_0$ (3 s.f.)

37 a $y = \ln(\sin x)$

$\dfrac{dy}{dx} = \cos x \times \dfrac{1}{\sin x} = \cot x$

At a stationary point $\dfrac{dy}{dx} = 0$

$\cot x = 0 \Rightarrow x = \dfrac{\pi}{2}$

(in the interval $0 < x < \pi$)

When $x = \dfrac{\pi}{2}$, $y = \ln\left(\sin\dfrac{\pi}{2}\right) = \ln 1 = 0$

$\therefore$ stationary point is at $\left(\dfrac{\pi}{2}, 0\right)$.

b $\dfrac{d^2y}{dx^2} = -\text{cosec}^2 x$

$\text{cosec}^2 x = \dfrac{1}{\sin^2 x} > 0$ for all $0 < x < \pi$

$\therefore \dfrac{d^2y}{dx^2} < 0$ for all $0 < x < \pi$

Hence the curve C is concave for all values of x in its domain.

38 a $m = 40\,e^{-0.244t}$

After 9 months, $t = 0.75$, so

$m = 40\,e^{-0.244\times0.75} = 40\,e^{-0.183} = 33.31...$

b $\dfrac{dm}{dt} = -0.244 \times 40\,e^{-0.244t} = -9.76\,e^{-0.244t}$

c The negative sign indicates that the mass is decreasing.

39 a $f(x) = \dfrac{\cos 2x}{e^x}$

$f'(x) = \dfrac{-2e^x \sin 2x - e^x \cos 2x}{e^{2x}}$

$= -\dfrac{2\sin 2x + \cos 2x}{e^x}$

At A and B, $f'(x) = 0$

$2\sin 2x + \cos 2x = 0$

$2\tan 2x + 1 = 0$

$\tan 2x = -0.5$

$2x = 2.678$ or 5.820

$x = 1.339$ or 2.910

(in the interval $0 \leqslant x \leqslant \pi$)

$x = 1.339 \Rightarrow y = f(x) = -0.2344$

$x = 2.910 \Rightarrow y = f(x) = 0.04874$

Therefore, to 3 significant figures:
coordinates of A are (1.34, −0.234);
coordinates of B are (2.91, 0.0487).

b The curve of $y = 2 + 4f(x-4)$ is a transformation of $f(x)$, obtained via a translation of 4 units to the right, a stretch by a factor of 4 in the y-direction, and then a translation of 2 units upwards.

Turning points are:
minimum $(1.34 + 4, -0.234 \times 4 + 2)$ and
maximum $(2.91 + 4, 0.0487 \times 4 + 2)$,
i.e. minimum $(5.34, 1.06)$
and maximum $(6.91, 2.19)$.

c $f''(x)$

$= -\dfrac{e^x(4\cos 2x - 2\sin 2x) - e^x(2\sin 2x + \cos 2x)}{e^{2x}}$

$= \dfrac{4\sin 2x - 3\cos 2x}{e^x}$

f(x) is concave when $f''(x) \leqslant 0$

$f''(x) = 0$ when

$4\sin 2x - 3\cos 2x = 0$

$\tan 2x = \dfrac{3}{4}$

$2x = 0.644$ or 3.785

$x = 0.322$ or 1.893

The curve has a minimum point and hence is convex between these values, so it is concave for
$0 \leqslant x \leqslant 0.322$ and $1.892 \leqslant x \leqslant \pi$.

Challenge

a $y = 2\sin 2t,\ x = 5\cos\left(t + \frac{\pi}{12}\right)$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 4\cos 2t,\ \frac{\mathrm{d}x}{\mathrm{d}t} = -5\sin\left(t + \frac{\pi}{12}\right)$$

$$\therefore\ \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4\cos 2t}{5\sin\left(t + \frac{\pi}{12}\right)}$$

b $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $4\cos 2t = 0$

$$2t = \frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{5\pi}{2} \text{ or } \frac{7\pi}{2}$$

$$t = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4} \text{ or } \frac{7\pi}{4}$$

(in the interval $0 \leqslant x \leqslant 2\pi$)

$$t = \frac{\pi}{4} \Rightarrow x = 5\cos\left(\frac{\pi}{3}\right) = \frac{5}{2}$$

and $y = 2\sin\frac{\pi}{2} = 2$, i.e. point $\left(\frac{5}{2}, 2\right)$

$$t = \frac{3\pi}{4} \Rightarrow x = 5\cos\left(\frac{5\pi}{6}\right) = -\frac{5\sqrt{3}}{2}$$

and $y = 2\sin\frac{3\pi}{2} = -2$, i.e. point $\left(-\frac{5\sqrt{3}}{2}, -2\right)$

$$t = \frac{5\pi}{4} \Rightarrow x = 5\cos\left(\frac{4\pi}{3}\right) = -\frac{5}{2}$$

and $y = 2\sin\frac{5\pi}{2} = 2$, i.e. point $\left(-\frac{5}{2}, 2\right)$

$$t = \frac{7\pi}{4} \Rightarrow x = 5\cos\left(\frac{11\pi}{6}\right) = \frac{5\sqrt{3}}{2}$$

and $y = 2\sin\frac{7\pi}{2} = -2$, i.e. point $\left(\frac{5\sqrt{3}}{2}, -2\right)$

c The curve cuts the x-axis when $y = 0$,
i.e. when $2\sin 2t = 0$

$$2t = 0,\ \pi,\ 2\pi,\ 3\pi,\ 4\pi$$

$$t = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi$$

$t = 0 \Rightarrow x = 5\cos\frac{\pi}{12} = 4.83$, i.e. $(4.83, 0)$

with gradient $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-4}{5\sin\frac{\pi}{12}} = -3.09$

$t = \frac{\pi}{2} \Rightarrow x = 5\cos\frac{7\pi}{12} = -1.29$, i.e. $(-1.29, 0)$

with gradient $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4}{5\sin\frac{7\pi}{12}} = 0.828$

$t = \pi \Rightarrow x = 5\cos\frac{13\pi}{12} = -4.83$, i.e. $(-4.83, 0)$

with gradient $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-4}{5\sin\frac{13\pi}{12}} = 3.09$

$t = \frac{3\pi}{2} \Rightarrow x = 5\cos\frac{19\pi}{12} = 1.29$, i.e. $(1.29, 0)$

with gradient $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4}{5\sin\frac{19\pi}{12}} = -0.828$

The curve cuts the y-axis when $x = 0$.

i.e. when $5\cos\left(t + \frac{\pi}{12}\right) = 0$

$$t + \frac{\pi}{12} = \frac{\pi}{2},\ \frac{3\pi}{2}$$

$$t = \frac{5\pi}{12},\ \frac{17\pi}{12}$$

$t = \frac{5\pi}{12} \Rightarrow y = 2\sin\frac{5\pi}{6} = 1$, i.e. $(0, 1)$

with gradient $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-4\cos\frac{5\pi}{6}}{5\sin\frac{\pi}{2}} = 0.693$

$t = \frac{17\pi}{12} \Rightarrow y = 2\sin\frac{17\pi}{6} = 1$, i.e. $(0, 1)$

with gradient $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-4\cos\frac{17\pi}{6}}{5\sin\frac{3\pi}{2}} = -0.693$

So the curve cuts the y-axis twice at (0, 1) with gradients 0.693 and −0.693.

d $\frac{\mathrm{d}x}{\mathrm{d}y} = -\frac{5\sin\left(t+\frac{\pi}{12}\right)}{4\cos 2t}$

$\frac{\mathrm{d}x}{\mathrm{d}y} = 0$ when $\sin\left(t+\frac{\pi}{12}\right) = 0$

$t+\frac{\pi}{12} = \pi, 2\pi$

$t = \frac{11\pi}{12}, \frac{23\pi}{12}$

$t = \frac{11\pi}{12} \Rightarrow y = 2\sin\frac{11\pi}{6} = -1$

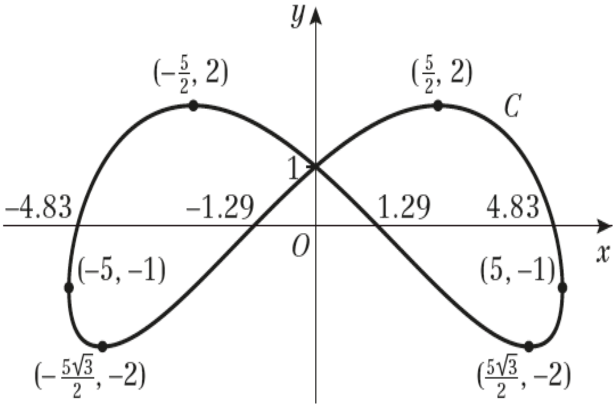
and $x = 5\cos\left(\frac{11\pi}{12}+\frac{\pi}{12}\right) = -5$

$t = \frac{23\pi}{12} \Rightarrow y = 2\sin\frac{23\pi}{6} = -1$

and $x = 5\cos\left(\frac{23\pi}{12}+\frac{\pi}{12}\right) = 5$

So points where curve is vertical are
$(-5, -1)$ and $(5, -1)$.

e



Numerical methods 10A

1 a $f(x) = x^3 - x + 5$

$f(-2) = -8 + 2 + 5 = -1 < 0$

$f(-1) = -1 + 1 + 5 = 5 > 0$

There is a change of sign between −2 and −1 so there is at least one root in the interval $-2 < x < -1$.

b $f(x) = x^2 - \sqrt{x} - 10$

$f(3) = 9 - \sqrt{3} - 10 = -2.732... < 0$

$f(4) = 16 - \sqrt{4} - 10 = 4 > 0$

There is a change of sign between 3 and 4 so there is at least one root in the interval $3 < x < 4$.

c $f(x) = x^3 - \frac{1}{x} - 2$

$f(-0.5) = (-0.5)^3 + 2 - 2 = -0.125 < 0$

$f(-0.2) = (-0.2)^3 + 5 - 2 = 2.992 > 0$

There is a change of sign between −0.5 and −0.2 so there is at least one root in the interval $-0.5 < x < -0.2$.

d $f(x) = e^x - \ln x - 5$

$f(1.65) = e^{1.65} - \ln 1.65 - 5 = -0.293... < 0$

$f(1.75) = e^{1.75} - \ln 1.75 - 5 = 0.194... > 0$

There is a change of sign between 1.65 and 1.75 so there is at least one root in the interval $1.65 < x < 1.75$.

2 a $f(x) = 3 + x^2 - x^3$

$f(1.8) = 3 + 1.8^2 - 1.8^3 = 0.408 > 0$

$f(1.9) = 3 + 1.9^2 - 1.9^3 = -0.249 < 0$

There is a change of sign so there is a root, α, in the interval [1.8, 1.9].

b Choose interval [1.8635, 1.8645] to test for root.

$$f(1.8635) = 3 + 1.8635^2 - 1.8635^3$$
$$= 0.00138... > 0$$
$$f(1.8645) = 3 + 1.8645^2 - 1.8645^3$$
$$= -0.00531... < 0$$

There is a change of sign between 1.8635 and 1.8645, so $1.8635 < \alpha < 1.8645$, which gives $\alpha = 1.864$ correct to 3 d.p.

3 a $h(x) = \sqrt[3]{x} - \cos x - 1$

$h(1.4) = \sqrt[3]{1.4} - \cos 1.4 - 1 = -0.0512... < 0$

$h(1.5) = \sqrt[3]{1.5} - \cos 1.5 - 1 = 0.0739... > 0$

There is a change of sign so there is a root, α, in the interval [1.4, 1.5].

b Choose interval [1.4405, 1.4415] to test for root.

$$h(1.4405) = \sqrt[3]{1.4405} - \cos 1.4405 - 1$$
$$= -0.00055... < 0$$
$$h(1.4415) = \sqrt[3]{1.4415} - \cos 1.4415 - 1$$
$$= 0.00069... > 0$$

There is a change of sign between 1.4405 and 1.4415, so $1.4405 < \alpha < 1.4415$, which gives $\alpha = 1.441$ correct to 3 d.p.

4 a $f(x) = \sin x - \ln x$

$f(2.2) = \sin 2.2 - \ln 2.2 = 0.020... > 0$

$f(2.3) = \sin 2.3 - \ln 2.3 = -0.087... < 0$

There is a change of sign so there is a root, α, in the interval [2.2, 2.3].

b Choose interval [2.2185, 2.2195] to test for root.

$$f(2.2185) = \sin 2.2185 - \ln 2.2185$$
$$= 0.00064... > 0$$
$$f(2.2195) = \sin 2.2195 - \ln 2.2195$$
$$= -0.00041... < 0$$

There is a change of sign between 2.2185 and 2.2195, so $2.2185 < \alpha < 2.2195$, which gives $\alpha = 2.219$ correct to 3 d.p.

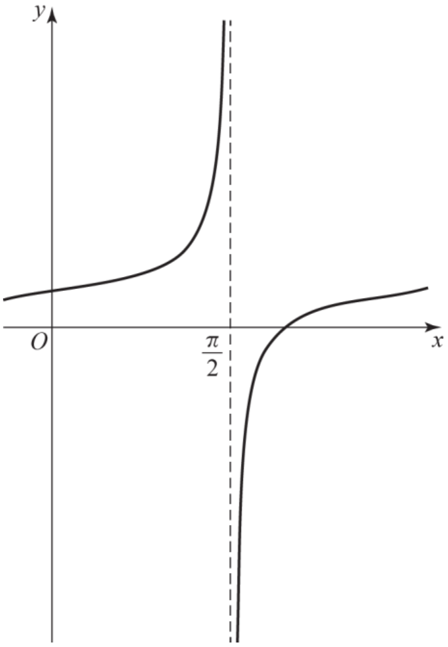
5 a $f(x) = 2 + \tan x$

$f(1.5) = 2 + \tan 1.5 = 16.1... > 0$

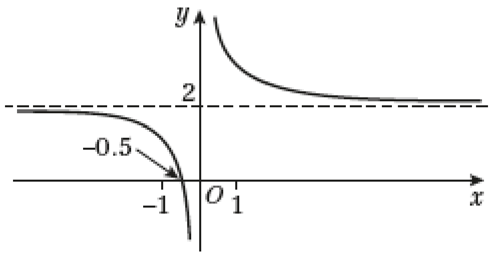
$f(1.6) = 2 + \tan 1.6 = -32.2... < 0$

So there is a change of sign in the interval [1.5, 1.6].

5 **b** A sketch shows there is a vertical asymptote in the graph of $y = \text{f}(x)$ at $x = \frac{\pi}{2} = 1.57...$ So there is no root in the interval [1.5, 1.6].



6 A sketch shows a root at −0.5.



Or $\text{f}(x) = 0$ when $\frac{1}{x} + 2 = 0 \Rightarrow x = -\frac{1}{2}$

which is in the interval [−1, 1].

7 **a** $\text{f}(x) = (105x^3 - 128x^2 + 49x - 6)\cos 2x$

$\text{f}(0.2) = (0.84 - 5.12 + 9.8 - 6)\cos 0.4$

$= -0.442... < 0$

$\text{f}(0.8) = (53.76 - 81.92 + 39.2 - 6)\cos 1.6$

$= -0.147... < 0$

b There is no sign change, so there are either no roots or an even number of roots in the interval [0.2, 0.8].

c $\text{f}(0.3) = (2.835 - 11.52 + 14.7 - 6)\cos 0.6$

$= 0.0123.. > 0$

and…

$\text{f}(0.4) = -0.111... < 0$

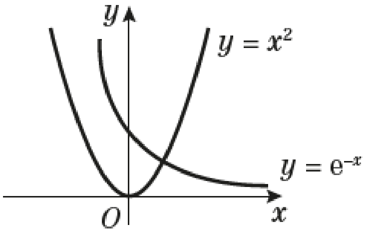
$\text{f}(0.5) = -0.202... < 0$

$\text{f}(0.6) = 0$

$\text{f}(0.7) = 0.271... > 0$

d From the changes in sign, there exists at least one root in each of the intervals $0.2 < x < 0.3$, $0.3 < x < 0.4$ and $0.7 < x < 0.8$. There is also a root at 0.6. Therefore there are at least four roots in the interval [0.2, 0.8].

8 **a**



b The curves meet where $e^{-x} = x^2$.
The curves meet at one point, so there is one value of x that satisfies the equation $e^{-x} = x^2$.
So $e^{-x} = x^2$ has one root.

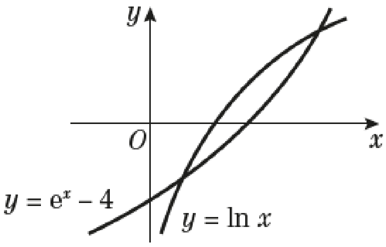
c $\text{f}(x) = e^{-x} - x^2$

$\text{f}(0.70) = e^{-0.70} - 0.70^2 = 0.0065...$

$\text{f}(0.71) = e^{-0.71} - 0.71^2 = 0.0124...$

There is a change of sign between 0.70 and 0.71 so there is at least one root in the interval $0.70 < x < 0.71$.

9 **a**



b The curves meet at two points, so there are two values of x that satisfy the equation $\ln x = e^x - 4$. So $\ln x = e^x - 4$ has two roots.

9 c $\mathrm{f}(x)=\ln x-\mathrm{e}^{x}+4$

$\mathrm{f}(1.4)=\ln 1.4-\mathrm{e}^{1.4}+4=0.281...$

$\mathrm{f}(1.5)=\ln 1.5-\mathrm{e}^{1.5}+4=-0.0762...$

There is a change of sign between 1.4 and 1.5 so there is at least one root in the interval $1.4 < x < 1.5$.

10 a $\mathrm{h}(x)=\sin 2x+\mathrm{e}^{4x}$

$\mathrm{h}'(x)=2\cos 2x+4\mathrm{e}^{4x}$

$$\begin{aligned}\mathrm{h}'(-0.9)&=2\cos(-1.8)+4\mathrm{e}^{-3.6}\\&=-0.345...<0\end{aligned}$$

$$\begin{aligned}\mathrm{h}'(-0.8)&=2\cos(-1.6)+4\mathrm{e}^{-3.2}\\&=0.104...>0\end{aligned}$$

The change in sign of $\mathrm{h}'(x)$ implies that the gradient changes from decreasing to increasing, so there is a turning point in the interval $-0.9 < x < -0.8$.

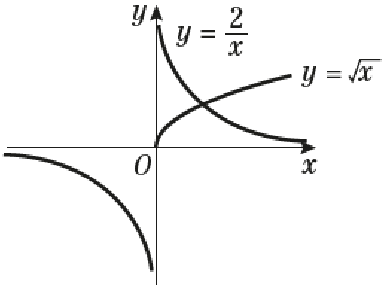
b Choose interval [−0.8225, −0.8235] to test for root.

$$\begin{aligned}\mathrm{h}'(-0.8235)&=2\cos(-1.647)+4\mathrm{e}^{-3.294}\\&=-0.00383...<0\end{aligned}$$

$$\begin{aligned}\mathrm{h}'(-0.8225)&=2\cos(-1.645)+4\mathrm{e}^{-3.29}\\&=0.000744...>0\end{aligned}$$

There is a change of sign between −0.8225 and −0.8235, so $-0.8225 < \alpha < -0.8235$, which gives $\alpha = -0.823$ correct to 3 d.p.

11 a



b The curves meet at one point, so there is one value of x that satisfies the equation $\sqrt{x}=\frac{2}{x}$. So $\sqrt{x}=\frac{2}{x}$ has **one** root.

c $\mathrm{f}(x)=\sqrt{x}-\frac{2}{x}$

$\mathrm{f}(1)=\sqrt{1}-\frac{2}{1}=-1$

$\mathrm{f}(2)=\sqrt{2}-\frac{2}{2}=0.414...$

There is a change of sign, so there is a root, r, between $x = 1$ and $x = 2$.

d $\sqrt{x}=\frac{2}{x}$

$x^{\frac{1}{2}}=\frac{2}{x}$

$x^{\frac{1}{2}}\times x=2$

$x^{\frac{1}{2}+1}=2$

$x^{\frac{3}{2}}=2$

$\left(x^{\frac{3}{2}}\right)^{2}=2^{2}$

$x^{3}=4$

So $p = 3$ and $q = 4$.

e $x^{\frac{3}{2}}=2$

$\Rightarrow x=2^{\frac{2}{3}}\left[=(2^{2})^{\frac{1}{3}}=4^{\frac{1}{3}}\right]$

12 a $\mathrm{f}(x)=x^{4}-21x-18$

$\mathrm{f}(-0.9)=0.6561+18.9-18=1.5561>0$

$\mathrm{f}(-0.8)=0.4096+16.8-18=-0.7904<0$

The change of sign between −0.9 and −0.8 implies there is at least one root in the interval [−0.9, −0.8].

12 b $\mathrm{f}'(x) = 4x^3 - 21$

$\mathrm{f}'(x) = 0 \Rightarrow 4x^3 = 21$

$x = \sqrt[3]{\dfrac{21}{4}} = 1.738...$

$\mathrm{f}(1.738) = 1.738^4 - 21 \times 1.738 - 18$

$= -45.373...$

Stationary point is (1.74, −45.37) to 2 d.p.

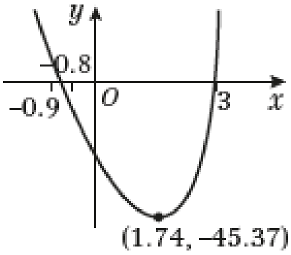
c $\mathrm{f}(x) = (x-3)(x^3 + ax^2 + bx + c)$

$\mathrm{f}(x) = x^4 + (a-3)x^3 + (b-3a)x^2 + (c-3b)x - 3c$

Comparing coefficients…

$a = 3, b = 9, c = 6$

d



Numerical methods 10B

1 a i $x^2-6x+2=0$

$6x=x^2+2$ Add 6*x* to each side.

$x=\frac{x^2+2}{6}$ Divide each side by 6.

ii $x^2-6x+2=0$

$x^2+2=6x$ Add 6*x* to each side.

$x^2=6x-2$ Subtract 2 from each side.

$x=\sqrt{6x-2}$ Take the square root of each side.

iii $x^2-6x+2=0$

$x^2+2=6x$ Add 6*x* to each side.

$x^2=6x-2$ Subtract 2 from each side.

$\frac{x^2}{x}=\frac{6x}{x}-\frac{2}{x}$ Divide each term by *x*.

$x=6-\frac{2}{x}$ Simplify.

b i $x_0=4\Rightarrow x_1=\frac{4^2+2}{6}=3$

$x_2=\frac{3^2+2}{6}=1.83333$

$x_3=\frac{1.83333^2+2}{6}=0.89352$

$x_4=\frac{0.89352^2+2}{6}=0.46640$

$x_5=\frac{0.46640^2+2}{6}=0.36959$

$x_6=\frac{0.36959^2+2}{6}=0.35610$

$x_7=\frac{0.35610^2+2}{6}=0.35447$

$x_8=\frac{0.35447^2+2}{6}=0.35428$

$x=0.354$ to 3 d.p.

ii $x_0=4\Rightarrow x_1=\sqrt{6x-2}=4.69042$

$x_2=\sqrt{6\times4.69042-2}=5.11297$

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$x_{15}=\sqrt{6\times5.64547-2}=5.6456$

x = 5.646 to 3 d.p.

iii $x_0=4\Rightarrow x_1=6-\frac{2}{x}=5.5$

$x_2=5.5-\frac{2}{x}=5.63636$

$x_3=5.63636-\frac{2}{x}=5.64516$

$x_4=5.64516-\frac{2}{x}=5.64571$

x = 5.646 to 3 d.p.

c $x^2-6x+2=0$

$x=\frac{6\pm\sqrt{36-8}}{2}=3\pm\sqrt{7}$

a = 3, b = 7

2 a i $\text{f}(x)=0$

$x^2-5x-3=0$

$x^2=5x+3$ Add 5*x* + 3 to each side.

$x=\sqrt{5x+3}$ Take the square root of each side.

ii $\text{f}(x)=0$

$x^2-5x-3=0$

$5x=x^2-3$ Add 5*x* to each side.

$x=\frac{x^2-3}{5}$ Divide each side by 5.

b i $x_0=5\Rightarrow x_1=\sqrt{5\times5+3}=\sqrt{28}=5.2915$

$x_2=\sqrt{5\times5.2915+3}=5.4275$

$x_3=\sqrt{5\times5.4275+3}=5.4900$

$x_4=\sqrt{5\times5.4900+3}=5.5180$

$x=5.5$ to 1 d.p.

2 b ii $x_0 = 5 \Rightarrow x_1 = \dfrac{5^2 - 3}{5} = 4.4$

$$x_2 = \frac{4.4^2 - 3}{5} = 3.272$$

$$x_3 = \frac{3.272^2 - 3}{5} = 1.5412$$

$$x_4 = \frac{1.5412^2 - 3}{5} = -0.1249$$

$$x_5 = \frac{(-0.1249)^2 - 3}{5} = -0.5969$$

$$x_6 = \frac{(-0.5969)^2 - 3}{5} = -0.5287$$

$x = -0.5$ to 1 d.p.

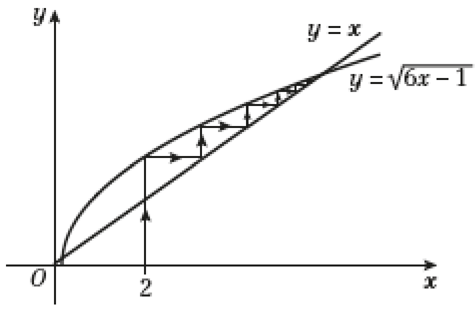
3 a $f(x) = 0$

$x^2 - 6x + 1 = 0$

$x^2 = 6x - 1$ Add 6*x* – 1 to each side.

$x = \sqrt{6x - 1}$ Take the square root of each side.

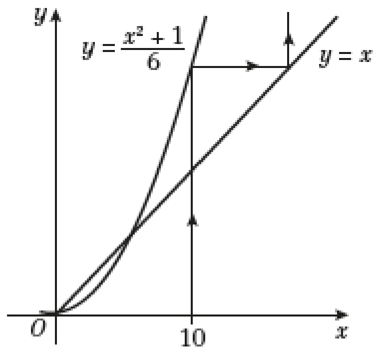
b



c The curve crosses the line twice so there are two roots.

d See diagram above.

e



4 a $f(x) = xe^{-x} - x + 2$

$$f(x) = 0 \Rightarrow e^{-x} = \frac{x-2}{x}$$

$$e^x = \frac{x}{x-2}$$

$$x = \ln\left|\frac{x}{x-2}\right|,\ x \neq 2$$

b $x_0 = -1 \Rightarrow x_1 = \ln\left|\dfrac{-1}{-3}\right| = -1.0986$

$$x_2 = \ln\left|\frac{-1.0986}{-1.0986 - 2}\right| = -1.0369$$

$$x_3 = \ln\left|\frac{-1.0369}{-1.0369 - 2}\right| = -1.0746$$

To 2 d.p., $x_1 = -1.10$, $x_2 = -1.04$, $x_3 = -1.07$.

5 a i $f(x) = 0$

$x^3 + 5x^2 - 2 = 0$

$x^3 + 5x^2 = 2$ Add 2 to each side.

$x^3 = 2 - 5x^2$ Subtract $5x^2$ from each side.

$x = \sqrt[3]{2 - 5x^2}$ Take the cube root of each side.

ii $f(x) = 0$

$x^3 + 5x^2 - 2 = 0$

$x^3 + 5x^2 = 2$ Add 2 to each side.

$x^3 = 2 - 5x^2$ Subtract $5x^2$ from each side.

$\dfrac{x^3}{x^2} = \dfrac{2}{x^2} - \dfrac{5x^2}{x^2}$ Divide each term by x^2.

$x = \dfrac{2}{x^2} - 5$ Simplify.

5 a iii $\mathrm{f}(x)=0$

$x^3+5x^2-2=0$

$x^3+5x^2=2$ Add 2 to each side.

$5x^2=2-x^3$ Subtract x^3 from each side.

$x^2=\frac{2-x^3}{5}$ Divide each side by 5.

$x=\sqrt{\frac{2-x^3}{5}}$ Take the square root of each side.

b $x_0=10\Rightarrow x_1=\frac{2}{10^2}-5=-4.98$

$x_2=\frac{2}{4.98^2}-5=-4.9194$

$x_3=\frac{2}{4.9194^2}-5=-4.9174$

$x_4=\frac{2}{4.9174^2}-5=-4.9173$

$x=-4.917$ to 3 d.p.

c $x_0=1\Rightarrow x_1=\sqrt{\frac{2-1^3}{5}}=0.4472$

$x_2=\sqrt{\frac{2-0.4472^3}{5}}=0.6182$

$x_3=\sqrt{\frac{2-0.6182^3}{5}}=0.5939$

$x_4=\sqrt{\frac{2-0.5939^3}{5}}=0.5984$

$x_5=\sqrt{\frac{2-0.5984^3}{5}}=0.5976$

$x_6=\sqrt{\frac{2-0.5976^3}{5}}=0.5978$

$x=0.598$ to 3 d.p.

d $x_0=2\Rightarrow x_1=\sqrt{\frac{2-2^3}{5}}=\sqrt{-\frac{6}{5}}$

This has no real number solutions so no iteration is possible.

6 a $x^4-3x^3-6=0$

$3x^3=x^4-6$ Add $3x^3$ to each side.

$\frac{3x^3}{3}=\frac{x^4}{3}-\frac{6}{3}$ Divide each term by 3.

$x^3=\frac{x^4}{3}-2$ Simplify.

$x=\sqrt[3]{\frac{x^4}{3}-2}$ Take the cube root of each side.

So $p=\frac{1}{3}$ and $q=-2$.

b $x_0=0\Rightarrow x_1=\sqrt[3]{\frac{1}{3}0^4-2}=-1.25992$

$x_2=\sqrt[3]{\frac{1}{3}\times 1.25992^4-2}=-1.05073$

$x_3=\sqrt[3]{\frac{1}{3}\times 1.05073^4-2}=-1.16807$

To 3 d.p., $x_1=-1.260$, $x_2=-1.051$, $x_3=-1.168$.

c $\mathrm{f}(-1.1315)$

$=(-1.1315)^4-3\times(-1.1315)^3-6$

$=-0.0148...$

$\mathrm{f}(-1.1325)$

$=(-1.1325)^4-3\times(-1.1325)^3-6$

$=-0.0024...$

There is a change of sign in this interval so α = 1.132 to 3 d.p.

7 **a** $\mathrm{f}(x)=3\cos(x^2)+x-2$

$\mathrm{f}(x)=0\Rightarrow 3\cos(x^2)=2-x$

$\cos(x^2)=\dfrac{2-x}{3}$

$x^2=\arccos\left(\dfrac{2-x}{3}\right)$

$x=\left(\arccos\left(\dfrac{2-x}{3}\right)\right)^{\frac{1}{2}}$

b $x_0=1\Rightarrow x_1=\left(\arccos\left(\dfrac{2-1}{3}\right)\right)^{\frac{1}{2}}=1.1094$

$x_2=\left(\arccos\left(\dfrac{2-1.1094}{3}\right)\right)^{\frac{1}{2}}=1.1267$

$x_3=\left(\arccos\left(\dfrac{2-1.1267}{3}\right)\right)^{\frac{1}{2}}=1.1293$

To 3 d.p., x_1 = 1.109, x_2 = 1.127, x_3 = 1.129.

c $\mathrm{f}(1.12975)$

$=3\cos(1.12975)^2+1.12975-2$

$=0.0004...$

$\mathrm{f}(1.12985)$

$=3\cos(1.12985)^2+1.12985-2$

$=-0.0001...$

There is a change of sign in this interval so α = 1.1298 to 4 d.p.

8 **a** $\mathrm{f}(x)=4\cot x-8x+3$

$\mathrm{f}(0.8)=4\cot 0.8-6.4+3=0.484...$

$\mathrm{f}(0.9)=4\cot 0.9-7.2+3=-1.025...$

There is a change of sign in the interval [0.8, 0.9], so there must be a root in this interval, since f is continuous over the interval.

b $\mathrm{f}(x)=0$

$4\cot x-8x+3=0$

$8x=4\cot x+3=4\dfrac{\cos x}{\sin x}+3$

$x=\dfrac{\cos x}{2\sin x}+\dfrac{3}{8}$

c $x_0=0.85\Rightarrow x_1=\dfrac{\cos 0.85}{2\sin 0.85}+\dfrac{3}{8}=0.8142$

$x_2=\dfrac{\cos 0.8142}{2\sin 0.8142}+\dfrac{3}{8}=0.8470$

$x_3=\dfrac{\cos 0.8470}{2\sin 0.8470}+\dfrac{3}{8}=0.8169$

d $\mathrm{f}(0.8305)=4\cot 0.8305-6.644+3$

$=0.0105...$

$\mathrm{f}(0.8315)=4\cot 0.8315-6.652+3$

$=-0.0047...$

There is a change of sign in this interval so α = 0.831 to 3 d.p.

9 **a** $\mathrm{g}(x)=\mathrm{e}^{x-1}+2x-15$

$\mathrm{g}(x)=0\Rightarrow\ \mathrm{e}^{x-1}=15-2x$

$x-1=\ln(15-2x),\ x<\dfrac{15}{2}$

$x=\ln(15-2x)+1,\ x<\dfrac{15}{2}$

b $x_0=3\Rightarrow x_1=\ln(15-2\times 3)+1=3.1972$

$x_2=\ln(15-2\times 3.1972)+1=3.1524$

$x_3=\ln(15-2\times 3.1524)+1=3.1628$

c $\mathrm{f}(3.155)=\mathrm{e}^{3.155-1}+2\times 3.155-15$

$=-0.062...$

$\mathrm{f}(3.165)=\mathrm{e}^{3.165-1}+2\times 3.165-15$

$=0.044...$

There is a change of sign in this interval so α = 3.16 to 2 d.p.

10 a $\mathrm{f}(x) = x\mathrm{e}^x - 4x$

At *A* and *B*, $\mathrm{f}(x) = 0$

$x\mathrm{e}^x - 4x = 0 \Rightarrow x(\mathrm{e}^x - 4) = 0$

$x = 0$ or $\ln 4$

Coordinates of *A* and *B* are
(0, 0) and (ln 4, 0).

b $\mathrm{f}'(x) = x\mathrm{e}^x + \mathrm{e}^x - 4 = \mathrm{e}^x(x+1) - 4$

c $\mathrm{f}'(0.7) = \mathrm{e}^{0.7}(0.7+1) - 4 = -0.5766...$

$\mathrm{f}'(0.8) = \mathrm{e}^{0.8}(0.8+1) - 4 = 0.0059...$

There is a change of sign in the interval, which implies f′(*x*) = 0, i.e. a stationary point, in this range.

d $\mathrm{f}'(x) = 0 \Rightarrow \mathrm{e}^x(x+1) - 4 = 0$

$$\mathrm{e}^x = \frac{4}{x+1}$$

$$x = \ln\left(\frac{4}{x+1}\right)$$

e $x_0 = 0 \Rightarrow x_1 = \ln\left(\frac{4}{0+1}\right) = 1.3863$

$$x_2 = \ln\left(\frac{4}{1.3863+1}\right) = 0.5166$$

$$x_3 = \ln\left(\frac{4}{0.5166+1}\right) = 0.9699$$

$$x_4 = \ln\left(\frac{4}{0.9699+1}\right) = 0.7083$$

To 3 d.p., $x_1 = 1.386$, $x_2 = 0.517$,
$x_3 = 0.970$, $x_4 = 0.708$.

Numerical methods 10C

1 a $f(x) = x^3 - 2x - 1$

$f(1) = -2$

$f(2) = 3$

There is a change of sign, so there is a root α in the interval [1, 2].

b $f(x) = x^3 - 2x - 1$

$f'(x) = 3x^2 - 2$

Using $x_0 = 1.5$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 1.5 - \frac{(-0.625)}{4.75}$$

$$x_1 = 1.6316$$

x_1 =1.632 correct to 3 d.p.

2 a $f(x) = x^2 - \frac{4}{x} + 6x - 10$

$$f'(x) = 2x + \frac{4}{x^2} + 6 = 2\left(x + \frac{2}{x^2} + 3\right)$$

b $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Using $x_0 = -0.4$

$$x_1 = -0.4 - \frac{0.4^2 - \frac{4}{-0.4} + 6 \times (-0.4) - 10}{2\left(-0.4 + \frac{2}{-0.4^2} + 3\right)}$$

$$= -0.4 - \frac{-2.24}{30.2}$$

$$= -0.4 + 0.07417...$$

$$= -0.3258...$$

$x_1 = -0.326$ correct to 3 d.p.

3 a $f(x) = x^{\frac{3}{2}} - e^{-x} + \frac{1}{\sqrt{x}} - 2$

A is a stationary point on the curve so $f'(q) = 0$. It is not possible to divide by zero using the Newton–Raphson method, so this value of x_0 cannot be used.

b $f'(x) = \frac{3}{2}x^{\frac{1}{2}} + e^{-x} - \frac{1}{2x^{\frac{3}{2}}}$

$$f(1.2) = 1.2^{\frac{3}{2}} - e^{-1.2} + \frac{1}{\sqrt{1.2}} - 2$$

$$= -0.07389...$$

$$f'(1.2) = \frac{3}{2}\sqrt{1.2} + e^{-1.2} - \frac{1}{2(1.2)^{\frac{3}{2}}}$$

$$= 1.56399...$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \Rightarrow$$

$$x_1 = 1.2 + \frac{0.07389...}{1.56399...} = 1.247 \text{ to 3 d.p.}$$

4 a $f(x) = 1 - x - \cos(x^2)$

$f(1.4) = 1 - 1.4 - \cos(1.4)^2 = -0.0205...$

$f(1.5) = 1 - 1.5 - \cos(1.5)^2 = 0.128...$

There is a change of sign in the interval [1.4, 1.5] so there must be a root α in this interval.

b $f'(x) = -1 + 2x\sin(x^2)$

$f'(1.4) = -1 + 2.8\sin 1.96 = 1.5905...$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \Rightarrow$$

$$x_1 = 1.4 + \frac{0.0205...}{1.5905...} = 1.413 \text{ correct to 3 d.p.}$$

c $f(1.4125) = -0.00076... < 0$

$f(1.4135) = 0.00081... > 0$

There is a sign change in the interval [1.4125, 1.4135] so x = 1.413 is correct to 3 d.p.

5 a $f(x) = x^2 - \frac{3}{x^2}$

$$f(1.3) = 1.69 - \frac{3}{1.69} = -0.0851...$$

$$f(1.4) = 1.96 - \frac{3}{1.96} = 0.429...$$

There is a change of sign in the interval [1.3, 1.4] so there must be a root α in this interval.

5 b $\mathrm{f}'(x) = 2x + \dfrac{6}{x^3}$

c $\mathrm{f}'(1.3) = 2.6 + \dfrac{6}{1.3^3} = 5.3309...$

$$x_1 = x_0 - \frac{\mathrm{f}(x_0)}{\mathrm{f}'(x_0)} \Rightarrow$$

$$x_1 = 1.3 + \frac{0.0851...}{5.3309...} = 1.316 \text{ to 3 d.p.}$$

6 a $\mathrm{f}(x) = x^2 \sin x - 2x + 1$

i $\mathrm{f}(0.6) = 0.36 \sin 0.6 - 1.2 + 1$

$= 0.0032...$

$\mathrm{f}(0.7) = 0.49 \sin 0.7 - 1.4 + 1$

$= -0.0843...$

ii $\mathrm{f}(1.2) = 1.44 \sin 1.2 - 2.4 + 1$

$= -0.0578...$

$\mathrm{f}(1.3) = 1.69 \sin 1.3 - 2.6 + 1$

$= 0.0284...$

iii $\mathrm{f}(2.4) = 5.76 \sin 2.4 - 4.8 + 1$

$= 0.0906...$

$\mathrm{f}(2.5) = 6.25 \sin 2.5 - 5 + 1$

$= -0.2595...$

There is a change of sign in all the intervals so there must be a root in each.

b There is a stationary point at $x = a$, so $\mathrm{f}'(x) = 0$ here. You cannot divide by zero in the Newton–Raphson formula so $x_0 = a$ cannot be used as a first approximation.

c $\mathrm{f}'(x) = x^2 \cos x + 2x \sin x - 2$

$\mathrm{f}'(2.4) = 5.76 \cos 2.4 + 4.8 \sin 2.4 - 2$

$= -3.0051...$

$$x_1 = x_0 - \frac{\mathrm{f}(x_0)}{\mathrm{f}'(x_0)} \Rightarrow$$

$$x_1 = 2.4 + \frac{0.0906...}{3.0051...} = 2.430 \text{ to 3 d.p.}$$

7 a $\mathrm{f}(x) = \ln(3x-4) - x^2 + 10$

$\mathrm{f}(3.4) = \ln 6.2 - 11.56 + 10 = 0.2645...$

$\mathrm{f}(3.5) = \ln 6.5 - 12.25 + 10 = -0.3781...$

There is a change of sign in the interval [3.4, 3.5] so there must be a root α in this interval.

b $\mathrm{f}'(x) = \dfrac{3}{3x-4} - 2x$

c $\mathrm{f}'(3.4) = \dfrac{3}{6.2} - 6.8 = -6.3161...$

$$x_1 = x_0 - \frac{\mathrm{f}(x_0)}{\mathrm{f}'(x_0)} \Rightarrow$$

$$x_1 = 3.4 + \frac{0.2645...}{6.3161...} = 3.442 \text{ to 3 d.p.}$$

Challenge

a From the graph, f(x) > 0 for all values of $x > 0$. Note also that $xe^{-x^2} > 0$ when $x > 0$.

So the same must be true for $x > \dfrac{1}{\sqrt{2}}$.

$$\mathrm{f}'(x) = \mathrm{e}^{-x^2}(1-2x^2) = 0 \Rightarrow x = \frac{1}{\sqrt{2}}$$

So $\mathrm{f}'(x) < 0$ for $x > \dfrac{1}{\sqrt{2}}$

$x_{n+1} = x_n - \dfrac{\mathrm{f}(x)}{\mathrm{f}'(x)}$ is an increasing sequence

as $\mathrm{f}(x) > 0$ and $\mathrm{f}'(x) < 0$ for $x > \dfrac{1}{\sqrt{2}}$.

Therefore the Newton–Raphson method will fail to converge.

Challenge

b $\mathrm{f}(-0.5)=\frac{1}{5}+(-0.5)\mathrm{e}^{-0.25}=-0.1894...$

$\mathrm{f}'(-0.5)=\mathrm{e}^{-0.25}(1-0.5)=0.3894...$

$$x_1=x_0-\frac{\mathrm{f}(x_0)}{\mathrm{f}'(x_0)}\Rightarrow$$

$$x_1=-0.5+\frac{0.1894...}{0.3894...}=-0.0136...$$

$$x_2=-0.0136...+\frac{0.1864...}{0.9994...}=-0.2001...$$

$$x_3=-0.2001...-\frac{0.0077...}{0.8838...}=-0.2088...$$

$$x_4=-0.2088...-\frac{0.0000...}{0.8737...}=-0.2089...$$

The root is −0.209 correct to 3 d.p.

Numerical methods 10D

1 a $M = E - 0.1\sin E$

When $M = \frac{\pi}{6}$, $\frac{\pi}{6} = E - 0.1\sin E$

$\text{f}(x) = x - 0.1\sin x - k$

If E is a root of $\text{f}(x)$

$\text{f}(E) = E - 0.1\sin E - k = 0$

$k = \frac{\pi}{6}$

b $\text{f}(E) = E - 0.1\sin E - \frac{\pi}{6}$

$\text{f}'(E) = 1 - 0.1\cos E$

$$x_1 = x_0 - \frac{\text{f}(x_0)}{\text{f}'(x_0)}$$

$x_0 = 0.6 \Rightarrow$

$$x_1 = 0.6 - \frac{0.6 - 0.1\times\sin 0.6 - \frac{\pi}{6}}{1 - 0.1\cos 0.6}$$

$$= 0.6 - \frac{0.019937...}{0.91746...} = 0.5782...$$

c $\text{f}(0.5775) = 0.5775 - 0.1\times\sin 0.5775 - \frac{\pi}{6}$

$= -0.00069...$

$\text{f}(0.5785) = 0.5785 - 0.1\times\sin 0.5785 - \frac{\pi}{6}$

$= 0.00022...$

There is a change of sign in this interval so $E = 0.578$ correct to 3 d.p.

2 a At A and B, $v = 0$.

$v = 0 \Rightarrow \left(10 - \frac{1}{2}(t+1)\right)\ln(t+1) = 0$

$\ln(t+1) = 0 \Rightarrow t = 0$

$10 - \frac{1}{2}(t+1) = 0 \Rightarrow t = 19$

So A is (0, 0) and B is (19, 0).

b $\text{f}'(t) = \left(10 - \frac{1}{2}(t+1)\right)\times\frac{1}{t+1} - \frac{1}{2}\ln(t+1)$

$\text{f}'(t) = \frac{10}{t+1} - \frac{1}{2}\left(\ln(t+1)+1\right)$

c $\text{f}'(5.8) = \frac{10}{5.8+1} - 0.5(\ln\left(5.8+1\right)+1)$

$= 0.0121...$

$\text{f}'(5.9) = \frac{10}{5.9+1} - 0.5(\ln\left(5.9+1\right)+1)$

$= -0.0164...$

The sign of the gradient changes in the interval [5.8, 5.9] so the x-coordinate of P is in this interval.

d At the stationary point $\text{f}'(t) = 0$.

$\frac{10}{t+1} - \frac{1}{2}(\ln(t+1)+1) = 0$

$\frac{10}{t+1} = \frac{1}{2}(\ln(t+1)+1)$

$\frac{20}{\ln(t+1)+1} = t+1$

$t = \frac{20}{1+\ln(t+1)} - 1$

e $t_1 = \frac{20}{1+\ln(t_0+1)} - 1 = \frac{20}{1+\ln 6} - 1 = 6.1639$

$t_2 = \frac{20}{1+\ln 7.1639} - 1 = 5.7361$

$t_3 = \frac{20}{1+\ln 6.7361} - 1 = 5.8787$

To 3 d.p. the values are $t_1 = 6.164$, $t_2 = 5.736$ and $t_3 = 5.879$.

3 a $\text{d}(x) = \text{e}^{-0.6x}(x^2 - 3x)$

$\text{d}(x) = 0 \Rightarrow x^2 - 3x = 0$

$x(x-3) = 0 \Rightarrow x = 0$ or 3

The stream is 3 metres wide so the function is only valid for $0 \leqslant x \leqslant 3$.

b $\text{d}'(x) = \text{e}^{-0.6x}(2x-3) - \frac{3}{5}\text{e}^{-0.6x}(x^2-3x)$

$= 2x\text{e}^{-0.6x} - 3\text{e}^{-0.6x} - \frac{3}{5}x^2\text{e}^{-0.6x} + \frac{9}{5}x\text{e}^{-0.6x}$

$= \text{e}^{-0.6x}\left(-\frac{3}{5}x^2 + \frac{19}{5}x - \frac{15}{5}\right)$

$\text{d}'(x) = -\frac{1}{5}\text{e}^{-0.6x}(3x^2 - 19x + 15)$

So $a = 3$, $b = -19$, $c = 15$.

3 c i $-\frac{1}{5}e^{-0.6x}(3x^2-19x+15)=0$

$$-\frac{1}{5}e^{-0.6x}\neq 0$$

$$\text{so } d'(x)=0\Rightarrow 3x^2-19x+15=0$$

$$3x^2=19x-15$$

$$x=\sqrt{\frac{19x-15}{3}}$$

ii $3x^2-19x+15=0$

$$19x=3x^2+15$$

$$x=\frac{3x^2+15}{19}$$

iii $3x^2=19x-15$

$$x=\frac{19x-15}{3x}$$

d For $x_0=1$ in equation from **c i**
Iterates to 5.409 after 21 iterations.

For $x_0=1$ in equation from **c iii**
Iterates to 5.409 after 8 iterations.

These are both outside the required range.

For $x_0=1$ in equation from **c ii**
Iterates to 0.924 after 6 iterations.

e $d(0.924)=e^{-0.6\times0.924}(0.924^2-3\times0.924)$

$$=-1.1018...$$

The maximum depth of the river is
1.10 m, correct to 2 d.p.

4 a $h(t)=40\sin\left(\frac{t}{10}\right)-9\cos\left(\frac{t}{10}\right)-0.5t^2+9$

$$h(t)=0\Rightarrow$$

$$40\sin\left(\frac{t}{10}\right)-9\cos\left(\frac{t}{10}\right)-0.5t^2+9=0$$

$$0.5t^2=40\sin\left(\frac{t}{10}\right)-9\cos\left(\frac{t}{10}\right)+9$$

$$t^2=18+80\sin\left(\frac{t}{10}\right)-18\cos\left(\frac{t}{10}\right)$$

$$t=\sqrt{18+80\sin\left(\frac{t}{10}\right)-18\cos\left(\frac{t}{10}\right)}$$

b $t_1=\sqrt{18+80\sin\left(\frac{8}{10}\right)-18\cos\left(\frac{8}{10}\right)}$

$$t_1=7.928$$

$$t_2=7.896$$

$$t_3=7.882$$

$$t_4=7.876$$

c $h'(t)=4\cos\left(\frac{t}{10}\right)+\frac{9}{10}\sin\left(\frac{t}{10}\right)-t$

d $h(8)=40\sin 0.8-9\cos 0.8-32+9$

$$=-0.5761$$

$$h'(8)=4\cos 0.8+0.9\sin 0.8-8$$

$$=-4.5676$$

Second approximation:

$$=8-\frac{h(8)}{h'(8)}=8-\frac{-0.5761}{-4.5676}=7.874 \text{ to 3 d.p.}$$

e Restrict the range of validity to $0\leqslant t\leqslant A$.

5 a $c(x)=5e^{-x}+4\sin\left(\frac{x}{2}\right)+\frac{x}{2}$

$$c'(x)=-5e^{-x}+2\cos\left(\frac{x}{2}\right)+\frac{1}{2}$$

b Turning points are when $c'(x)=0$

$$-5e^{-x}+2\cos\left(\frac{x}{2}\right)+\frac{1}{2}=0$$

i $2\cos\left(\frac{x}{2}\right)=5e^{-x}-\frac{1}{2}$

$$\cos\left(\frac{x}{2}\right)=\frac{5}{2}e^{-x}-\frac{1}{4}$$

$$x=2\arccos\left(\frac{5}{2}e^{-x}-\frac{1}{4}\right)$$

5 b ii

$$5e^{-x} = 2\cos\left(\frac{x}{2}\right) + \frac{1}{2}$$

$$5e^{-x} = \frac{4\cos\left(\frac{x}{2}\right) + 1}{2}$$

$$10e^{-x} = 4\cos\left(\frac{x}{2}\right) + 1$$

$$e^{-x} = \frac{4\cos\left(\frac{x}{2}\right) + 1}{10}$$

$$e^{x} = \frac{10}{4\cos\left(\frac{x}{2}\right) + 1}$$

$$x = \ln\left(\frac{10}{4\cos\left(\frac{x}{2}\right) + 1}\right)$$

c

$$x_1 = 2\arccos\left(\frac{5}{2}e^{-3} - \frac{1}{4}\right) = 3.393$$

$$x_2 = 2\arccos\left(\frac{5}{2}e^{-3.393} - \frac{1}{4}\right) = 3.475$$

$$x_3 = 2\arccos\left(\frac{5}{2}e^{-3.475} - \frac{1}{4}\right) = 3.489$$

$$x_4 = 2\arccos\left(\frac{5}{2}e^{-3.489} - \frac{1}{4}\right) = 3.491$$

d

$$x_1 = \ln\left(\frac{10}{4\cos\left(\frac{1}{2}\right) + 1}\right) = 0.796$$

$$x_2 = \ln\left(\frac{10}{4\cos\left(\frac{0.796}{2}\right) + 1}\right) = 0.758$$

$$x_3 = \ln\left(\frac{10}{4\cos\left(\frac{0.758}{2}\right) + 1}\right) = 0.752$$

$$x_3 = \ln\left(\frac{10}{4\cos\left(\frac{0.752}{2}\right) + 1}\right) = 0.751$$

e The model does support the assumption that the crime rate was increasing. The model shows that there is a minimum point 3/4 of the way through 2000 and a maximum point mid-way through 2003. So, the crime rate is increasing in the interval between October 2000 and June 2003.

Numerical methods, Mixed exercise 10

1 a $\mathrm{f}(x) = x^3 - 6x - 2$

$\mathrm{f}(x) = 0 \Rightarrow x^3 = 6x + 2$

$x^2 = 6 + \frac{2}{x}$

$x = \pm\sqrt{6 + \frac{2}{x}}$

$a = 6,\ b = 2$

b $x_{n+1} = \sqrt{6 + \frac{2}{x_n}}$

$x_0 = 2 \Rightarrow x_1 = \sqrt{6 + \frac{2}{2}} = \sqrt{7} = 2.64575...$

$x_2 = \sqrt{6 + \frac{2}{2.64575...}} = 2.59921...$

$x_3 = \sqrt{6 + \frac{2}{2.59921...}} = 2.60181...$

$x_4 = \sqrt{6 + \frac{2}{2.60181...}} = 2.60167...$

To 4 d.p., the values are $x_1 = 2.6458$, $x_2 = 2.5992$, $x_3 = 2.6018$, $x_4 = 2.6017$.

c $\mathrm{f}(2.6015) = 2.6015^3 - 6 \times 2.6015 - 2$

$= -0.0025...$

$\mathrm{f}(2.6025) = 2.6025^3 - 6 \times 2.6025 - 2$

$= 0.0117...$

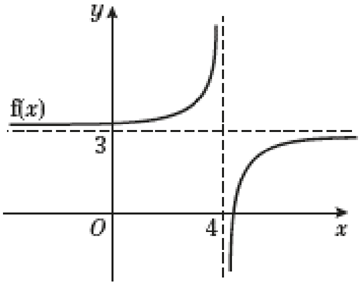
There is a change of sign in this interval so $\alpha = 2.602$ correct to 3 d.p.

2 a $\mathrm{f}(x) = \frac{1}{4 - x} + 3$

$\mathrm{f}(3.9) = \frac{1}{0.1} + 3 = 13$

$\mathrm{f}(4.1) = -\frac{1}{0.1} + 3 = -7$

b There is an asymptote at $x = 4$ which causes the change of sign, not a root.



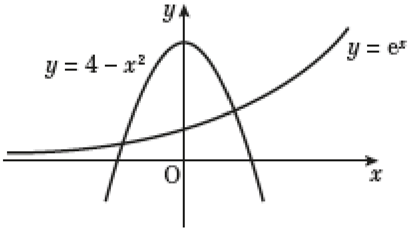
c $\mathrm{f}(x) = 0 \Rightarrow \frac{1}{4 - x} + 3 = 0$

$\frac{1}{x - 4} = 3$

$1 = 3x - 12 \ \Rightarrow\ x = \frac{13}{3}$

So $\alpha = \frac{13}{3}$.

3 a



b There is one positive and one negative root of the equation $\mathrm{p}(x) = \mathrm{q}(x)$ at the points of intersection.

$\mathrm{p}(x) = \mathrm{q}(x) \Rightarrow 4 - x^2 = \mathrm{e}^x$

i.e. $x^2 + \mathrm{e}^x - 4 = 0$

c $x^2 = 4 - \mathrm{e}^x$

$x = \pm\left(4 - \mathrm{e}^x\right)^{\frac{1}{2}}$

d $x_{n+1} = \pm\left(4 - \mathrm{e}^{x_n}\right)^{\frac{1}{2}}$

$x_0 = -2 \Rightarrow x_1 = -\left(4 - \mathrm{e}^{-2}\right)^{\frac{1}{2}} = -1.96587...$

$x_2 = -\left(4 - \mathrm{e}^{-1.96587...}\right)^{\frac{1}{2}} = -1.96467...$

$x_3 = -\left(4 - \mathrm{e}^{-1.96467...}\right)^{\frac{1}{2}} = -1.96463...$

$x_4 = -\left(4 - \mathrm{e}^{-1.96463...}\right)^{\frac{1}{2}} = -1.96463...$

To 4 d.p., the values are $x_1 = -1.9659$, $x_2 = -1.9647$, $x_3 = -1.9646$, $x_4 = -1.9646$.

e $x_0 = 1.4 \Rightarrow 4 - \mathrm{e}^{1.4} < 0$

There can be no square root of a negative number.

4 a $g(x) = x^5 - 5x - 6$

$g(1) = 1 - 5 - 6 = -10$

$g(2) = 32 - 10 - 6 = 16$

There is a change of sign in the interval, so there must be a root in the interval, since f is continuous over the interval.

b $g(x) = 0 \Rightarrow x^5 = 5x + 6$

$x = (5x+6)^{\frac{1}{5}}$

$p = 5, q = 6, r = 5$

c $x_{n+1} = (5x_n + 6)^{\frac{1}{5}}$

$x_0 = 1 \Rightarrow x_1 = (5+6)^{\frac{1}{5}} = 1.61539...$

$x_2 = (5\times1.61539... + 6)^{\frac{1}{5}} = 1.69707...$

$x_3 = (5\times1.69707... + 6)^{\frac{1}{5}} = 1.70681...$

To 4 d.p., the values are $x_1 = 1.6154$, $x_2 = 1.6971$, $x_3 = 1.7068$.

d $g(1.7075) = 1.7075^5 - 5\times1.7075 - 6$

$= -0.0229...$

$g(1.7085) = 1.7085^5 - 5\times1.7085 - 6$

$= 0.0146...$

The sign change implies there is a root in this interval so $\alpha = 1.708$ correct to 3 d.p.

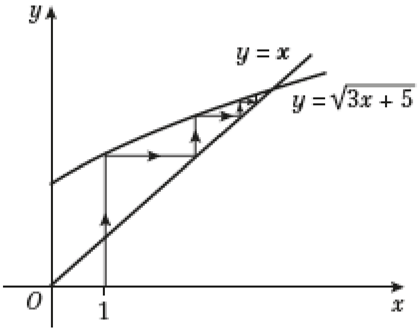
5 a $g(x) = x^2 - 3x - 5$

$g(x) = 0 \Rightarrow x^2 - 3x - 5 = 0$

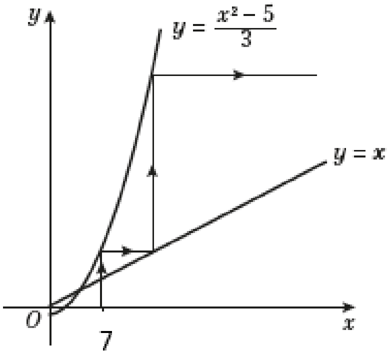
$x^2 = 3x + 5$

$x = \sqrt{3x+5}$

b, c



d



$g(x) = 0 \Rightarrow x^2 - 3x - 5 = 0$

$3x = x^2 - 5$

$x = \frac{x^2 - 5}{3}$

6 a $f(x) = 5x - 4\sin x - 2$

$f(1.1) = 5(1.1) - 4\sin(1.1) - 2$

$= -0.0648...$

$f(1.15) = 5(1.15) - 4\sin(1.15) - 2$

$= -0.0989...$

f(1.1) < 0 and f(1.15) > 0 so there is a change of sign, which implies there is a root between $x = 1.1$ and $x = 1.15$.

b $5x - 4\sin x - 2 = 0$

$5x - 2 = 4\sin x$ Add 4 sin x to each side.

$5x = 4\sin x + 2$ Add 2 to each side.

$\frac{5x}{5} = \frac{4\sin x}{5} + \frac{2}{5}$ Divide each term by 5.

$x = \frac{4}{5}\sin x + \frac{2}{5}$ Simplify.

So $p = \frac{4}{5}$ and $q = \frac{2}{5}$.

c $x_0 = 1.1 \Rightarrow$

$x_1 = 0.8\sin(1.1) + 0.4 = 1.1129...$

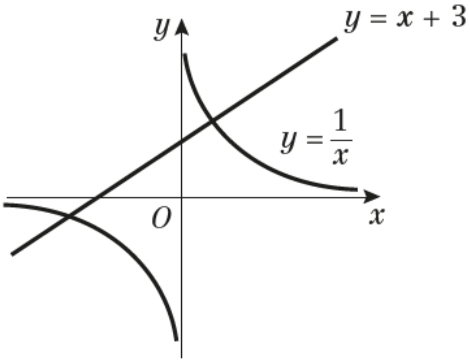
$x_2 = 0.8\sin(1.1129...) + 0.4 = 1.1176...$

$x_3 = 0.8\sin(1.1176...) + 0.4 = 1.1192...$

$x_4 = 0.8\sin(1.1192...) + 0.4 = 1.1198...$

To 3 d.p., the values are $x_1 = 1.113$, $x_2 = 1.118$, $x_3 = 1.119$, $x_4 = 1.120$.

7 a



b The line meets the curve at two points, so there are two values of x that satisfy the equation $\frac{1}{x}=x+3$.

So $\frac{1}{x}=x+3$ has two roots.

c $\frac{1}{x}=x+3 \Rightarrow 0=x+3-\frac{1}{x}$

Let $\mathrm{f}(x)=x+3-\frac{1}{x}$

$\mathrm{f}(0.30)=(0.30)+3-\frac{1}{0.30}=-0.0333...$

$\mathrm{f}(0.31)=(0.31)+3-\frac{1}{0.31}=0.0841...$

f(0.30) < 0 and f(0.31) > 0 so there is a change of sign, which implies there is a root between $x = 0.30$ and $x = 0.31$.

d $\frac{1}{x}=x+3$

$\frac{1}{x}\times x=x\times x+3\times x$ Multiply by x.

$1=x^2+3x$

So $x^2+3x-1=0$

e Using $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ with

$a=1, b=3, c=-1$

$$x=\frac{-(3)\pm\sqrt{(3)^2-4(1)(-1)}}{2(1)}$$

$$=\frac{-3\pm\sqrt{9+4}}{2}=\frac{-3\pm\sqrt{13}}{2}$$

So $x=\frac{-3+\sqrt{13}}{2}=0.3027...$

The positive root is 0.303 to 3 d.p.

8 a $\mathrm{g}(x)=x^3-7x^2+2x+4$

$\mathrm{g}'(x)=3x^2-14x+2$

b Using $x_0 = 6.6$,

$$x_1=x_0-\frac{\mathrm{g}(x_0)}{\mathrm{g}'(x_0)}$$

$$=6.6-\frac{\mathrm{g}(6.6)}{\mathrm{g}'(6.6)}$$

$$=6.6-\frac{6.6^3-7(6.6^2)+2(6.6)+4}{3(6.6^2)-14(6.6)+2}$$

$=6.606$ correct to 3 d.p.

c $\mathrm{g}(1)=0 \Rightarrow x-1$ is a factor of $\mathrm{g}(x)$

$\mathrm{g}(x)=(x-1)(x^2-6x-4)$

$(x-1)(x^2-6x-4)=0$

Other two roots of g(x) are given by

$$\frac{6\pm\sqrt{36+16}}{2}=\frac{6\pm\sqrt{52}}{2}=3\pm\sqrt{13}$$

d Percentage error:

$$\frac{6.606-\left(3+\sqrt{13}\right)}{3+\sqrt{13}}\times100=0.007\%$$

9 a $\mathrm{f}(x)=2\sec x+2x-3$

$\mathrm{f}(0.4)=2\sec 0.4+0.8-3=-0.0285...$

$\mathrm{f}(0.5)=2\sec 0.5+1-3=0.2789...$

The sign change implies there is a root in this interval.

9 b $f'(x) = 2\sec x \tan x + 2$

Using $x_0 = 0.4$,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.4 - \frac{f(0.4)}{f'(0.4)}$$

$$= 0.4 - \frac{-0.0285...}{2\sec 0.4 \tan 0.4 + 2}$$

$$= 0.4097...$$

$\alpha = 0.410$ correct to 3 d.p.

c $f(-1.1895) =$
$2\sec(-1.1895) + 2\times(-1.1895) - 3$
$= -0.0044...$

$f(-1.1905) =$
$2\sec(-1.1905) + 2\times(-1.1905) - 3$
$= 0.0069...$

There is a change of sign in this interval, so there is a root $\beta = 1.191$ correct to 3 d.p.

10 a $e^{0.8x} - \frac{1}{3-2x} = 0$

$e^{0.8x} = \frac{1}{3-2x}$ Add $\frac{1}{3-2x}$ to each side.

$(3-2x)e^{0.8x} = \frac{1}{3-2x}\times(3-2x)$

Multiply each side by (3 − 2x).

$(3-2x)e^{0.8x} = 1$ Simplify.

$\frac{(3-2x)e^{0.8x}}{e^{0.8x}} = \frac{1}{e^{0.8x}}$ Divide each side by $e^{0.8x}$.

$3-2x = e^{-0.8x}$ Simplify (remember $\frac{1}{e^a} = e^{-a}$).

$3 = e^{-0.8x} + 2x$ Add 2x to each side.

$2x = 3 - e^{-0.8x}$ Subtract $e^{-0.8x}$ from each side.

$\frac{2x}{2} = \frac{3}{2} - \frac{e^{-0.8x}}{2}$ Divide each term by 2.

$x = 1.5 - 0.5e^{-0.8x}$ Simplify.

b $x_0 = 1.3$

$x_1 = 1.5 - 0.5e^{-0.8(1.3)} = 1.32327...$
$x_2 = 1.5 - 0.5e^{-0.8(1.32327..)} = 1.32653...$
$x_3 = 1.5 - 0.5e^{-0.8(1.32653...)} = 1.32698...$

So $x_3 = 1.327$ correct to 3 d.p.

10 c $\mathrm{e}^{0.8x} - \dfrac{1}{3-2x} = 0$

$\mathrm{e}^{0.8x} = \dfrac{1}{3-2x}$ Add $\dfrac{1}{3-2x}$ to each side.

$0.8x = \ln\left(\dfrac{1}{3-2x}\right)$ Take logs.

$0.8x = -\ln(3-2x)$ Simplify using $\ln\left(\dfrac{1}{c}\right) = -\ln c$.

$\dfrac{0.8x}{0.8} = -\dfrac{\ln(3-2x)}{0.8}$ Divide each side by 0.8.

$x = -1.25\ln(3-2x)$ Simplify $\left(\dfrac{1}{0.8} = 1.25\right)$.

So $p = -1.25$

d $x_0 = -2.6$

$$\begin{aligned} x_1 &= -1.25\ln[3-2(-2.6)] \\ &= -2.63016... \\ x_2 &= -1.25\ln[3-2(-2.63016...)] \\ &= -2.63933... \\ x_3 &= -1.25\ln[3-2(-2.63933...)] \\ &= -2.64210... \end{aligned}$$

So $x_3 = -2.64$ (2 d.p.)

11 a $y = x^x \Rightarrow \ln y = x\ln x$

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = x \times \frac{1}{x} + \ln x = 1 + \ln x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y(1+\ln x) = x^x(1+\ln x)$$

b $\mathrm{f}(x) = x^x - 2$

$\mathrm{f}(1.4) = 1.4^{1.4} - 2 = -0.3983...$

$\mathrm{f}(1.6) = 1.6^{1.6} - 2 = 0.1212...$

The sign change implies there is a root in this interval.

c $\mathrm{f}'(x) = x^x(1+\ln x)$

$\mathrm{f}(1.5) = 1.5^{1.5} - 2 = -0.16288...$

$\mathrm{f}'(1.5) = 1.5^{1.5}(1+\ln 1.5) = 2.58200...$

$$\begin{aligned} x_1 &= x_0 - \frac{\mathrm{f}(x_0)}{\mathrm{f}'(x_0)} \\ &= 1.5 - \frac{\mathrm{f}(1.5)}{\mathrm{f}'(1.5)} \\ &= 1.5 + \frac{0.16288...}{2.58200...} \\ &= 1.56308... \end{aligned}$$

To a second approximation, $\alpha = 1.5631$, to 4 d.p.

d $\mathrm{f}(1.55955) = 1.55955^{1.55955} - 2$
$= -0.00017...$

$\mathrm{f}(1.55965) = 1.55965^{1.55965} - 2$
$= 0.00011...$

There is a change of sign in this interval so α = 1.5596 to 4 d.p.

12 a $\mathrm{f}(x) = \cos(4x) - \dfrac{1}{2}x$

$\mathrm{f}(1.3) = \cos 5.2 - 0.65 = -0.181...$

$\mathrm{f}(1.4) = \cos 5.6 - 0.7 = 0.0755...$

The sign change implies there is a root in this interval.

b $\mathrm{f}'(x) = -4\sin(4x) - \dfrac{1}{2}$

At B, $\mathrm{f}'(x) = 0$

$\sin 4x = -\dfrac{1}{8} \Rightarrow$

$4x = -0.1253...,\ 3.2669...,\ 6.1579...,$ etc

From the graph $0 < x < 1$ so $4x = 3.2669$

So $x = 0.81673...$

$\mathrm{f}(0.81673...) = \cos(3.2669...) - 0.40803...$
$= -1.4005...$

B has coordinates (0.817, −1.401).

12 c $x_{n+1} = \frac{1}{4}\arccos\left(\frac{1}{2}x_n\right)$

$x_0 = 0.4 \Rightarrow x_1 = \frac{1}{4}\arccos(0.2) = 0.34235...$

$x_2 = \frac{1}{4}\arccos(0.17117...) = 0.34969...$

$x_3 = \frac{1}{4}\arccos(0.17484...) = 0.34876...$

$x_4 = \frac{1}{4}\arccos(0.17438...) = 0.34887...$

To 4 d.p., the values are $x_1 = 0.3424$, $x_2 = 0.3497$, $x_3 = 3488$, $x_4 = 0.3489$.

d $f(1.7) = \cos 6.8 - 0.85 = 0.01939...$

$f'(1.7) = -4\sin 6.8 - \frac{1}{2} = -2.4765...$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 1.7 - \frac{f(1.7)}{f'(1.7)}$$
$$= 1.7 + \frac{0.01939...}{2.4765...}$$
$$= 1.7078...$$

To 3 d.p., the second approximation is 1.708.

12 e $f(1.7075) = \cos 6.83 - 0.85375$
$= 0.000435...$
$f(1.7085) = \cos 6.834 - 0.85425$
$= -0.00215...$

There is a change of sign so there is a root of 1.708 correct to 3 decimal places in this interval.

Challenge

a $f(x) = x^6 + x^3 - 7x^2 - x + 3$

$f'(x) = 6x^5 + 3x^2 - 14x - 1$

$f''(x) = 30x^4 + 6x - 14$

i $f''(x) = 0 \Rightarrow 6x = 14 - 30x^4$

$3x = 7 - 15x^4$

$$x = \frac{7 - 15x^4}{3}$$

ii $f''(x) = 0 \Rightarrow 6x = 14 - 30x^4$

$15x^4 + 3x = 7$

$x(15x^3 + 3) = 7$

$$x = \frac{7}{15x^3 + 3}$$

iii $f''(x) = 0 \Rightarrow 6x = 14 - 30x^4$

$15x^4 = 7 - 3x$

$$x^4 = \frac{7 - 3x}{15}$$

$$x = \sqrt[4]{\frac{7 - 3x}{15}}$$

b As B is a point of inflection $f''(x) = 0$.

Using $x_0 = 1$ in part **iii**

$$x_1 = \sqrt[4]{\frac{4}{15}} = 0.7186...$$

$$x_2 = \sqrt[4]{\frac{7 - 3\times 0.7186...}{15}} = 0.7538...$$

$$x_3 = \sqrt[4]{\frac{7 - 3\times 0.7538...}{15}} = 0.7496...$$

$$x_4 = \sqrt[4]{\frac{7 - 3\times 0.7496...}{15}} = 0.7501...$$

$$x_5 = \sqrt[4]{\frac{7 - 3\times 0.7501...}{15}} = 0.7501...$$

Correct to 3 d.p., an approximation for the x-coordinate of B is 0.750.

c A has a negative x-coordinate. Formula **iii** gives the positive fourth root, so cannot be used to find a negative root.

d As A is a point of inflection, $\mathrm{f}''(x) = 0$.

$\mathrm{f}''(0) = -14$

$\mathrm{f}''(-1) = 30(-1^4) + 6(-1) - 14 = 10$

There is a change of sign, so the x-coordinate of the root A lies in the interval [−1, 0].

$\mathrm{f}'''(x) = 120x^3 + 6$

Using the Newton–Raphson formula:

$$x_1 = x_0 - \frac{\mathrm{f}''(x_0)}{\mathrm{f}'''(x_0)}$$

Using $x_0 = -0.9$

$$\begin{aligned}
x_1 &= -0.9 - \frac{\mathrm{f}''(-0.9)}{\mathrm{f}'''(-0.9)} \\
&= -0.9 - \frac{30(-0.9)^4 + 6(-0.9) - 14}{120(-0.9)^3 + 6} \\
&= -0.9 - \frac{19.683 - 5.4 - 14}{-87.48 + 6} \\
&= -0.9 - \frac{0.283}{81.48} = -0.89652...
\end{aligned}$$

$$\begin{aligned}
x_2 &= -0.89652... - \frac{\mathrm{f}''(-0.89652...)}{\mathrm{f}'''(-0.89652...)} \\
&= -0.89650...
\end{aligned}$$

The x-coordinate of A is −0.897 correct to 3 d.p.

Integration 11A

1 a $\int\left(3\sec^2 x+\frac{5}{x}+\frac{2}{x^2}\right)\mathrm{d}x$

$=\int\left(3\sec^2 x+\frac{5}{x}+2x^{-2}\right)\mathrm{d}x$

$=3\tan x+5\,1\mathrm{n}\,|x|-\frac{2}{x}+c$

b $\int(5\mathrm{e}^x-4\sin x+2x^3)\mathrm{d}x$

$=5\mathrm{e}^x+4\cos x+\frac{2x^4}{4}+c$

$=5\mathrm{e}^x+4\cos x+\frac{x^4}{2}+c$

c $\int 2(\sin x-\cos x+x)\mathrm{d}x$

$=\int(2\sin x-2\cos x+2x)\mathrm{d}x$

$=-2\cos x-2\sin x+x^2+c$

d $\int\left(3\sec x\tan x-\frac{2}{x}\right)\mathrm{d}x$

$=3\sec x-2\,1\mathrm{n}\,|x|+c$

e $\int\left(5\mathrm{e}^x+4\cos x-\frac{2}{x^2}\right)\mathrm{d}x$

$=\int(5\mathrm{e}^x+4\cos x-2x^{-2})\mathrm{d}x$

$=5\mathrm{e}^x+4\sin x+\frac{2}{x}+c$

f $\int\left(\frac{1}{2x}+2\operatorname{cosec}^2 x\right)\mathrm{d}x$

$=\int\left(\frac{1}{2}\times\frac{1}{x}+2\operatorname{cosec}^2 x\right)\mathrm{d}x$

$=\frac{1}{2}1\mathrm{n}\,|x|-2\cot x+c$

g $\int\left(\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}\right)\mathrm{d}x$

$=\int\left(\frac{1}{x}+x^{-2}+x^{-3}\right)\mathrm{d}x$

$=1\mathrm{n}\,|x|+\frac{x^{-1}}{-1}+\frac{x^{-2}}{-2}+c$

$=1\mathrm{n}\,|x|-\frac{1}{x}-\frac{1}{2x^2}+c$

h $\int(\mathrm{e}^x+\sin x+\cos x)\mathrm{d}x$

$=\mathrm{e}^x-\cos x+\sin x+c$

i $\int(2\operatorname{cosec} x\cot x-\sec^2 x)\mathrm{d}x$

$=-2\operatorname{cosec} x-\tan x+c$

j $\int\left(\mathrm{e}^x+\frac{1}{x}-\operatorname{cosec}^2 x\right)\mathrm{d}x$

$=\mathrm{e}^x+1\mathrm{n}\,|x|+\cot x+c$

2 a $\int\left(\frac{1}{\cos^2 x}+\frac{1}{x^2}\right)\mathrm{d}x$

$=\int(\sec^2 x+x^{-2})\mathrm{d}x$

$=\tan x-\frac{1}{x}+c$

b $\int\left(\frac{\sin x}{\cos^2 x}+2\mathrm{e}^x\right)\mathrm{d}x$

$=\int(\tan x\sec x+2\mathrm{e}^x)\mathrm{d}x$

$=\sec x+2\mathrm{e}^x+c$

c $\int\left(\frac{1+\cos x}{\sin^2 x}+\frac{1+x}{x^2}\right)\mathrm{d}x$

$=\int(\operatorname{cosec}^2 x+\cot x\operatorname{cosec} x+x^{-2}+x^{-1})\mathrm{d}x$

$=-\cot x-\operatorname{cosec} x-\frac{1}{x}+1\mathrm{n}\,|x|+c$

2 d $\int\left(\frac{1}{\sin^2 x}+\frac{1}{x}\right)\mathrm{d}x$

$=\int\left(\operatorname{cosec}^2 x+\frac{1}{x}\right)\mathrm{d}x$

$=-\cot x+1\mathrm{n}\,|x|+c$

2 e $\int \sin x(1+\sec^2 x)\mathrm{d}x$

$=\int (\sin x+\sin x\sec^2 x)\mathrm{d}x$

$=\int (\sin x+\tan x\sec x)\mathrm{d}x$

$=-\cos x+\sec x+c$

f $\int \cos x(1+\operatorname{cosec}^2 x)\mathrm{d}x$

$=\int (\cos x+\cos x\operatorname{cosec}^2 x)\mathrm{d}x$

$=\int (\cos x+\cot x\operatorname{cosec} x)\mathrm{d}x$

$=\sin x-\operatorname{cosec} x+c$

g $\int \operatorname{cosec}^2 x(1+\tan^2 x)\mathrm{d}x$

$=\int (\operatorname{cosec}^2 x+\operatorname{cosec}^2 x\tan^2 x)\mathrm{d}x$

$=\int (\operatorname{cosec}^2 x+\sec^2 x)\mathrm{d}x$

$=-\cot x+\tan x+c$

h $\int \sec^2 x(1-\cot^2 x)\mathrm{d}x$

$=\int (\sec^2 x-\sec^2 x\cot^2 x)\mathrm{d}x$

$=\int (\sec^2 x-\operatorname{cosec}^2 x)\mathrm{d}x$

$=\tan x+\cot x+c$

i $\int \sec^2 x(1+\mathrm{e}^x\cos^2 x)\mathrm{d}x$

$=\int (\sec^2 x+\mathrm{e}^x\cos^2 x\sec^2 x)\mathrm{d}x$

$=\int (\sec^2 x+\mathrm{e}^x)\mathrm{d}x$

$=\tan x+\mathrm{e}^x+c$

j $\int\left(\dfrac{1+\sin x}{\cos^2 x}+\cos^2 x\sec x\right)\mathrm{d}x$

$=\int (\sec^2 x+\tan x\sec x+\cos x)\mathrm{d}x$

$=\tan x+\sec x+\sin x+c$

3 a $\int_3^7 2\mathrm{e}^x\mathrm{d}x=\left[2\mathrm{e}^x\right]_3^7=2\mathrm{e}^7-2\mathrm{e}^3$

b $\int_1^6\dfrac{1+x}{x^3}\mathrm{d}x=\int_1^6\left(\dfrac{1}{x^3}+\dfrac{1}{x^2}\right)\mathrm{d}x$

$=\left[-\dfrac{1}{2x^2}-\dfrac{1}{x}\right]_1^6=\left(-\dfrac{1}{72}-\dfrac{1}{6}\right)-\left(-\dfrac{1}{2}-1\right)$

$=-\dfrac{13}{72}+\dfrac{108}{72}=\dfrac{95}{72}$

c $\int_{\frac{\pi}{2}}^{\pi}-5\sin x\mathrm{d}x=\left[5\cos x\right]_{\frac{\pi}{2}}^{\pi}$

$=5\cos\pi-5\cos\dfrac{\pi}{2}=-5-0=-5$

d $\int_{-\frac{\pi}{4}}^{0}\sec x(\sec x+\tan x)\mathrm{d}x$

$=\int_{-\frac{\pi}{4}}^{0}(\sec^2 x+\sec x\tan x)\mathrm{d}x$

$=\left[\tan x+\sec x\right]_{-\frac{\pi}{4}}^{0}=(0+1)-\left(-1+\sqrt{2}\right)$

$=2-\sqrt{2}$

4 $\int_a^{2a}\dfrac{3x-1}{x}\mathrm{d}x=\int_a^{2a}3-\dfrac{1}{x}\mathrm{d}x=\left[3x-\ln|x|\right]_a^{2a}$

$=(6a-\ln 2a)-(3a-\ln a)$ (a is positive)

$=3a-\ln 2a+\ln a$

$=3a-(\ln 2+\ln a)+\ln a$

$=3a-\ln 2$

$=3a+\ln\left(\dfrac{1}{2}\right)=6+\ln\left(\dfrac{1}{2}\right)$

so $a=2$.

5 $\int_{\ln 1}^{\ln a}\mathrm{e}^x+\mathrm{e}^{-x}\mathrm{d}x=\left[\mathrm{e}^x-\mathrm{e}^{-x}\right]_{\ln 1}^{\ln a}$

$=(\mathrm{e}^{\ln a}-\mathrm{e}^{-\ln a})-(\mathrm{e}^{\ln 1}-\mathrm{e}^{-\ln 1})$

$=\left(a-\dfrac{1}{a}\right)-(1-1)$

So $a-\dfrac{1}{a}=\dfrac{48}{7}$

$7a^2-48a-7=0$

$(7a+1)(a-7)=0$

$a=7$ since $a>0$.

6 $\int_2^b(3\mathrm{e}^x+6\mathrm{e}^{-2x})\mathrm{d}x=\left[3\mathrm{e}^x-3\mathrm{e}^{-2x}\right]_2^b$

$=3((\mathrm{e}^b-\mathrm{e}^{-2b})-(\mathrm{e}^2-\mathrm{e}^{-4}))=0$

$(\mathrm{e}^b-\mathrm{e}^{-2b})-(\mathrm{e}^2-\mathrm{e}^{-4})=0$

so $b=2$.

7 a $\mathrm{f}(x)=\dfrac{1}{8}x^{\frac{3}{2}}-\dfrac{4}{x}=0$

$\dfrac{1}{8}x^{\frac{5}{2}}-4=0$ since $x\neq 0$

$x^{\frac{5}{2}}=32\Rightarrow x=4$

7 b $\int\left(\frac{1}{8}x^{\frac{3}{2}}-\frac{4}{x}\right)\mathrm{d}x=\frac{x^{\frac{5}{2}}}{20}-4\ln|x|+c$

c $\int_1^4 \mathrm{f}(x)\mathrm{d}x=\left[\frac{x^{\frac{5}{2}}}{20}-4\ln|x|\right]_1^4$

$=\left(\frac{32}{20}-4\ln 4\right)-\left(\frac{1}{20}-4\ln 1\right)$

$=\frac{31}{20}-4\ln 4$

Integration 11B

1 a $\int \sin(2x+1)\mathrm{d}x = -\frac{1}{2}\cos(2x+1)+c$

b $\int 3\mathrm{e}^{2x}\mathrm{d}x = \frac{3}{2}\mathrm{e}^{2x}+c$

c $\int 4\mathrm{e}^{x+5}\mathrm{d}x = 4\mathrm{e}^{x+5}+c$

d $\int \cos(1-2x)\mathrm{d}x = -\frac{1}{2}\sin(1-2x)+c$

OR Let $y=\sin(1-2x)$

then $\frac{\mathrm{d}y}{\mathrm{d}x} = \cos(1-2x)\times(-2)$ (by chain rule)

$\therefore \int \cos(1-2x)\mathrm{d}x = -\frac{1}{2}\sin(1-2x)+c$

e $\int \operatorname{cosec}^2 3x\,\mathrm{d}x = -\frac{1}{3}\cot 3x + c$

f $\int \sec 4x\tan 4x\,\mathrm{d}x = \frac{1}{4}\sec 4x + c$

g $\int 3\sin\left(\frac{1}{2}x+1\right)\mathrm{d}x = -6\cos\left(\frac{1}{2}x+1\right)+c$

h $\int \sec^2(2-x)\mathrm{d}x = -\tan(2-x)+c$

OR Let $y=\tan(2-x)$

then $\frac{\mathrm{d}y}{\mathrm{d}x} = \sec^2(2-x)\times(-1)$ (by chain rule)

$\therefore \int \sec^2(2-x)\mathrm{d}x = -\tan(2-x)+c$

i $\int \operatorname{cosec} 2x\cot 2x\,\mathrm{d}x = -\frac{1}{2}\operatorname{cosec} 2x + c$

j $\int(\cos 3x - \sin 3x)\mathrm{d}x$

$= \frac{1}{3}\sin 3x + \frac{1}{3}\cos 3x + c$

$= \frac{1}{3}(\sin 3x + \cos 3x)+c$

2 a $\int\left(\mathrm{e}^{2x} - \frac{1}{2}\sin(2x-1)\right)\mathrm{d}x$

$= \frac{1}{2}\mathrm{e}^{2x} + \frac{1}{4}\cos(2x-1)+c$

b $\int(\mathrm{e}^{x}+1)^2\,\mathrm{d}x$

$= \int(\mathrm{e}^{2x}+2\mathrm{e}^{x}+1)\mathrm{d}x$

$= \frac{1}{2}\mathrm{e}^{2x}+2\mathrm{e}^{x}+x+c$

c $\int \sec^2 2x(1+\sin 2x)\mathrm{d}x$

$= \int(\sec^2 2x + \sec^2 2x\sin 2x)\mathrm{d}x$

$= \int(\sec^2 2x + \sec 2x\tan 2x)\mathrm{d}x$

$= \frac{1}{2}\tan 2x + \frac{1}{2}\sec 2x + c$

d $\int \frac{3-2\cos\left(\frac{1}{2}x\right)}{\sin^2\left(\frac{1}{2}x\right)}\mathrm{d}x$

$= \int\left(3\operatorname{cosec}^2\frac{1}{2}x - 2\operatorname{cosec}\frac{1}{2}x\cot\frac{1}{2}x\right)\mathrm{d}x$

$= -6\cot\left(\frac{1}{2}x\right)+4\operatorname{cosec}\left(\frac{1}{2}x\right)+c$

2 e $\int(\mathrm{e}^{3-x}+\sin(3-x)+\cos(3-x))\mathrm{d}x$

$= -\mathrm{e}^{3-x}+\cos(3-x)-\sin(3-x)+c$

Note: extra minus signs from $-x$ terms and chain rule.

3 a $\int\frac{1}{2x+1}\mathrm{d}x = \frac{1}{2}\ln|2x+1|+c$

b $\int\frac{1}{(2x+1)^2}\mathrm{d}x$

$= \int(2x+1)^{-2}\mathrm{d}x$

$= \frac{(2x+1)^{-1}}{-1}\times\frac{1}{2}+c$

$= -\frac{1}{2(2x+1)}+c$

c $\int (2x+1)^2\,dx$

$$= \frac{(2x+1)^3}{3}\times\frac{1}{2}+c$$

$$= \frac{(2x+1)^3}{6}+c$$

d $\int \frac{3}{4x-1}\,dx = \frac{3}{4}\ln|4x-1|+c$

e $\int \frac{3}{1-4x}\,dx$

$$= -\int \frac{3}{4x-1}\,dx$$

$$= -\frac{3}{4}\ln|4x-1|+c$$

OR Let $y = \ln|1-4x|$

then $\frac{dy}{dx} = \frac{1}{1-4x}\times(-4)$ (by chain rule)

$\therefore \int \frac{3}{1-4x}\,dx = -\frac{3}{4}\ln|1-4x|+c$

Note: $\ln|1-4x| = \ln|4x-1|$ because of $|\;|$ sign.

f $\int \frac{3}{(1-4x)^2}\,dx$

$$= \int 3(1-4x)^{-2}\,dx$$

$$= \frac{3}{-4}\times\frac{(1-4x)^{-1}}{-1}+c$$

$$= \frac{3}{4(1-4x)}+c$$

g $\int (3x+2)^5\,dx = \frac{(3x+2)^6}{18}+c$

h $\int \frac{3}{(1-2x)^3}\,dx = \frac{3}{-2}\times\frac{(1-2x)^{-2}}{-2}+c$

$$= \frac{3}{4(1-2x)^2}+c$$

OR Let $y = (1-2x)^{-2}$

then $\frac{dy}{dx} = -2(1-2x)^{-3}\times(-2)$
(by chain rule)

$\therefore \int \frac{3}{(1-2x)^3}\,dx = \frac{3}{4}(1-2x)^{-2}+c$

4 a $\int \left(3\sin(2x+1)+\frac{4}{2x+1}\right)dx$

$$= -\frac{3}{2}\cos(2x+1)+\frac{4}{2}\ln|2x+1|+c$$

$$= -\frac{3}{2}\cos(2x+1)+2\ln|2x+1|+c$$

b $\int \left(e^{5x}+(1-x)^5\right)dx = \int e^{5x}dx+\int (1-x)^5dx$

$$= \frac{1}{5}e^{5x}-\frac{1}{6}(1-x)^6+c$$

OR Let $y = (1-x)^6$

then $\frac{dy}{dx} = 6(1-x)^5\times(-1)$ (by chain rule)

$\therefore \int (1-x)^5\,dx = -\frac{1}{6}(1-x)^6+c$

c $\int \left(\frac{1}{\sin^2 2x}+\frac{1}{1+2x}+\frac{1}{(1+2x)^2}\right)dx$

$$= \int \left(\operatorname{cosec}^2 2x+\frac{1}{1+2x}+(1+2x)^{-2}\right)dx$$

$$= -\frac{1}{2}\cot 2x+\frac{1}{2}\ln|1+2x|$$

$$+\frac{(1+2x)^{-1}}{-1}\times\frac{1}{2}+c$$

$$= -\frac{1}{2}\cot 2x+\frac{1}{2}\ln|1+2x|-\frac{1}{2(1+2x)}+c$$

d $\int\left((3x+2)^2+\frac{1}{(3x+2)^2}\right)\mathrm{d}x$

$$=\int\left((3x+2)^2+(3x+2)^{-2}\right)\mathrm{d}x$$

$$=\frac{(3x+2)^3}{9}-\frac{(3x+2)^{-1}}{3}+c$$

$$=\frac{(3x+2)^3}{9}-\frac{1}{3(3x+2)}+c$$

5 a $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\cos(\pi-2x)\mathrm{d}x=\left[-\frac{1}{2}\sin(\pi-2x)\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$

$$=\left(-\frac{1}{2}\sin\left(-\frac{\pi}{2}\right)\right)-\left(-\frac{1}{2}\sin\frac{\pi}{2}\right)$$

$$=\frac{1}{2}+\frac{1}{2}=1$$

b $\int_{\frac{1}{2}}^{1}\frac{12}{(3-2x)^4}\mathrm{d}x$

Consider $y=\frac{1}{(3-2x)^3}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{6}{(3-2x)^4}$$

So $\int_{\frac{1}{2}}^{1}\frac{12}{(3-2x)^4}\mathrm{d}x=\left[\frac{2}{(3-2x)^3}\right]_{\frac{1}{2}}^{1}$

$$=2-\frac{1}{4}=\frac{7}{4}$$

c $\int_{\frac{2\pi}{9}}^{\frac{5\pi}{18}}\sec^2(\pi-3x)\mathrm{d}x=\left[-\frac{1}{3}\tan(\pi-3x)\right]_{\frac{2\pi}{9}}^{\frac{5\pi}{18}}$

$$=\left(-\frac{1}{3}\tan\left(\pi-\frac{15\pi}{18}\right)\right)-\left(-\frac{1}{3}\tan\left(\pi-\frac{6\pi}{9}\right)\right)$$

$$=\left(-\frac{1}{3}\tan\frac{\pi}{6}\right)-\left(-\frac{1}{3}\tan\frac{\pi}{3}\right)$$

$$=\left(-\frac{1}{3}\times\frac{1}{\sqrt{3}}\right)-\left(-\frac{1}{3}\times\sqrt{3}\right)$$

$$=-\frac{\sqrt{3}}{9}+\frac{\sqrt{3}}{3}=\frac{2\sqrt{3}}{9}$$

d $\int_{2}^{3}\frac{5}{7-2x}\mathrm{d}x=\left[-\frac{5}{2}\ln|7-2x|\right]_{2}^{3}$

$$=\left(-\frac{5}{2}\ln 1\right)-\left(-\frac{5}{2}\ln 3\right)$$

$$=\frac{5}{2}\ln 3$$

6 $\int_{3}^{b}(2x-6)^2\mathrm{d}x=\int_{3}^{b}(4x^2-24x+36)\mathrm{d}x$

$$\left[\frac{4x^3}{3}-12x^2+36x\right]_{3}^{b}=36$$

$$\left(\frac{4b^3}{3}-12b^2+36b\right)-(36-108+108)=36$$

$$\frac{4b^3}{3}-12b^2+36b-72=0$$

$$b^3-9b^2+27b-54=0$$

$$(b-6)(b^2-3b+9)=0$$

$b=6$ since $b^2-3b+9>0$.

7 $\int_{\mathrm{e}^2}^{\mathrm{e}^8}\frac{\mathrm{d}x}{kx}=\left[\frac{1}{k}\ln x\right]_{\mathrm{e}^2}^{\mathrm{e}^8}=\frac{1}{4}$

$$\frac{8}{k}-\frac{2}{k}=\frac{1}{4}$$

$$k=32-8=24$$

8 $\int_{\frac{\pi}{4k}}^{\frac{\pi}{3k}}(1-\pi\sin kx)\mathrm{d}x=\left[x+\frac{\pi}{k}\cos kx\right]_{\frac{\pi}{4k}}^{\frac{\pi}{3k}}$

$$=\left(\frac{\pi}{3k}+\frac{\pi}{k}\cos\frac{\pi}{3}\right)-\left(\frac{\pi}{4k}+\frac{\pi}{k}\cos\frac{\pi}{4}\right)$$

$$=\left(\frac{\pi}{3k}+\frac{\pi}{2k}\right)-\left(\frac{\pi}{4k}+\frac{\pi}{\sqrt{2}k}\right)$$

$$=\frac{\pi}{k}\left(\frac{1}{3}+\frac{1}{2}\right)-\left(\frac{1}{4}+\frac{1}{\sqrt{2}}\right)=\frac{\pi}{k}\left(\frac{7}{12}-\frac{\sqrt{2}}{2}\right)$$

$$\frac{\pi}{k}\left(\frac{7}{12}-\frac{\sqrt{2}}{2}\right)=\pi(7-6\sqrt{2})$$

$$\frac{\pi}{k}\left(\frac{7-6\sqrt{2}}{12}\right)=\pi(7-6\sqrt{2})$$

$$k=\frac{1}{12}$$

Challenge

$$\int_5^{11} \frac{1}{ax+b}\,\mathrm{d}x = \left[\frac{1}{a}\ln|ax+b| + \frac{1}{a}\ln k\right]_5^{11}$$

where $\frac{1}{a}\ln k$ is a constant

$$= \frac{1}{a}\left[\ln k|ax+b|\right]_5^{11}$$

$$= \frac{1}{a}(\ln k|11a+b| - \ln k|5a+b|)$$

$$= \frac{1}{a}(\ln k|11a+b| - k\ln|5a+b|)$$

So $\ln k|11a+b| - \ln k|5a+b| = \ln\left(\frac{41}{17}\right)$

$$\ln\left|\frac{11a+b}{5a+b}\right| = \ln\left(\frac{41}{17}\right)$$

$$\frac{11a+b}{5a+b} = \pm\frac{41}{17}$$

Case 1:

$$\frac{11a+b}{5a+b} = \frac{41}{17}$$

$187a + 17b = 205a + 41b$

$18a = -24b$

$3a = -4b$

So a must be a multiple of 4 between 0 and 10.

$a = 4 \Rightarrow b = -3$

$a = 8 \Rightarrow b = -6$

Case 2:

$$\frac{11a+b}{5a+b} = -\frac{41}{17}$$

$187a + 17b = -205a - 41b$

$392a = -58b$

$b = -\frac{196}{29}a$

But this cannot be an integer, since $a < 29$, so case 2 gives no possible solutions.

Therefore the only two possible solutions are $a = 4$, $b = -3$ and $a = 8$, $b = -6$.

Integration 11C

1 a $\int \cot^2 x \, \mathrm{d}x = \int (\operatorname{cosec}^2 x - 1)\mathrm{d}x$

$= -\cot x - x + c$

b $\int \cos^2 x \, \mathrm{d}x = \int \frac{1}{2}(1+\cos 2x)\,\mathrm{d}x$

$= \frac{1}{2}x + \frac{1}{4}\sin 2x + c$

c $\int \sin 2x \cos 2x \, \mathrm{d}x = \int \frac{1}{2}\sin 4x \, \mathrm{d}x$

$= -\frac{1}{8}\cos 4x + c$

d $\int (1+\sin x)^2 \mathrm{d}x = \int (1 + 2\sin x + \sin^2 x)\mathrm{d}x$

But $\cos 2x = 1 - 2\sin^2 x$

$\therefore \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

$\therefore \int (1+\sin x)^2 \mathrm{d}x = \int \left(\frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x\right)\mathrm{d}x$

$= \frac{3}{2}x - 2\cos x - \frac{1}{4}\sin 2x + c$

e $\int \tan^2 3x \, \mathrm{d}x = \int (\sec^2 3x - 1)\mathrm{d}x$

$= \frac{1}{3}\tan 3x - x + c$

f $\int (\cot x - \operatorname{cosec} x)^2 \, \mathrm{d}x$

$= \int (\cot^2 x - 2\cot x \operatorname{cosec} x + \operatorname{cosec}^2 x)\mathrm{d}x$

$= \int (2\operatorname{cosec}^2 x - 1 - 2\cot x \operatorname{cosec} x)\,\mathrm{d}x$

$= -2\cot x - x + 2\operatorname{cosec} x + c$

g $\int (\sin x + \cos x)^2 \mathrm{d}x$

$= \int (\sin^2 x + 2\sin x \cos x + \cos^2 x)\mathrm{d}x$

$= \int (1 + \sin 2x)\mathrm{d}x$

$= x - \frac{1}{2}\cos 2x + c$

h $\int \sin^2 x \cos^2 x \, \mathrm{d}x = \int \left(\frac{1}{2}\sin 2x\right)^2 \, \mathrm{d}x$

$= \int \frac{1}{4}\sin^2 2x \, \mathrm{d}x$

$= \int \frac{1}{4}\left(\frac{1}{2} - \frac{1}{2}\cos 4x\right)\mathrm{d}x$

$= \int \left(\frac{1}{8} - \frac{1}{8}\cos 4x\right)\mathrm{d}x$

$= \frac{1}{8}x - \frac{1}{32}\sin 4x + c$

i $\frac{1}{\sin^2 x \cos^2 x} = \frac{1}{\left(\frac{1}{2}\sin 2x\right)^2} = 4\operatorname{cosec}^2 2x$

$\therefore \int \frac{1}{\sin^2 x \cos^2 x}\mathrm{d}x = \int 4\operatorname{cosec}^2 2x \, \mathrm{d}x$

$= -2\cot 2x + c$

j $\int (\cos 2x - 1)^2 \mathrm{d}x$

$= \int (\cos^2 2x - 2\cos 2x + 1)\mathrm{d}x$

$= \int \left(\frac{1}{2}\cos 4x + \frac{1}{2} - 2\cos 2x + 1\right)\mathrm{d}x$

$= \int \left(\frac{1}{2}\cos 4x + \frac{3}{2} - 2\cos 2x\right)\mathrm{d}x$

$= \frac{1}{8}\sin 4x + \frac{3}{2}x - \sin 2x + c$

2 a $\int \left(\frac{1-\sin x}{\cos^2 x}\right)\mathrm{d}x = \int (\sec^2 x - \tan x \sec x)\mathrm{d}x$

$= \tan x - \sec x + c$

b $\int \left(\frac{1+\cos x}{\sin^2 x}\right)\mathrm{d}x = \int (\operatorname{cosec}^2 x + \cot x \operatorname{cosec} x)\mathrm{d}x$

$= -\cot x - \operatorname{cosec} x + c$

c $\int \frac{\cos 2x}{\cos^2 x}\mathrm{d}x = \int \frac{2\cos^2 x - 1}{\cos^2 x}\mathrm{d}x$

$= \int (2 - \sec^2 x)\mathrm{d}x$

$= 2x - \tan x + c$

d $$\int \frac{\cos^2 x}{\sin^2 x}\mathrm{d}x = \int \cot^2 x \,\mathrm{d}x$$
$$= \int (\operatorname{cosec}^2 x - 1)\mathrm{d}x$$
$$= -\cot x - x + c$$

e $$I = \int \frac{(1+\cos x)^2}{\sin^2 x}\mathrm{d}x = \int \frac{1+2\cos x+\cos^2 x}{\sin^2 x}\mathrm{d}x$$
$$= \int (\operatorname{cosec}^2 x + 2\cot x \operatorname{cosec} x + \cot^2 x)\mathrm{d}x$$

But $\operatorname{cosec}^2 x = 1 + \cot^2 x$

$\Rightarrow \cot^2 x = \operatorname{cosec}^2 x - 1$

$$\therefore I = \int (2\operatorname{cosec}^2 x - 1 + 2\cot x \operatorname{cosec} x)\mathrm{d}x$$
$$= -2\cot x - x - 2\operatorname{cosec} x + c$$

f $$\int (\cot x - \tan x)^2 \mathrm{d}x$$
$$= \int (\cot^2 x - 2\cot x \tan x + \tan^2 x)\mathrm{d}x$$
$$= \int (\operatorname{cosec}^2 x - 1 - 2 + \sec^2 x - 1)\mathrm{d}x$$
$$= \int (\operatorname{cosec}^2 x - 4 + \sec^2 x)\mathrm{d}x$$
$$= -\cot x - 4x + \tan x + c$$

g $$\int (\cos x - \sin x)^2 \mathrm{d}x$$
$$= \int (\cos^2 x - 2\cos x \sin x + \sin^2 x)\mathrm{d}x$$
$$= \int (1 - \sin 2x)\mathrm{d}x$$
$$= x + \frac{1}{2}\cos 2x + c$$

h $$\int (\cos x - \sec x)^2 \mathrm{d}x$$
$$= \int (\cos^2 x - 2\cos x \sec x + \sec^2 x)\mathrm{d}x$$
$$= \int \left(\frac{1}{2}\cos 2x + \frac{1}{2} - 2 + \sec^2 x\right)\mathrm{d}x$$
$$= \int \left(\frac{1}{2}\cos 2x - \frac{3}{2} + \sec^2 x\right)\mathrm{d}x$$
$$= \frac{1}{4}\sin 2x - \frac{3}{2}x + \tan x + c$$

i $$\int \frac{\cos 2x}{1-\cos^2 2x}\mathrm{d}x = \int \frac{\cos 2x}{\sin^2 2x}\mathrm{d}x$$
$$= \int \cot 2x \operatorname{cosec} 2x \,\mathrm{d}x$$
$$= -\frac{1}{2}\operatorname{cosec}\, 2x + c$$

3 $$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 x\mathrm{d}x = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 - \cos 2x)\mathrm{d}x$$
$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$
$$= \frac{1}{2}\left(\left(\frac{\pi}{2} - 0\right) - \left(\frac{\pi}{4} - \frac{1}{2}\right)\right) = \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right)$$
$$= \frac{2+\pi}{8}$$

4 a $$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin^2 x \cos^2 x}\mathrm{d}x = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{4}{\sin^2 2x}\mathrm{d}x$$
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 4\operatorname{cosec}^2 2x\mathrm{d}x = \left[-2\cot 2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$
$$= \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$$

b $$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\sin x - \operatorname{cosec} x)^2 \mathrm{d}x$$
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\sin^2 x - 2 + \operatorname{cosec}^2 x)\mathrm{d}x$$
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\frac{1}{2}(1 - \cos 2x) - 2 + \operatorname{cosec}^2 x\right)\mathrm{d}x$$
$$= \left[\frac{x}{2} - \frac{1}{4}\sin 2x - 2x - \cot x\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$
$$= \left(\frac{\pi}{8} - \frac{1}{4} - \frac{\pi}{2} - 1\right) - \left(\frac{\pi}{12} - \frac{\sqrt{3}}{8} - \frac{\pi}{3} - \sqrt{3}\right)$$
$$= \frac{27\sqrt{3} - 30 - 3\pi}{24}$$
$$= \frac{9\sqrt{3} - 10 - \pi}{8}$$

c $$\int_0^{\frac{\pi}{4}} \frac{(1+\sin x)^2}{\cos^2 x}\mathrm{d}x$$
$$= \int_0^{\frac{\pi}{4}} \frac{(1 + 2\sin x + \sin^2 x)}{\cos^2 x}\mathrm{d}x$$
$$= \int_0^{\frac{\pi}{4}} \left(2\sec^2 x + 2\sec x \tan x - 1\right)\mathrm{d}x$$
$$= \left[2\tan x + 2\sec x - x\right]_0^{\frac{\pi}{4}}$$
$$= \left(2 + 2\sqrt{2} - \frac{\pi}{4}\right) - 2 = 2\sqrt{2} - \frac{\pi}{4}$$

d $$\int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \frac{\sin 2x}{1-\sin^2 2x}\,\mathrm{d}x$$
$$= \int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \frac{\sin 2x}{\cos^2 2x}\,\mathrm{d}x$$
$$= \int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \sec 2x \tan 2x\,\mathrm{d}x$$
$$= \left[\frac{1}{2}\sec 2x\right]_{\frac{3\pi}{8}}^{\frac{\pi}{2}}$$
$$= -\frac{1}{2} + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}-1}{2}$$

5 **a** $$\sin(3x+2x) = \sin 3x \cos 2x + \cos 3x \sin 2x$$
$$\sin(3x-2x) = \sin 3x \cos 2x - \cos 3x \sin 2x$$

Adding the above,
$$\sin 5x + \sin x = 2\sin 3x \cos 2x$$

b $$\int \sin 3x \cos 2x\,\mathrm{d}x = \frac{1}{2}\int (\sin 5x + \sin x)\,\mathrm{d}x$$
$$= \frac{1}{2}\left(-\frac{1}{5}\cos 5x - \cos x\right) + c$$
$$= -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + c$$

6 **a** $$\mathrm{f}(x) = 5\sin^2 x + 7\cos^2 x$$
$$= 5\sin^2 x + 7 - 7\sin^2 x$$
$$= 7 - 2\sin^2 x$$
$$= 7 - 2\left(\frac{1}{2}(1-\cos 2x)\right)$$
$$= 7 - 1 + 2\cos 2x$$
$$= \cos 2x + 6$$

6 **b** $$\int_0^{\frac{\pi}{4}} \mathrm{f}(x)\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} (\cos 2x + 6)\,\mathrm{d}x$$
$$= \left[\frac{1}{2}\sin 2x + 6x\right]_0^{\frac{\pi}{4}}$$
$$= \frac{1}{2}(1+3\pi)$$

7 **a** $$\cos^4 x \equiv \left(\cos^2 x\right)^2 \equiv \left(\frac{1}{2}(\cos 2x + 1)\right)^2$$
$$\equiv \frac{1}{4}\left(\cos^2 2x + 2\cos 2x + 1\right)$$
$$\equiv \frac{1}{4}\left(\frac{1}{2}\left(\cos 4x + 1\right) + 2\cos 2x + 1\right)$$
$$\equiv \frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x + \frac{3}{8}$$

b $$\int \cos^4 x\,\mathrm{d}x = \int\left(\frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x + \frac{3}{8}\right)\mathrm{d}x$$
$$= \frac{1}{32}\sin 4x + \frac{1}{4}\sin 2x + \frac{3x}{8} + c$$

Integration 11D

1 **a** $y = \ln|x^2+4|$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^2+4} \times 2x \qquad \text{(chain rule)}$$

$$\therefore \int \frac{x}{x^2+4}\,\mathrm{d}x = \frac{1}{2}\ln|x^2+4| + c$$

b $y = \ln|\mathrm{e}^{2x}+1|$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{e}^{2x}+1} \times \mathrm{e}^{2x} \times 2 \qquad \text{(chain rule)}$$

$$\therefore \int \frac{\mathrm{e}^{2x}}{\mathrm{e}^{2x}+1}\,\mathrm{d}x = \frac{1}{2}\ln|\mathrm{e}^{2x}+1| + c$$

c $y = (x^2+4)^{-2}$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -2(x^2+4)^{-3} \times 2x \qquad \text{(chain rule)}$$

$$\therefore \int \frac{x}{(x^2+4)^3}\,\mathrm{d}x = -\frac{1}{4}(x^2+4)^{-2} + c$$

$$\text{or } -\frac{1}{4(x^2+4)^2} + c$$

d $y = (\mathrm{e}^{2x}+1)^{-2}$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -2(\mathrm{e}^{2x}+1)^{-3} \times \mathrm{e}^{2x} \times 2 \qquad \text{(chain rule)}$$

$$\therefore \int \frac{\mathrm{e}^{2x}}{(\mathrm{e}^{2x}+1)^3}\,\mathrm{d}x = -\frac{1}{4}(\mathrm{e}^{2x}+1)^{-2} + c$$

$$\text{or } -\frac{1}{4(\mathrm{e}^{2x}+1)^2} + c$$

e $y = \ln|3+\sin 2x|$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3+\sin 2x} \times \cos 2x \times 2 \qquad \text{(chain rule)}$$

$$\therefore \int \frac{\cos 2x}{3+\sin 2x}\,\mathrm{d}x = \frac{1}{2}\ln|3+\sin 2x| + c$$

f $y = (3+\cos 2x)^{-2}$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -2(3+\cos 2x)^{-3} \times (-\sin 2x) \times 2$$

(chain rule)

$$\therefore \int \frac{\sin 2x}{(3+\cos 2x)^3}\,\mathrm{d}x = \frac{1}{4}(3+\cos 2x)^{-2} + c$$

$$\text{or } \frac{1}{4(3+\cos 2x)^2} + c$$

g $y = \mathrm{e}^{x^2}$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{x^2} \times 2x \qquad \text{(chain rule)}$$

$$\therefore \int x\mathrm{e}^{x^2}\,\mathrm{d}x = \frac{1}{2}\mathrm{e}^{x^2} + c$$

h $y = (1+\sin 2x)^5$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 5(1+\sin 2x)^4 \times \cos 2x \times 2$$

(chain rule)

$$\therefore \int \cos 2x(1+\sin 2x)^4\,\mathrm{d}x = \frac{1}{10}(1+\sin 2x)^5 + c$$

i $y = \tan^3 x$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3\tan^2 x \times \sec^2 x \quad \text{(chain rule)}$$

$$\therefore \int \sec^2 x \tan^2 x\,\mathrm{d}x = \frac{1}{3}\tan^3 x + c$$

j $\sec^2 x(1+\tan^2 x) = \sec^2 x + \sec^2 x\tan^2 x$

$$\therefore \int \sec^2 x(1+\tan^2 x)\,\mathrm{d}x$$

$$= \int \sec^2 x + \sec^2 x\tan^2 x\,\mathrm{d}x$$

$$= \tan x + \frac{1}{3}\tan^3 x + c$$

2 a $y=(x^2+2x+3)^5$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=5(x^2+2x+3)^4\times(2x+2)$$

$$=5(x^2+2x+3)^4\times 2(x+1)$$

$$\therefore \int (x+1)(x^2+2x+3)^4\,\mathrm{d}x$$

$$=\frac{1}{10}(x^2+2x+3)^5+c$$

b $y=\cot^2 2x$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=2\cot 2x\times(-\mathrm{cosec}^2\,2x)\times 2$$

$$=-4\,\mathrm{cosec}^2\,2x\cot 2x$$

$$\therefore \int \mathrm{cosec}^2\,2x\cot 2x\,\mathrm{d}x=-\frac{1}{4}\cot^2 2x+c$$

c $y=\sin^6 3x$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=6\sin^5 3x\times\cos 3x\times 3$$

$$\therefore \int \sin^5 3x\cos 3x\,\mathrm{d}x=\frac{1}{18}\sin^6 3x+c$$

d $y=\mathrm{e}^{\sin x}$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^{\sin x}\times\cos x$$

$$\therefore \int \cos x\,\mathrm{e}^{\sin x}\,\mathrm{d}x=\mathrm{e}^{\sin x}+c$$

e $y=\text{1n}\left|\mathrm{e}^{2x}+3\right|$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\mathrm{e}^{2x}+3}\times\mathrm{e}^{2x}\times 2$$

$$\therefore \int \frac{\mathrm{e}^{2x}}{\mathrm{e}^{2x}+3}\,\mathrm{d}x=\frac{1}{2}\text{1n}\left|\mathrm{e}^{2x}+3\right|+c$$

f $y=(x^2+1)^{\frac{5}{2}}$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{5}{2}(x^2+1)^{\frac{3}{2}}\times 2x=5x(x^2+1)^{\frac{3}{2}}$$

$$\therefore \int x(x^2+1)^{\frac{3}{2}}\,\mathrm{d}x=\frac{1}{5}(x^2+1)^{\frac{5}{2}}+c$$

g $y=(x^2+x+5)^{\frac{3}{2}}$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3}{2}(x^2+x+5)^{\frac{1}{2}}\times(2x+1)$$

$$\therefore \int (2x+1)\sqrt{x^2+x+5}\,\mathrm{d}x=\frac{2}{3}(x^2+x+5)^{\frac{3}{2}}+c$$

h $y=(x^2+x+5)^{\frac{1}{2}}$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}(x^2+x+5)^{-\frac{1}{2}}\times(2x+1)$$

$$=\frac{1}{2}\frac{(2x+1)}{\sqrt{x^2+x+5}}$$

$$\therefore \int \frac{2x+1}{\sqrt{x^2+x+5}}\,\mathrm{d}x=2(x^2+x+5)^{\frac{1}{2}}+c$$

i $y=(\cos 2x+3)^{\frac{1}{2}}$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}(\cos 2x+3)^{-\frac{1}{2}}\times(-\sin 2x)\times 2$$

$$=-\frac{\sin 2x}{\sqrt{\cos 2x+3}}$$

$$=-\frac{2\sin x\cos x}{\sqrt{\cos 2x+3}}$$

$$\therefore \int \frac{\sin x\cos x}{\sqrt{\cos 2x+3}}\,\mathrm{d}x=-\frac{1}{2}(\cos 2x+3)^{\frac{1}{2}}+c$$

j $y=\text{1n}\left|\cos 2x+3\right|$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\cos 2x+3}\times(-\sin 2x)\times 2$$

$$=-\frac{2\sin 2x}{\cos 2x+3}$$

$$=-\frac{4\sin x\cos x}{\cos 2x+3}$$

$$\therefore \int \frac{\sin x\cos x}{\cos 2x+3}\,\mathrm{d}x=-\frac{1}{4}\text{1n}\left|\cos 2x+3\right|+c$$

3 a Let $I=\int_0^3(3x^2+10x)\sqrt{x^3+5x^2+9}\,\mathrm{d}x$

Consider $y=(x^3+5x^2+9)^{\frac{3}{2}}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3}{2}(3x^2+10x)(x^3+5x^2+9)^{\frac{1}{2}}$$

$$\text{So } I=\left[\frac{2}{3}(x^3+5x^2+9)^{\frac{3}{2}}\right]_0^3$$

$$=486-18=468$$

3 b Let $I=\int_{\frac{\pi}{9}}^{\frac{2\pi}{9}}\frac{6\sin 3x}{1-\cos 3x}\,\mathrm{d}x$

Consider $y=\ln|1-\cos 3x|$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3\sin 3x}{(1-\cos 3x)}$$

So $I=\left[2\ln|1-\cos 3x|\right]_{\frac{\pi}{9}}^{\frac{2\pi}{9}}$

$$=2\left(\ln\frac{3}{2}-\ln\frac{1}{2}\right)=2\ln 3$$

c Let $I=\int_4^7\frac{x}{x^2-1}\,\mathrm{d}x$

Consider $y=\ln|x^2-1|$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x}{x^2-1}$$

So $I=\left[\frac{1}{2}\ln|x^2-1|\right]_4^7$

$$=\frac{1}{2}(\ln 48-\ln 15)$$

$$=\frac{1}{2}\ln\frac{48}{15}=\frac{1}{2}\ln\frac{16}{5}$$

d Let $I=\int_0^{\frac{\pi}{4}}\sec^2 xe^{4\tan x}\,\mathrm{d}x$

Consider $y=\mathrm{e}^{4\tan x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=4\sec^2 x\mathrm{e}^{4\tan x}$$

So $I=\left[\frac{1}{4}\mathrm{e}^{4\tan x}\right]_0^{\frac{\pi}{4}}$

$$=\frac{1}{4}\mathrm{e}^4-\frac{1}{4}\mathrm{e}^0=\frac{1}{4}(\mathrm{e}^4-1)$$

4 Let $I=\int_0^k kx^2\mathrm{e}^{x^3}\,\mathrm{d}x$

Consider $y=\mathrm{e}^{x^3}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=3x^2\mathrm{e}^{x^3}$$

So $I=\left[\frac{k}{3}\mathrm{e}^{x^3}\right]_0^k$

$$=\frac{k}{3}(\mathrm{e}^{k^3}-1)=\frac{2}{3}(\mathrm{e}^8-1)$$

$k=2$

5 Let $I=\int_0^\theta 4\sin 2x\cos^4 2x\,\mathrm{d}x$

Consider $y=\cos^5 2x$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-10\sin 2x\cos^4 2x$$

So $I=\left[-\frac{2}{5}\cos^5 2x\right]_0^\theta$

$$=\left(-\frac{2}{5}\cos^5 2\theta\right)+\frac{2}{5}=\frac{4}{5}$$

$$\cos^5 2\theta=-1\Rightarrow\cos 2\theta=-1$$

$$2\theta=\pi\Rightarrow\theta=\frac{\pi}{2}$$

6 a $\int\cot x\,\mathrm{d}x=\int\frac{\cos x}{\sin x}\,\mathrm{d}x$

Consider $y=\ln|\sin x|$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\cos x}{\sin x}$$

So $\int\cot x\,\mathrm{d}x=\ln|\sin x|+c$

b $\int\tan x\,\mathrm{d}x\equiv\int\frac{\sin x}{\cos x}\,\mathrm{d}x$

Consider $y=\ln|\cos x|$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{\sin x}{\cos x}$$

So $\int\tan x\,\mathrm{d}x=-\ln|\cos x|+c$

$$\equiv\ln\left|\frac{1}{\cos x}\right|+c$$

$$\equiv\ln|\sec x|+c$$

Integration 11E

1 a $\int x\sqrt{1+x}\ \mathrm{d}x$

Let $u = 1+x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 1$

So dx can be replaced by du.

$$\text{So } I = \int (u-1)u^{\frac{1}{2}}\ \mathrm{d}u$$

$$= \int (u^{\frac{3}{2}} - u^{\frac{1}{2}})\ \mathrm{d}u$$

$$= \frac{2u^{\frac{5}{2}}}{5} - \frac{2u^{\frac{3}{2}}}{3} + c$$

$$= \frac{2(1+x)^{\frac{5}{2}}}{5} - \frac{2(1+x)^{\frac{3}{2}}}{3} + c$$

b $\int \frac{1+\sin x}{\cos x}\ \mathrm{d}x$

Let $u = \sin x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = \cos x$

So dx can be replaced by $\frac{\mathrm{d}u}{\cos x}$.

$$\text{So } I = \int \frac{1+u}{\cos^2 x}\ \mathrm{d}u$$

$$= \int \frac{1+u}{1-\sin^2 x}\ \mathrm{d}u$$

$$= \int \frac{1+u}{1-u^2}\ \mathrm{d}u$$

$$= \int \frac{1}{1-u}\ \mathrm{d}u$$

$$= -\ln|1-u| + c$$

$$= -\ln|1-\sin x| + c$$

c $\int \sin^3 x\ \mathrm{d}x$

Let $u = \cos x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$

So $\sin x\mathrm{d}x$ can be replaced by $-\mathrm{d}u$.

Now $\sin^3 x = \sin x(1-\cos^2 x)$,

$$\text{so } I = \int \sin x(1-\cos^2 x)\ \mathrm{d}x$$

$$= \int (u^2 - 1)\ \mathrm{d}u$$

$$= \frac{u^3}{3} - u + c$$

$$= \frac{\cos^3 x}{3} - \cos x + c$$

d $\int \frac{2}{\sqrt{x}(x-4)}\ \mathrm{d}x$

Let $u = \sqrt{x}$

$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\sqrt{x}}$

So $\frac{\mathrm{d}x}{\sqrt{x}}$ can be replaced by 2du.

$$\text{So } I = \int \frac{4}{(u^2-4)}\ \mathrm{d}u$$

$$= \int \frac{4}{(u-2)(u+2)}\ \mathrm{d}u$$

$$= \int \left(\frac{1}{u-2} - \frac{1}{u+2}\right)\ \mathrm{d}u$$

$$= \ln|u-2| - \ln|u+2| + c$$

$$= \ln\left|\frac{\sqrt{x}-2}{\sqrt{x}+2}\right| + c$$

1 e $\int \sec^2 x \tan x\sqrt{1+\tan x}\ \mathrm{d}x$

Let $u^2 = 1+\tan x$

$2u\dfrac{\mathrm{d}u}{\mathrm{d}x} = \sec^2 x$

So $\sec^2 x\mathrm{d}x$ can be replaced by $2u\mathrm{d}u$.

So $I = \int 2u(u^2-1)u\ \mathrm{d}u$

$= \int \left(2u^4 - 2u^2\right)\ \mathrm{d}u$

$= \dfrac{2u^5}{5} - \dfrac{2u^3}{3} + c$

$= \dfrac{2(1+\tan x)^{\frac{5}{2}}}{5} - \dfrac{2(1+\tan x)^{\frac{3}{2}}}{3} + c$

f $\int \sec^4 x\ \mathrm{d}x$

Let $u = \tan x$

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = \sec^2 x$

So $\sec^2 x\mathrm{d}x$ can be replaced by $\mathrm{d}u$.

Now $\sec^2 x = 1+\tan^2 x$

So $I = \int (1+\tan^2 x)\sec^2 x\ \mathrm{d}x$

$= \int (1+u^2)\ \mathrm{d}u$

$= u + \dfrac{u^3}{3} + c$

$= \tan x + \dfrac{\tan^3 x}{3} + c$

2 a $\int_0^5 x\sqrt{x+4}\ \mathrm{d}x$

Let $u = x+4$

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = 1$

So $\mathrm{d}x$ can be replaced by $\mathrm{d}u$.

x	u
5	9
0	4

So $I = \int_4^9 (u-4)\sqrt{u}\ \mathrm{d}u$

$= \int_4^9 (u^{\frac{3}{2}} - 4u^{\frac{1}{2}})\ \mathrm{d}u$

$= \left[\dfrac{2}{5}u^{\frac{5}{2}} - \dfrac{8}{3}u^{\frac{3}{2}}\right]_4^9$

$= \left(\dfrac{486}{5} - \dfrac{216}{3}\right) - \left(\dfrac{64}{5} - \dfrac{64}{3}\right)$

$= \dfrac{506}{15}$

b $\int_0^2 x(2+x)^3\ \mathrm{d}x$

Let $u = 2+x$

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = 1$

So $\mathrm{d}x$ can be replaced by $\mathrm{d}u$.

x	u
2	4
0	2

So $I = \int_2^4 (u-2)u^3\ \mathrm{d}u$

$= \left[\dfrac{u^5}{5} - \dfrac{u^4}{2}\right]_2^4$

$= \left(\dfrac{1024}{5} - \dfrac{256}{2}\right) - \left(\dfrac{32}{5} - \dfrac{16}{2}\right)$

$= \dfrac{768}{10} + \dfrac{16}{10} = \dfrac{784}{10} = \dfrac{392}{5}$

2 **c** $\int_0^{\frac{\pi}{2}} \sin x\sqrt{3\cos x+1}\ \mathrm{d}x$

Let $u=\cos x$

$\frac{\mathrm{d}u}{\mathrm{d}x}=-\sin x$

So $\sin x\mathrm{d}x$ can be replaced by $-\mathrm{d}u$.

x	u
$\frac{\pi}{2}$	0
0	1

So $I=\int_1^0 -(3u+1)^{\frac{1}{2}}\ \mathrm{d}u$

Consider $y=(3u+1)^{\frac{3}{2}}$

$\frac{\mathrm{d}y}{\mathrm{d}u}=\frac{9}{2}(3u+1)^{\frac{1}{2}}$

$I=\left[-\frac{2}{9}(3u+1)^{\frac{1}{2}}\right]_1^0$

$=-\frac{2}{9}+\frac{16}{9}=\frac{14}{9}$

d $\int_0^{\frac{\pi}{3}} \sec x\tan x\sqrt{\sec x+2}\ \mathrm{d}x$

Let $u=\sec x$

$\frac{\mathrm{d}u}{\mathrm{d}x}=\sec x\tan x$

So $\sec x\tan x\mathrm{d}x$ can be replaced by $\mathrm{d}u$.

x	u
$\frac{\pi}{3}$	2
0	1

So $I=\int_1^2 (u+2)^{\frac{1}{2}}\ \mathrm{d}u$

$=\left[\frac{2}{3}(u+2)^{\frac{3}{2}}\right]_1^2$

$=\frac{16}{3}-2\sqrt{3}$

2 **e** $\int_1^4 \frac{\mathrm{d}x}{\sqrt{x}(4x-1)}$

Let $u=\sqrt{x}$

$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2\sqrt{x}}$

So $\frac{\mathrm{d}x}{\sqrt{x}}$ can be replaced by $2\mathrm{d}u$.

x	u
4	2
1	1

So $I=\int_1^2 \frac{2}{(4u^2-1)}\ \mathrm{d}u$

$=\int_1^2\left(\frac{1}{2u-1}-\frac{1}{2u+1}\right)\mathrm{d}u$

$=\left[\frac{1}{2}\ln|2u-1|-\frac{1}{2}\ln|2u+1|\right]_1^2$

$=\frac{1}{2}\ln 3-\frac{1}{2}\ln 5-\frac{1}{2}\ln 1+\frac{1}{2}\ln 3$

$=\ln 3-\frac{1}{2}\ln 5$

$=\frac{1}{2}\ln 9-\frac{1}{2}\ln 5$

$=\frac{1}{2}\ln\frac{9}{5}$

3 **a** $\int x(3+2x)^5\ \mathrm{d}x$

Let $u=3+2x$

$\frac{\mathrm{d}u}{\mathrm{d}x}=2$

So $2\mathrm{d}x$ can be replaced by $\mathrm{d}u$.

$I=\int\frac{(u-3)}{4}u^5\ \mathrm{d}u$

$=\int\frac{u^6-3u^5}{4}\ \mathrm{d}u$

$=\frac{u^7}{28}-\frac{3u^6}{24}+c$

$=\frac{(3+2x)^7}{28}-\frac{(3+2x)^6}{8}+c$

3 **b** $\int \frac{x}{\sqrt{1+x}}\,\mathrm{d}x$

Let $u = 1+x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 1$

So dx can be replaced by du.

$$I = \int \frac{(u-1)}{u^{\frac{1}{2}}}\,\mathrm{d}u$$

$$I = \int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)\mathrm{d}u$$

$$= \frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + c$$

$$= \frac{2}{3}(1+x)^{\frac{3}{2}} - 2\sqrt{1+x} + c$$

c $\int \frac{\sqrt{x^2+4}}{x}\,\mathrm{d}x$

Let $u = \sqrt{x^2+4}$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = x(x^2+4)^{-\frac{1}{2}} = \frac{x}{u}$$

$$I = \int \frac{u}{x} \times \frac{u}{x}\,\mathrm{d}u$$

$$= \int \frac{u^2}{x^2}\,\mathrm{d}u = \int \frac{u^2}{u^2-4}\,\mathrm{d}u$$

$$= \int \left(1 + \frac{1}{u-2} - \frac{1}{u+2}\right)\mathrm{d}u$$

$$= u + \left(\ln|u-2| - \ln|u+2|\right) + c$$

$$= \sqrt{x^2+4} + \ln\left|\frac{\sqrt{x^2+4}-2}{\sqrt{x^2+4}+2}\right| + c$$

4 **a** $\int_2^7 x\sqrt{2+x}\,\mathrm{d}x$

Let $u = 2+x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 1$

So dx can be replaced by du.

x	u
7	9
2	4

$$\text{So } I = \int_4^9 (u-2)\sqrt{u}\,\mathrm{d}u$$

$$= \left[\frac{2}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}}\right]_4^9$$

$$= \left(\frac{486}{5} - \frac{108}{3}\right) - \left(\frac{64}{5} - \frac{32}{3}\right)$$

$$= \frac{886}{15}$$

b $\int_2^5 \frac{1}{1+\sqrt{x-1}}\,\mathrm{d}x$

Let $u = \sqrt{x-1}$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\sqrt{x-1}} = \frac{1}{2u}$$

So dx can be replaced by $2u$du.

x	u
5	2
2	1

$$\text{So } I = \int_1^2 \frac{2u}{1+u}\,\mathrm{d}u$$

$$I = \int_1^2 2\left(1 - \frac{1}{1+u}\right)\mathrm{d}u$$

$$= \left[2u - 2\ln|1+u|\right]_1^2$$

$$= (4 - 2\ln 3) - (2 - 2\ln 2)$$

$$= 2 + 2\ln 2 - 2\ln 3$$

$$= 2 + 2\ln\frac{2}{3}$$

4 c $\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1+\cos\theta}\,\mathrm{d}\theta$

Let $u = 1+\cos\theta$

$$\frac{\mathrm{d}u}{\mathrm{d}\theta} = -\sin\theta$$

So $\mathrm{d}\theta$ can be replaced by $\frac{\mathrm{d}u}{\sin\theta}$.

θ	u
$\frac{\pi}{2}$	1
1	2

So $I = -\int_2^1 \frac{2\sin\theta\cos\theta}{u}\,\frac{\mathrm{d}u}{\sin\theta}$

$$= \int_2^1 \frac{2(1-u)}{u}\,\mathrm{d}u$$

$$= \left[2\ln|u| - 2u\right]_2^1$$

$$= -2-(2\ln 2-4)$$

$$= 2-2\ln 2$$

5 $\int_6^{20} \frac{8x}{\sqrt{4x+1}}\,\mathrm{d}x$

Let $u^2 = 4x+1$

$$2u\frac{\mathrm{d}u}{\mathrm{d}x} = 4$$

So $\mathrm{d}x$ can be replaced by $\frac{u}{2}\mathrm{d}u$.

x	u
20	9
6	5

So $I = \int_5^9 \frac{2(u^2-1)}{u}\,\frac{u}{2}\mathrm{d}u$

$$= \left[\frac{u^3}{3} - u\right]_5^9$$

$$= \left(\frac{792}{3} - 9\right) - \left(\frac{125}{3} - 5\right)$$

$$= \frac{592}{3}$$

6 $\int_{\ln 3}^{\ln 4} \frac{\mathrm{e}^{4x}}{\mathrm{e}^x - 2}\,\mathrm{d}x$

Let $u^2 = \mathrm{e}^x - 2$

$$2u\frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x = u^2+2$$

So $\mathrm{d}x$ can be replaced by $\frac{2u}{u^2+2}\mathrm{d}u$.

x	u
ln 4	$\sqrt{2}$
ln 3	1

$$I = \int_1^{\sqrt{2}} \frac{2(u^2+2)^3}{u}\,\mathrm{d}u$$

$$I = \int_1^{\sqrt{2}} \left(2u^5 + 12u^3 + 24u + \frac{16}{u}\right)\mathrm{d}u$$

$$= \left[\frac{1}{3}u^6 + 3u^4 + 12u^2 + 16\ln|u|\right]_1^{\sqrt{2}}$$

$$= \frac{70}{3} + 8\ln 2$$

$a = 70, b = 3, c = 8, d = 2$

7 $-\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$

Let $x = \cos\theta$

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\sin\theta$$

So $\mathrm{d}x$ can be replaced by $-\sin\theta\ \mathrm{d}\theta$.

$$\sqrt{1-x^2} = \sqrt{1-\cos^2\theta} = \sin\theta$$

$$I = -\int \frac{1}{\sin\theta}(-\sin\theta)\ \mathrm{d}\theta = \int 1\ \mathrm{d}\theta$$

$$= \theta + c$$

$$= \arccos x + c$$

8 $\int_0^{\frac{\pi}{3}} \sin^3 x \cos^2 x \, \mathrm{d}x$

Let $u = \cos x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$

So $\mathrm{d}x$ can be replaced by $-\frac{\mathrm{d}u}{\sin x}$.

x	u
$\frac{\pi}{3}$	$\frac{1}{2}$
0	1

$$I = \int_1^{\frac{1}{2}} -(1-u^2)u^2 \, \mathrm{d}u$$

$$I = \int_1^{\frac{1}{2}} (u^2-1)u^2 \, \mathrm{d}u$$

$$= \int_1^{\frac{1}{2}} (u^4-u^2) \, \mathrm{d}u$$

$$= \left[\frac{u^5}{5} - \frac{u^3}{3}\right]_1^{\frac{1}{2}}$$

$$= \left(\frac{1}{160} - \frac{1}{24}\right) - \left(\frac{1}{5} - \frac{1}{3}\right)$$

$$= \frac{47}{480}$$

9 $I = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} x^2\sqrt{1-x^2} \, \mathrm{d}x$

Let $x = \sin\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = \cos\theta$

x	θ
$\frac{\sqrt{3}}{2}$	$\frac{\pi}{3}$
$\frac{1}{2}$	$\frac{\pi}{6}$

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2\theta \cos^2\theta \, \mathrm{d}\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{4}\sin^2 2\theta \, \mathrm{d}\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{8}(1-\cos 4\theta) \, \mathrm{d}\theta$$

$$= \left[\frac{1}{8}\left(\theta - \frac{1}{4}\sin 4\theta\right)\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{1}{8}\left(\left(\frac{\pi}{3} + \frac{\sqrt{3}}{8}\right) - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right)\right)$$

$$= \frac{1}{8}\left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right)$$

$$= \frac{2\pi + 3\sqrt{3}}{96}$$

Challenge

$$\int \frac{1}{x^2\sqrt{9-x^2}}\,\mathrm{d}x$$

Let $x = 3\sin u$

$$\frac{\mathrm{d}x}{\mathrm{d}u} = 3\cos u$$

So $\mathrm{d}x$ can be replaced by $3\cos u\ \mathrm{d}u$.

$$\int \frac{3\cos u}{9\sin^2 u\sqrt{9-9\sin^2 u}}\,\mathrm{d}u$$

$$= \int \frac{3\cos u}{9\sin^2 u 3\cos u}\,\mathrm{d}u$$

$$= \frac{1}{9}\int \cos\mathrm{ec}^2 u\ \mathrm{d}u$$

$$= -\frac{1}{9}\cot u + c$$

$$I = -\frac{\cos u}{9\sin u} + c$$

$$\cos u = \sqrt{1-\sin^2 u} = \sqrt{1-\frac{x^2}{9}} = \frac{\sqrt{9-x^2}}{3}$$

$$I = -\frac{\frac{\sqrt{9-x^2}}{3}}{\frac{9x}{3}} + c$$

$$= -\frac{\sqrt{9-x^2}}{9x} + c$$

Integration 11F

1 a $u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x \Rightarrow v = -\cos x$$

$$\therefore \int x \sin x \,\mathrm{d}x = -x\cos x - \int -\cos x \times 1 \,\mathrm{d}x$$

$$= -x\cos x + \int \cos x \,\mathrm{d}x$$

$$= -x\cos x + \sin x + c$$

b $u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x \Rightarrow v = \mathrm{e}^x$$

$$\therefore \int x\mathrm{e}^x \,\mathrm{d}x = x\mathrm{e}^x - \int \mathrm{e}^x \times 1 \,\mathrm{d}x$$

$$= x\mathrm{e}^x - \mathrm{e}^x + c$$

c $u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sec^2 x \Rightarrow v = \tan x$$

$$\therefore \int x\sec^2 x \,\mathrm{d}x = x\tan x - \int \tan x \times 1 \,\mathrm{d}x$$

$$= x\tan x - \ln|\sec x| + c$$

d $u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sec x \tan x \Rightarrow v = \sec x$$

$$\therefore \int x\sec x \tan x \,\mathrm{d}x = x\sec x - \int \sec x \times 1 \,\mathrm{d}x$$

$$= x\sec x - \ln|\sec x + \tan x| + c$$

e $u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \operatorname{cosec}^2 x \Rightarrow v = -\cot x$$

$$\therefore \int \frac{x}{\sin^2 x} \,\mathrm{d}x = \int x\operatorname{cosec}^2 x \,\mathrm{d}x$$

$$= -x\cot x - \int -\cot x \times 1 \,\mathrm{d}x$$

$$= -x\cot x + \int \cot x \,\mathrm{d}x$$

$$= -x\cot x + \ln|\sin x| + c$$

2 a $u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = 3 \Rightarrow v = 3x$$

$$\therefore \int 3\ln x \,\mathrm{d}x = 3x\ln x - \int 3x \times \frac{1}{x} \,\mathrm{d}x$$

$$= 3x\ln x - \int 3 \,\mathrm{d}x$$

$$= 3x\ln x - 3x + c$$

b $I = \int x\ln x \,\mathrm{d}x$

Let $u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$

$$\frac{dv}{dx} = x \Rightarrow v = \frac{x^2}{2}$$

$$I = \frac{x^2}{2}\ln x - \int \frac{x^2}{2} \times \frac{1}{x} \,\mathrm{d}x$$

$$= \frac{x^2}{2}\ln x - \int \frac{x}{2} \,\mathrm{d}x$$

$$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$$

c $u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = x^{-3} \Rightarrow v = \frac{x^{-2}}{-2}$$

$$\therefore \int \frac{1\mathrm{n}\, x}{x^3} \,\mathrm{d}x = -\frac{1}{2x^2}\ln x - \int -\frac{1}{2x^2} \times \frac{1}{x} \,\mathrm{d}x$$

$$= -\frac{1\mathrm{n}\, x}{2x^2} + \int \frac{1}{2}x^{-3} \mathrm{d}x$$

$$= -\frac{1\mathrm{n}\, x}{2x^2} + \frac{x^{-2}}{2 \times (-2)} + c$$

$$= -\frac{1\mathrm{n}\, x}{2x^2} - \frac{1}{4x^2} + c$$

2 d $u=(\ln x)^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=2\ln x\times\frac{1}{x}$

$\frac{\mathrm{d}v}{\mathrm{d}x}=1\Rightarrow v=x$

$\therefore I=\int(\ln x)^2\mathrm{d}x=x(\ln x)^2$

$-\int x\times 2\ln x\times\frac{1}{x}\mathrm{d}x$

$=x(\ln x)^2-\int 2\ln x\,\mathrm{d}x$

Let $J=\int 2\ln x\,\mathrm{d}x$

$u=\ln x\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x}$

$\frac{\mathrm{d}v}{\mathrm{d}x}=2\Rightarrow v=2x$

$\therefore J=2x\ln x-\int 2x\times\frac{1}{x}\mathrm{d}x=2x\ln x-2x+c$

$\therefore I=x(\ln x)^2-2x\ln x+2x+c$

e $u=\ln x\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x}$

$\frac{\mathrm{d}v}{\mathrm{d}x}=x^2+1\Rightarrow v=\frac{x^3}{3}+x$

$\therefore\int(x^2+1)\ln x\ \mathrm{d}x=\ln x\left(\frac{x^3}{3}+x\right)$

$-\int\left(\frac{x^3}{3}+x\right)\times\frac{1}{x}\mathrm{d}x$

$=\left(\frac{x^3}{3}+x\right)\ln x-\int\left(\frac{x^2}{3}+1\right)\mathrm{d}x$

$=\left(\frac{x^3}{3}+x\right)\ln x-\frac{x^3}{9}-x+c$

3 a $u=x^2\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=2x$

$\frac{\mathrm{d}v}{\mathrm{d}x}=\mathrm{e}^{-x}\Rightarrow v=-\mathrm{e}^{-x}$

$\therefore I=\int x^2\mathrm{e}^{-x}\mathrm{d}x=-x^2\mathrm{e}^{-x}-\int-\mathrm{e}^{-x}\times 2x\ \mathrm{d}x$

$=-x^2\mathrm{e}^{-x}+\int 2x\mathrm{e}^{-x}\ \mathrm{d}x$

Let $J=\int 2x\mathrm{e}^{-x}\ \mathrm{d}x$

$u=2x\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=2$

$\frac{\mathrm{d}v}{\mathrm{d}x}=\mathrm{e}^{-x}\Rightarrow v=-\mathrm{e}^{-x}$

$\therefore J=-\mathrm{e}^{-x}2x-\int(-\mathrm{e}^{-x})\times 2\mathrm{d}x$

$=2x\mathrm{e}^{-x}+\int 2\mathrm{e}^{-x}\ \mathrm{d}x$

$=-2x\mathrm{e}^{-x}-2\mathrm{e}^{-x}+c$

$\therefore I=-x^2\ \mathrm{e}^{-x}-2x\mathrm{e}^{-x}-2\mathrm{e}^{-x}+c$

b $u=x^2\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=2x$

$\frac{\mathrm{d}v}{\mathrm{d}x}=\cos x\Rightarrow v=\sin x$

$\therefore\ I=\int x^2\cos x\,\mathrm{d}x=x^2\sin x-\int 2x\sin x\,\mathrm{d}x$

Let $J=\int 2x\sin x\,\mathrm{d}x$

$u=2x\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=2$

$\frac{\mathrm{d}v}{\mathrm{d}x}=\sin x\Rightarrow v=-\cos x$

$\therefore\ J=-2x\cos x-\int(-\cos x)\times 2\,\mathrm{d}x$

$=-2x\cos x+\int 2\cos x\,\mathrm{d}x$

$=-2x\cos x+2\sin x+c$

$\therefore\ I=x^2\sin x+2x\cos x-2\sin x+c$

3 c $u = 12x^2 \Rightarrow \frac{du}{dx} = 24x$

$$\frac{dv}{dx} = (3+2x)^5 \Rightarrow v = \frac{(3+2x)^6}{12}$$

$$\therefore I = \int 12x^2(3+2x)^5\,dx = 12x^2\frac{(3+2x)^6}{12}$$

$$-\int 24x\frac{(3+2x)^6}{12}\,dx$$

$$= x^2(3+2x)^6 - \int 2x(3+2x)^6\,dx$$

Let $J = \int 2x(3+2x)^6\,dx$

$$u = 2x \Rightarrow \frac{du}{dx} = 2$$

$$v = \frac{(3+2x)^7}{14} \Rightarrow \frac{dv}{dx} = (3+2x)^6$$

$$\therefore J = 2x\frac{(3+2x)^7}{14} - \int\frac{(3+2x)^7}{14}\times 2\,dx$$

$$= x\frac{(3+2x)^7}{7} - \int\frac{(3+2x)^7}{7}\,dx$$

$$= x\frac{(3+2x)^7}{7} - \frac{(3+2x)^8}{7\times 16} + c$$

$$\therefore I = x^2(3+2x)^6 - x\frac{(3+2x)^7}{7} + \frac{(3+2x)^8}{112} + c$$

d $u = 2x^2 \Rightarrow \frac{du}{dx} = 4x$

$$\frac{dv}{dx} = \sin 2x \Rightarrow v = -\frac{1}{2}\cos 2x$$

$$\therefore I = \int 2x^2\sin 2x\,dx = -\frac{2x^2}{2}\cos 2x$$

$$-\int\left(-\frac{1}{2}\cos 2x\right)\times 4x\,dx$$

$$= -x^2\cos 2x + \int 2x\cos 2x\,dx$$

Let $J = \int 2x\cos 2x\,dx$

$$u = x \Rightarrow \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = 2\cos 2x \Rightarrow v = \sin 2x$$

$$\therefore J = x\sin 2x - \int\sin 2x\,dx$$

$$= x\sin 2x + \frac{1}{2}\cos 2x + c$$

$$\therefore I = -x^2\cos 2x + x\sin 2x + \frac{1}{2}\cos 2x + c$$

e $\int 2x^2\sec^2 x\tan x\,dx$

Let $u = 2x^2 \Rightarrow \frac{du}{dx} = 4x$

$$\frac{dv}{dx} = \sec^2 x\tan x \Rightarrow v = \frac{1}{2}\sec^2 x$$

$$I = x^2\sec^2 x - 2\int x\sec^2 x\,dx$$

Now let $u = x \Rightarrow \frac{du}{dx} = 1$

$$\frac{dv}{dx} = \sec^2 x \Rightarrow v = \tan x$$

$$I = x^2\sec^2 x - 2\left(x\tan x - \int\tan x\,dx\right)$$

$$= x^2\sec^2 x - 2x\tan x + 2\ln|\sec x| + c$$

4 a $u = x \Rightarrow \frac{du}{dx} = 1$

$$\frac{dv}{dx} = e^{2x} \Rightarrow v = \frac{1}{2}e^{2x}$$

$$\therefore \int_0^{\ln 2} xe^{2x}\,dx = \left[\frac{1}{2}e^{2x}\times x\right]_0^{\ln 2} - \int_0^{\ln 2}\frac{1}{2}e^{2x}\,dx$$

$$= \left(\frac{1}{2}e^{2\ln 2}\ln 2\right) - (0) - \left[\frac{1}{4}e^{2x}\right]_0^{\ln 2}$$

$$= \frac{4}{2}\ln 2 - \left(\left(\frac{1}{4}e^{2\ln 2}\right) - \left(\frac{1}{4}e^{0}\right)\right)$$

$$= 2\ln 2 - \frac{4}{4} + \frac{1}{4}$$

$$= 2\ln 2 - \frac{3}{4}$$

4 b $u = x \Rightarrow \frac{du}{dx} = 1$

$$\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$$

$$\therefore \int_0^{\frac{\pi}{2}} x\sin x\,dx = [-x\cos x]_0^{\frac{\pi}{2}}$$

$$-\int_0^{\frac{\pi}{2}} (-\cos x)dx$$

$$= \left(-\frac{\pi}{2}\cos\frac{\pi}{2}\right) - (0) + \int_0^{\frac{\pi}{2}} (-\cos x)dx$$

$$= 0 + [\sin x]_0^{\frac{\pi}{2}}$$

$$= \left(\sin\frac{\pi}{2}\right) - (\sin 0)$$

$$= 1$$

c $u = x \Rightarrow \frac{du}{dx} = 1$

$$\frac{dv}{dx} = \cos x \Rightarrow v = \sin x$$

$$\therefore \int_0^{\frac{\pi}{2}} x\cos x\,dx = [x\sin x]_0^{\frac{\pi}{2}}$$

$$-\int_0^{\frac{\pi}{2}} \sin x\,dx$$

$$= \left(\frac{\pi}{2}\sin\frac{\pi}{2}\right) - (0) - [-\cos x]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} + \left(\cos\frac{\pi}{2}\right) - (\cos 0)$$

$$= \frac{\pi}{2} - 1$$

d $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$$\frac{dv}{dx} = x^{-2} \Rightarrow v = -x^{-1}$$

$$\therefore \int_1^2 \frac{\ln x}{x^2}dx = \left[-\frac{\ln x}{x}\right]_1^2 - \int_1^2 \frac{1}{x}\times\left(-x^{-1}\right)dx$$

$$= \left(-\frac{\ln 2}{2}\right) - \left(-\frac{\ln 1}{1}\right) + \int_1^2 \frac{1}{x^2}dx$$

$$= -\frac{1}{2}\ln 2 + [-x^{-1}]_1^2$$

$$= -\frac{1}{2}\ln 2 + \left(-\frac{1}{2}\right) - \left(-\frac{1}{1}\right)$$

$$= \frac{1}{2}(1 - \ln 2)$$

e $u = x \Rightarrow \frac{du}{dx} = 1$

$$\frac{dv}{dx} = 4(1+x)^3 \Rightarrow v = (1+x)^4$$

$$\therefore \int_0^1 4x(1+x)^3 dx = [x(1+x)^4]_0^1 - \int_0^1 (1+x)^4 dx$$

$$= (1\times 2^4) - (0) - \left[\frac{(1+x)^5}{5}\right]_0^1$$

$$= 16 - \left(\left(\frac{2^5}{5}\right) - \left(\frac{1}{5}\right)\right)$$

$$= 16 - \frac{31}{5}$$

$$= 16 - 6.2$$

$$= 9.8$$

f $\int_0^{\pi} x\cos\frac{1}{4}x\,dx$

Let $u = x \Rightarrow \frac{du}{dx} = 1$

$$\frac{dv}{dx} = \cos\frac{1}{4}x \Rightarrow v = 4\sin\frac{1}{4}x$$

$$I = \left[4x\sin\frac{1}{4}x\right]_0^{\pi} - 4\int_0^{\pi} \sin\frac{1}{4}x\,dx$$

$$= \frac{4\pi}{\sqrt{2}} - 4\left[-4\cos\frac{1}{4}x\right]_0^{\pi}$$

$$= \frac{4\pi}{\sqrt{2}} - 4\left(-4\cos\frac{\pi}{4} + 4\right)$$

$$= 2\sqrt{2}\pi + 8\sqrt{2} - 16$$

4 g $u=\ln|\sec x|\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=\tan x$

$$\frac{\mathrm{d}v}{\mathrm{d}x}=\sin x\Rightarrow v=-\cos x$$

$$\therefore\int_0^{\frac{\pi}{3}}\sin x\ln|\sec x|\,\mathrm{d}x=\Big[-\cos x\ln|\sec x|\Big]_0^{\frac{\pi}{3}}$$

$$+\int_0^{\frac{\pi}{3}}\cos x\tan x\,\mathrm{d}x$$

$$=\left(-\cos\frac{\pi}{3}\ln\left|\sec\frac{\pi}{3}\right|\right)-(-\cos 0\ \ln|\sec 0|)$$

$$+\int_0^{\frac{\pi}{3}}\sin x\,\mathrm{d}x$$

$$=-\frac{1}{2}\ln 2+0+[-\cos]_0^{\frac{\pi}{3}}$$

$$=-\frac{1}{2}\ln 2+\left(-\frac{1}{2}\right)-(-1)=-\frac{1}{2}\ln 2+\frac{1}{2}$$

5 a $I=\int x\cos 4x\ \mathrm{d}x$

Let $u=x\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=1$

$$\frac{\mathrm{d}v}{\mathrm{d}x}=\cos 4x\Rightarrow v=\frac{1}{4}\sin x$$

$$I=\frac{x}{4}\sin 4x-\int\frac{1}{4}\sin 4x\ \mathrm{d}x$$

$$I=\frac{x}{4}\sin 4x+\frac{1}{16}\cos 4x+c$$

$$I=\frac{1}{16}\big(4x\sin 4x+\cos 4x\big)+c$$

b $I=\int x^2\sin 4x\ \mathrm{d}x$

Let $u=x^2\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=2x$

$$\frac{\mathrm{d}v}{\mathrm{d}x}=\sin 4x\Rightarrow v=-\frac{1}{4}\cos 4x$$

$$I=-\frac{x^2}{4}\cos 4x+\int\frac{x}{2}\cos 4x\ \mathrm{d}x$$

$$=-\frac{x^2}{4}\cos 4x+\frac{1}{32}\big(4x\sin 4x+\cos 4x\big)+c$$

$$=\frac{1}{32}\big((1-8x^2)\cos 4x+4x\sin 4x\big)+c$$

6 a $\int\sqrt{8-x}\ \mathrm{d}x=\int(8-x)^{\frac{1}{2}}\mathrm{d}x$

$$=-\frac{2}{3}(8-x)^{\frac{3}{2}}+c$$

b $I=\int(x-2)\sqrt{8-x}\ \mathrm{d}x$

Let $u=x-2\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=1$

$$\frac{\mathrm{d}v}{\mathrm{d}x}=\sqrt{8-x}\Rightarrow v=-\frac{2}{3}(8-x)^{\frac{3}{2}}$$

$$I=-\frac{2}{3}(x-2)(8-x)^{\frac{3}{2}}+\frac{2}{3}\int(8-x)^{\frac{3}{2}}\ \mathrm{d}x$$

$$=-\frac{2}{3}(x-2)(8-x)^{\frac{3}{2}}-\frac{4}{15}(8-x)^{\frac{5}{2}}+c$$

$$=-\frac{2}{3}(x-2)(8-x)^{\frac{3}{2}}-\frac{4}{15}(8-x)^{\frac{3}{2}}(8-x)+c$$

$$=\frac{2}{3}(2-x)(8-x)^{\frac{3}{2}}+\frac{4}{15}(8-x)^{\frac{3}{2}}(x-8)+c$$

$$=\frac{2}{15}(8-x)^{\frac{3}{2}}(5(2-x)+2(x-8))+c$$

$$=\frac{2}{15}(8-x)^{\frac{3}{2}}(-3x-6)+c$$

$$=-\frac{2}{5}(8-x)^{\frac{3}{2}}(x+2)+c$$

c $\int_4^7(x-2)\sqrt{8-x}\ \mathrm{d}x$

$$=\left[-\frac{2}{5}(8-x)^{\frac{3}{2}}(x+2)\right]_4^7$$

$$=-\frac{2}{5}\times 9+\frac{2}{5}\times 48$$

$$=\frac{78}{5}=15.6$$

7 a $\int\sec^2 3x\ \mathrm{d}x$

$$=\frac{1}{3}\tan 3x+c$$

7 b $I = \int x \sec^2 3x \, dx$

Let $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sec^2 3x \Rightarrow v = \frac{1}{3} \tan 3x$

$I = \frac{x}{3} \tan 3x - \frac{1}{3} \int \tan 3x \, dx$

$I = \frac{x}{3} \tan 3x - \frac{1}{9} \ln |\sec 3x| + c$

c $\int_{\frac{\pi}{18}}^{\frac{\pi}{9}} x \sec^2 x \, dx$

$= \left[\frac{x}{3} \tan 3x - \frac{1}{9} \ln |\sec 3x| \right]_{\frac{\pi}{18}}^{\frac{\pi}{9}}$

$= \left(\frac{\pi}{27} \sqrt{3} - \frac{1}{9} \ln 2 \right) - \left(\frac{\pi}{54} \times \frac{1}{\sqrt{3}} - \frac{1}{9} \ln \frac{2}{\sqrt{3}} \right)$

$= \left(\frac{\sqrt{3}\pi}{27} - \frac{1}{9} \ln 2 \right) - \left(\frac{\sqrt{3}\pi}{162} - \frac{1}{9} \ln \frac{2}{\sqrt{3}} \right)$

$= \frac{5\sqrt{3}\pi}{162} - \frac{1}{9} \ln 2 + \frac{1}{9} \ln 2 - \frac{1}{9} \ln \sqrt{3}$

$= \frac{5\sqrt{3}\pi}{162} - \frac{1}{18} \ln 3$

$p = \frac{5\sqrt{3}}{162}, \ q = \frac{1}{18}$

Integration 11G

1 a $\frac{3x+5}{(x+1)(x+2)} \equiv \frac{A}{x+1} + \frac{B}{x+2}$

$$\Rightarrow 3x+5 \equiv A(x+2)+B(x+1)$$
$$x=-1 \Rightarrow 2 = A$$
$$x=-2 \Rightarrow -1=-B \Rightarrow B=1$$
$$\therefore \int \frac{3x+5}{(x+1)(x+2)}\,\mathrm{d}x = \int \left(\frac{2}{x+1}+\frac{1}{x+2}\right)\mathrm{d}x$$
$$= 2\ln|x+1| + \ln|x+2| + c$$
$$= \ln\left(|x+1|^2\right) + \ln|x+2| + c$$
$$= \ln\left|(x+1)^2(x+2)\right| + c$$

b $\frac{3x-1}{(2x+1)(x-2)} \equiv \frac{A}{2x+1} + \frac{B}{x-2}$

$$\Rightarrow 3x-1 \equiv A(x-2)+B(2x+1)$$
$$x=2 \Rightarrow 5=5B \Rightarrow B=1$$
$$x=-\frac{1}{2} \Rightarrow -\frac{5}{2}=-\frac{5}{2}A \Rightarrow A=1$$
$$\therefore \int \frac{3x-1}{(2x+1)(x-2)}\,\mathrm{d}x = \int \left(\frac{1}{2x+1}+\frac{1}{x-2}\right)\mathrm{d}x$$
$$= \frac{1}{2}\ln|2x+1| + \ln|x-2| + c$$
$$= \ln\left|(x-2)\sqrt{2x+1}\right| + c$$

c $\frac{2x-6}{(x+3)(x-1)} \equiv \frac{A}{x+3} + \frac{B}{x-1}$

$$\Rightarrow 2x-6 \equiv A(x-1)+B(x+3)$$
$$x=1 \Rightarrow -4=4B \Rightarrow B=-1$$
$$x=-3 \Rightarrow -12=-4A \Rightarrow A=3$$
$$\therefore \int \frac{2x-6}{(x+3)(x-1)}\,\mathrm{d}x = \int \left(\frac{3}{x+3}-\frac{1}{x-1}\right)\mathrm{d}x$$
$$= 3\ln|x+3| - \ln|x-1| + c$$
$$= \ln\left|\frac{(x+3)^3}{x-1}\right| + c$$

d $\frac{3}{(2+x)(1-x)} \equiv \frac{A}{(2+x)} + \frac{B}{1-x}$

$$\Rightarrow 3 \equiv A(1-x)+B(2+x)$$
$$x=1 \Rightarrow 3=3B \Rightarrow B=1$$
$$x=-2 \Rightarrow 3=3A \Rightarrow A=1$$
$$\therefore \int \frac{3}{(2+x)(1-x)}\,\mathrm{d}x = \int \left(\frac{1}{(2+x)}+\frac{1}{1-x}\right)\mathrm{d}x$$
$$= \ln|2+x| - \ln|1-x| + c$$
$$= \ln\left|\frac{2+x}{1-x}\right| + c$$

2 a $\frac{2(x^2+3x-1)}{(x+1)(2x-1)} \equiv 1 + \frac{A}{x+1} + \frac{B}{2x-1}$

$$\Rightarrow 2x^2+6x-2 \equiv (x+1)(2x-1) + A(2x-1)+B(x+1)$$
$$x=-1 \Rightarrow -6=-3A \Rightarrow A=2$$
$$x=\frac{1}{2} \Rightarrow \frac{3}{2}=\frac{3}{2}B \Rightarrow B=1$$
$$\therefore \int \frac{2(x^2+3x-1)}{(x+1)(2x-1)}\,\mathrm{d}x = \int \left(1+\frac{2}{x+1}+\frac{1}{2x-1}\right)\mathrm{d}x$$
$$= x + 2\ln|x+1| + \frac{1}{2}\ln|2x-1| + c$$
$$= x + \ln\left|(x+1)^2\sqrt{2x-1}\right| + c$$

2 **b** $\dfrac{x^3+2x^2+2}{x(x+1)} \Rightarrow$

$$\begin{array}{r|l} & x+1 \\ \cline{2-2} x^2+x & x^3+2x^2+2 \\ & \underline{x^3+x^2} \\ & \qquad x^2+2 \\ & \qquad \underline{x^2+x} \\ & \qquad 2-x \end{array}$$

$$\frac{x^3+2x^2+2}{x(x+1)} \equiv x+1+\frac{2-x}{x(x+1)}$$

$$\equiv x+1+\frac{A}{x}+\frac{B}{x+1}$$

$$\Rightarrow x^3+2x^2+2 \equiv (x+1)x(x+1)$$
$$+A(x+1)+Bx$$

$$x=0 \Rightarrow 2=A \Rightarrow A=2$$
$$x=-1 \Rightarrow 3=-B \Rightarrow B=-3$$

$$\therefore \int \frac{x^3+2x^2+2}{x(x+1)}\,\mathrm{d}x = \int\left(x+1+\frac{2}{x}-\frac{3}{x+1}\right)\mathrm{d}x$$

$$=\frac{x^2}{2}+x+2\ln|x|-3\ln|x+1|+c$$

$$=\frac{x^2}{2}+x+\ln\left|\frac{x^2}{(x+1)^3}\right|+c$$

c $\dfrac{x^2}{x^2-4} \equiv 1+\dfrac{A}{x-2}+\dfrac{B}{x+2}$

$$\Rightarrow x^2 \equiv (x-2)(x+2)+A(x+2)+B(x-2)$$

$$x=2 \Rightarrow 4=4A \Rightarrow A=1$$
$$x=-2 \Rightarrow 4=-4B \Rightarrow B=-1$$

$$\therefore \int \frac{x^2}{x^2-4}\,\mathrm{d}x = \int\left(1+\frac{1}{x-2}-\frac{1}{x+2}\right)\mathrm{d}x$$

$$=x+\ln|x-2|-\ln|x+2|+c$$

$$=x+\ln\left|\frac{x-2}{x+2}\right|+c$$

d $\dfrac{x^2+x+2}{3-2x-x^2} \equiv \dfrac{x^2+x+2}{(3+x)(1-x)}$

$$\equiv -1+\frac{A}{3+x}+\frac{B}{1-x}$$

$$\Rightarrow x^2+x+2 \equiv -1(3+x)(1-x)$$
$$+A(1-x)+B(3+x)$$

$$x=1 \Rightarrow 4=4B \Rightarrow B=1$$
$$x=-3 \Rightarrow 8=4A \Rightarrow A=2$$

$$\therefore \int \frac{x^2+x+2}{3-2x-x^2}\,\mathrm{d}x = \int\left(-1+\frac{2}{3+x}+\frac{1}{1-x}\right)\mathrm{d}x$$

$$=-x+2\ln|3+x|-\ln|1-x|+c$$

$$=-x+\ln\left|\frac{(3+x)^2}{1-x}\right|+c$$

3 **a** $\mathrm{f}(x)=\dfrac{4}{(2x+1)(1-2x)}$

$$\frac{4}{(2x+1)(1-2x)}=\frac{A}{2x+1}+\frac{B}{1-2x}$$

$$4=A(1-2x)+B(2x+1)$$

Let $x=\dfrac{1}{2}: 4=2B \Rightarrow B=2$

Let $x=-\dfrac{1}{2}: 4=2A \Rightarrow A=2$

b $\displaystyle\int \mathrm{f}(x)\,\mathrm{d}x=\int\left(\frac{2}{(2x+1)}+\frac{2}{(1-2x)}\right)\mathrm{d}x$

$$=\ln|2x+1|-\ln|1-2x|+c$$

$$=\ln\left|\frac{2x+1}{1-2x}\right|+c$$

c $\displaystyle\int_1^2 \mathrm{f}(x)\,\mathrm{d}x=\left[\ln\left|\frac{2x+1}{1-2x}\right|\right]_1^2$

$$=\ln\frac{5}{3}-\ln 3=\ln\frac{5}{9}$$

$$k=\frac{5}{9}$$

4 a $\mathrm{f}(x)=\dfrac{17-5x}{(3+2x)(2-x)^2}$

$$\frac{17-5x}{(3+2x)(2-x)^2}=\frac{A}{3+2x}+\frac{B}{(2-x)^2}+\frac{C}{2-x}$$

$$17-5x=A(2-x)^2+B(3+2x)+C(3+2x)(2-x)$$

Let $x=2:7=7B\Rightarrow B=1$

Let $x=-\dfrac{3}{2}:17+\dfrac{15}{2}=\dfrac{49}{4}A\Rightarrow A=2$

Let $x=0:17=4A+3B+6C$

$\Rightarrow 17=8+3+6C\Rightarrow C=1$

$$\mathrm{f}(x)=\frac{2}{3+2x}+\frac{1}{(2-x)^2}+\frac{1}{2-x}$$

b $\displaystyle\int_0^1\left(\frac{2}{3+2x}+\frac{1}{2-x}+\frac{1}{(2-x)^2}\right)\mathrm{d}x$

$$=\left[\ln|3+2x|-\ln|2-x|+\frac{1}{(2-x)}\right]_0^1$$

$$=(\ln 5-\ln 1+1)-\left(\ln 3-\ln 2+\frac{1}{2}\right)$$

$$=\frac{1}{2}+\ln\frac{10}{3}$$

5 a $\mathrm{f}(x)=\dfrac{9x^2+4}{9x^2-4}$

Dividing gives:

$$\mathrm{f}(x)=1+\frac{8}{9x^2-4}$$

$$=1+\frac{8}{(3x+2)(3x-2)}$$

$$\frac{8}{(3x+2)(3x-2)}=\frac{B}{3x-2}+\frac{C}{3x+2}$$

$$8=B(3x+2)+C(3x-2)$$

Let $x=-\dfrac{2}{3}:8=-4C\Rightarrow C=-2$

Let $x=\dfrac{2}{3}:8=4B\Rightarrow B=2$

$A=1, B=2, C=-2$

b $\displaystyle\int_{-\frac{1}{3}}^{\frac{1}{3}}\left(1+\frac{2}{3x-2}-\frac{2}{3x+2}\right)\mathrm{d}x$

$$=\left[x+\frac{2}{3}\ln|3x-2|-\frac{2}{3}\ln|3x+2|\right]_{-\frac{1}{3}}^{\frac{1}{3}}$$

$$=\left(\frac{1}{3}-\frac{2}{3}\ln 3\right)-\left(-\frac{1}{3}+\frac{2}{3}\ln 3\right)$$

$$=\frac{2}{3}-\frac{4}{3}\ln 3$$

$$a=\frac{2}{3}, b=-\frac{4}{3}, c=3$$

6 a $\mathrm{f}(x)=\dfrac{6+3x-x^2}{x^3+2x^2}=\dfrac{6+3x-x^2}{x^2(x+2)}$

$$\frac{6+3x-x^2}{x^2(x+2)}=\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x+2}$$

$$6+3x-x^2=A(x+2)+Bx(x+2)+Cx^2$$

Let $x=0:6=2A\Rightarrow A=3$

Let $x=-2:-4=4C\Rightarrow C=-1$

Let $x=1:8=3A+3B+C\Rightarrow B=0$

$$\mathrm{f}(x)=\frac{3}{x^2}-\frac{1}{x+2}$$

b $\displaystyle\int_2^4\frac{6+3x-x^2}{x^3+2x^2}\,\mathrm{d}x$

$$=\int_2^4\left(\frac{3}{x^2}-\frac{1}{x+2}\right)\mathrm{d}x$$

$$=\left[-\frac{3}{x}-\ln|x+2|\right]_2^4$$

$$=\left(-\frac{3}{4}-\ln 6\right)-\left(-\frac{3}{2}-\ln 4\right)$$

$$=\frac{3}{4}+\ln\frac{2}{3}$$

$$a=\frac{3}{4}, b=\frac{2}{3}$$

7 a Let $\mathrm{f}(x)=\dfrac{32x^2+4}{(4x+1)(4x-1)}$

Dividing:

$$\frac{32x^2+4}{(4x+1)(4x-1)}=2+\frac{6}{(4x+1)(4x-1)}$$

$$\Rightarrow A=2$$

$$\frac{6}{(4x+1)(4x-1)}=\frac{B}{4x+1}+\frac{C}{4x-1}$$

$$6=B(4x-1)+C(4x+1)$$

Let $x=\frac{1}{4}: 6=2C\Rightarrow C=3$

Let $x=-\frac{1}{4}: 6=-2B\Rightarrow B=-3$

$$\mathrm{f}(x)=2-\frac{3}{4x+1}+\frac{3}{4x-1}$$

b $\displaystyle\int_1^2 \mathrm{f}(x)\,\mathrm{d}x=\int_1^2\left(2-\frac{3}{4x+1}+\frac{3}{4x-1}\right)\mathrm{d}x$

$$=\left[2x-\frac{3}{4}\ln|4x+1|+\frac{3}{4}\ln|4x-1|\right]_1^2$$

$$=\left(4-\frac{3}{4}\ln 9+\frac{3}{4}\ln 7\right)-\left(2-\frac{3}{4}\ln 5+\frac{3}{4}\ln 3\right)$$

$$=2+\frac{3}{4}(-\ln 9+\ln 7+\ln 5-\ln 3)$$

$$=2+\frac{3}{4}\ln\frac{35}{27}\text{, so }k=\frac{3}{4}, m=\frac{35}{27}$$

Integration 11H

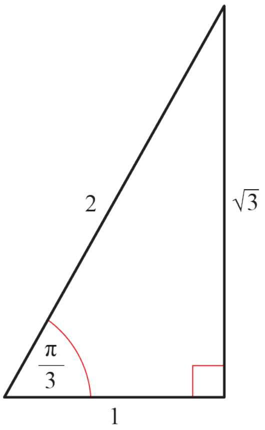
1 **a** Area $= \int_0^1 \frac{2}{1+x}\,\mathrm{d}x = \left[2\ln|1+x|\right]_0^1$

$= (2\ln 2) - (2\ln 1)$

$\therefore$ Area $= 2\ln 2$

b Area $= \int_0^{\frac{\pi}{3}} \sec x\,\mathrm{d}x$

$= \left[\ln|\sec x + \tan x|\right]_0^{\frac{\pi}{3}}$



$= [\ln(2+\sqrt{3})] - [\ln(1)]$

$\therefore$ Area $= \ln(2+\sqrt{3})$

c Area $= \int_1^2 \ln x\,\mathrm{d}x$

$u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$

$\frac{\mathrm{d}v}{\mathrm{d}x} = 1 \Rightarrow v = x$

$\therefore$ Area $= [x\ln x]_1^2 - \int_1^2 x \times \frac{1}{x}\,\mathrm{d}x$

$= (2\ln 2) - (0) - [x]_1^2$

$= 2\ln 2 - 1$

d Area $= \int_0^{\frac{\pi}{4}} \sec x \tan x\,\mathrm{d}x$

$= [\sec x]_0^{\frac{\pi}{4}}$

$= (\sqrt{2}) - (1)$

$\therefore$ Area $= \sqrt{2} - 1$

e Area $= \int_0^2 x\sqrt{4-x^2}\,\mathrm{d}x$

Let $y = (4-x^2)^{\frac{3}{2}}$

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2}(4-x^2)^{\frac{1}{2}} \times (-2x) = -3x(4-x^2)^{\frac{1}{2}}$

$\therefore$ Area $= \left[-\frac{1}{3}(4-x^2)^{\frac{3}{2}}\right]_0^2$

$= (0) - \left(-\frac{1}{3} \times 2^3\right) = \frac{8}{3}$

2 **a** $\mathrm{f}(x) = \frac{4x-1}{(x+2)(2x+1)}$

$\frac{4x-1}{(x+2)(2x+1)} = \frac{A}{x+2} + \frac{B}{2x+1}$

$4x - 1 = A(2x+1) + B(x+2)$

$x = -2 \Rightarrow -9 = -3A \Rightarrow A = 3$

$x = -\frac{1}{2} \Rightarrow -3 = \frac{3}{2}B \Rightarrow B = -2$

$\mathrm{f}(x) = \frac{3}{x+2} - \frac{2}{2x+1}$

$\int_0^2 \left(\frac{3}{x+2} - \frac{2}{2x+1}\right)\mathrm{d}x$

$= \left[3\ln(x+2) - \ln(2x+1)\right]_0^2$

$= 3\ln 4 - \ln 5 - 3\ln 2 + \ln 1$

$= \ln 64 - \ln 5 - \ln 8$

$= \ln\frac{8}{5}$

2 b $\frac{x}{(x+1)^2} \equiv \frac{A}{(x+1)^2} + \frac{B}{x+1}$

$\Rightarrow x \equiv A + B(x+1)$

Compare coefficient of x: $1 = B \Rightarrow B = 1$

Compare constants: $0 = A + B \Rightarrow A = -1$

$\therefore$ area $= \int_0^2 \frac{x}{(x+1)^2}\,dx$

$= \int_0^2 \left(\frac{1}{x+1} - \frac{1}{(x+1)^2}\right) dx$

$= \left[\ln|x+1| + \frac{1}{x+1}\right]_0^2$

$= \left(\ln 3 + \frac{1}{3}\right) - (\ln 1 + 1)$

$= \ln 3 - \frac{2}{3}$

c Area $= \int_0^{\frac{\pi}{2}} x \sin x\,dx$

$u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$

$\therefore$ area $= [-x\cos x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos x)\,dx$

$= \left(-\frac{\pi}{2}\cos\frac{\pi}{2}\right) - (0) + \int_0^{\frac{\pi}{2}} \cos x\,dx$

$= 0 + [\sin x]_0^{\frac{\pi}{2}}$

$= \left(\sin\frac{\pi}{2} - 0\right)$

$= 1$

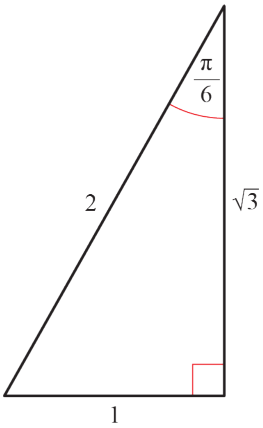
d Area $= \int_0^{\frac{\pi}{6}} \cos x\sqrt{2\sin x + 1}\,dx$

Let $y = (2\sin x + 1)^{\frac{3}{2}}$

$\Rightarrow \frac{dy}{dx} = \frac{3}{2}(2\sin x + 1)^{\frac{1}{2}} \times 2\cos x$

$= 3\cos x(2\sin x + 1)^{\frac{1}{2}}$

$\therefore$ area $= \left[\frac{1}{3}(2\sin x + 1)^{\frac{3}{2}}\right]_0^{\frac{\pi}{6}}$



$= \left(\frac{1}{3}2^{\frac{3}{2}}\right) - \left(\frac{1}{3}1^{\frac{3}{2}}\right)$

$= \frac{2\sqrt{2}}{3} - \frac{1}{3}$

$= \frac{2\sqrt{2} - 1}{3}$

2 e Area $= \int_0^{\ln 2} x\mathrm{e}^{-x}\mathrm{d}x$

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{-x} \Rightarrow v = \mathrm{e}^{-x}$$

$$\therefore \text{ area } = \left[-x\mathrm{e}^{-x}\right]_0^{\ln 2} - \int_0^{\ln 2} (-\mathrm{e}^{-x})\mathrm{d}x$$

$$= (-\ln\ 2 \times \mathrm{e}^{-\ln 2}) - (0) + \int_0^{\ln 2} \mathrm{e}^{-x}\mathrm{d}x$$

$$= -\ln 2 \times \frac{1}{2} + \left[-\mathrm{e}^{-x}\right]_0^{\ln 2}$$

$$= -\frac{1}{2}\ln 2 + (-\mathrm{e}^{-\ln 2}) - (-\mathrm{e}^{-0})$$

$$= -\frac{1}{2}\ln 2 - \frac{1}{2} + 1$$

$$= \frac{1}{2}(1 - \ln 2)$$

3 Area $= \int_1^2 \frac{4x+3}{(x+2)(2x-1)}\ \mathrm{d}x$

$$\frac{4x+3}{(x+2)(2x-1)} = \frac{A}{x+2} + \frac{B}{2x-1}$$

$$4x+3 = A(2x-1) + B(x+2)$$

Let $x = -2 : -5 = -5A \Rightarrow A = 1$

Let $x = \frac{1}{2} : 5 = \frac{5}{2}B \Rightarrow B = 2$

Area $= \int_1^2 \left(\frac{1}{x+2} + \frac{2}{2x-1}\right)\ \mathrm{d}x$

$$= \left[\ln|x+2| + \ln|2x-1|\right]_1^2$$

$$= (\ln 4 + \ln 3) - (\ln 3 + \ln 1)$$

$$= \ln 4$$

4 Area $= \int_2^4 \left(\mathrm{e}^{0.5x} + \frac{1}{x}\right)\ \mathrm{d}x$

$$= \left[2\mathrm{e}^{0.5x} + \ln|x|\right]_2^4$$

$$= (2\mathrm{e}^2 + \ln 4) - (2\mathrm{e} + \ln 2)$$

$$= 2\mathrm{e}^2 - 2\mathrm{e} + \ln\frac{4}{2}$$

$$= 2\mathrm{e}^2 - 2\mathrm{e} + \ln 2$$

5 a $\mathrm{g}(x) = 0 \Rightarrow A(0,0), B(\pi,0), C(2\pi,0)$

b $I = \int x\sin x\ \mathrm{d}x$

Let $u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x \Rightarrow v = -\cos x$$

$$I = -x\cos x + \int \cos x\ \mathrm{d}x$$

$$= -x\cos x + \sin x + c$$

Area between A and B:

$$\left[-x\cos x + \sin x\right]_0^{\pi} = \pi$$

Area between B and C:

$$\left[-x\cos x + \sin x\right]_{\pi}^{2\pi} = -2\pi - \pi = -3\pi$$

Total area $= \pi + 3\pi = 4\pi$

6 a $I = \int x^2 \ln x\ \mathrm{d}x$

Let $u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = x^2 \Rightarrow v = \frac{x^3}{3}$$

$$I = \frac{x^3}{3}\ln x - \int \frac{x^3}{3} \times \frac{1}{x}\ \mathrm{d}x$$

$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + c$$

b $x^2 \ln x = 0 \Rightarrow x = 0$ or 1

Area between $x = 0$ and $x = 1$:

$$\left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]_0^1 = -\frac{1}{9}$$

Area between $x = 1$ and $x = 2$:

$$\left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]_1^2 = \left(\frac{8}{3}\ln 2 - \frac{8}{9}\right) + \frac{1}{9}$$

$$= \frac{8}{3}\ln 2 - \frac{7}{9}$$

Total area $= \frac{8}{3}\ln 2 - \frac{7}{9} + \frac{1}{9} = \frac{8}{3}\ln 2 - \frac{2}{3}$

$$= \frac{2}{3}(4\ln 2 - 1)$$

7 a $y = 3\cos x\sqrt{\sin x + 1}$

Curve crosses the x axis when $y = 0$.

$\cos x = 0$ or $\sin x = -1$

$$x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$$

$$A\left(-\frac{\pi}{2}, 0\right), B\left(\frac{\pi}{2}, 0\right), C\left(\frac{3\pi}{2}, 0\right)$$

Curve crosses the y axis when $x = 0$.

So $D(0, 3)$.

b $I = \int 3\cos x\sqrt{\sin x + 1}\ \mathrm{d}x$

Let $u = \sin x + 1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x$

$$I = \int 3\sqrt{u}\ \mathrm{d}u = 2u^{\frac{3}{2}} + c$$

$$= 2(\sin x + 1)^{\frac{3}{2}} + c$$

c $R_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3\cos x\sqrt{\sin x + 1}\ \mathrm{d}x$

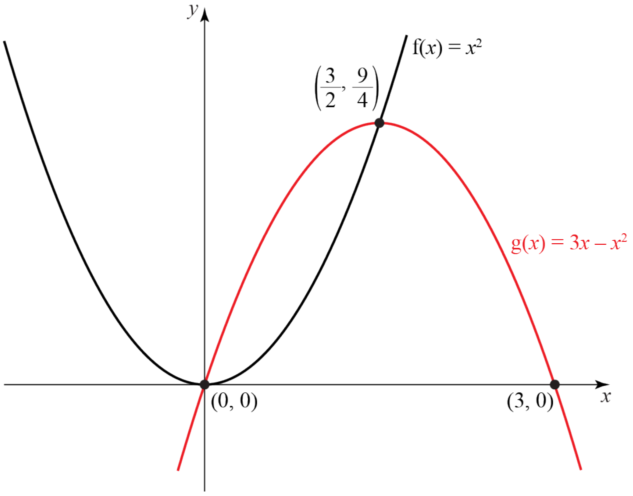
$$\left[2(\sin x + 1)^{\frac{3}{2}}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4\sqrt{2} = \sqrt{32}$$

$$R_2 = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 3\cos x\sqrt{\sin x + 1}\ \mathrm{d}x$$

$$\left[2(\sin x + 1)^{\frac{3}{2}}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = -4\sqrt{2} = -\sqrt{32}$$

$R_1 = R_2 = \sqrt{32}$, so $a = 32$.

8 a



$\mathrm{f}(x) = \mathrm{g}(x) \Rightarrow x^2 = 3x - x^2$

$2x^2 - 3x = 0$

$x(2x - 3) = 0$

$x = 0$ or $\frac{3}{2}$

$x = 0 \Rightarrow y = 0$

$x = \frac{3}{2} \Rightarrow y = \frac{9}{4}$

Points of intersection are

$(0, 0)$ and $\left(\frac{3}{2}, \frac{9}{4}\right)$

b Area under $\mathrm{f}(x)$ between 0 and $\frac{3}{2}$:

$$\int_0^{\frac{3}{2}} x^2\ \mathrm{d}x = \left[\frac{x^3}{3}\right]_0^{\frac{3}{2}} = \frac{27}{24} = \frac{9}{8}$$

Area under $\mathrm{g}(x)$ between 0 and $\frac{3}{2}$:

$$\int_0^{\frac{3}{2}} 3x - x^2\ \mathrm{d}x = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^{\frac{3}{2}}$$

$$= \frac{27}{8} - \frac{27}{24} = \frac{9}{4}$$

Area between the two curves

$$= \frac{9}{4} - \frac{9}{8} = \frac{9}{8}$$

9 **a** Points of intersection are when:

$$2\cos x + 2 = -2\cos x + 4$$
$$4\cos x = 2$$
$$\cos x = \frac{1}{2} \Rightarrow x = -\frac{\pi}{3}, \frac{\pi}{3} \text{ or } \frac{5\pi}{3}$$
$$x = -\frac{\pi}{3}, \frac{\pi}{3} \text{ or } \frac{5\pi}{3} \Rightarrow y = 3$$
$$A\left(-\frac{\pi}{3}, 3\right), B\left(\frac{\pi}{3}, 3\right), C\left(\frac{5\pi}{3}, 3\right)$$

b $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (2\cos x + 2)\,\mathrm{d}x$

$$= \left[2\sin x + 2x\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$
$$= \left(\sqrt{3} + \frac{2\pi}{3}\right) - \left(-\sqrt{3} - \frac{2\pi}{3}\right) = 2\sqrt{3} + \frac{4\pi}{3}$$
$$= 2\sqrt{3} + \frac{4\pi}{3}$$

$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (-2\cos x + 4)\,\mathrm{d}x$

$$= \left[-2\sin x + 4x\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$
$$= \left(-\sqrt{3} + \frac{4\pi}{3}\right) - \left(\sqrt{3} - \frac{4\pi}{3}\right)$$
$$= -2\sqrt{3} + \frac{8\pi}{3}$$

$$R_1 = 2\sqrt{3} + \frac{4\pi}{3} - \left(-2\sqrt{3} + \frac{8\pi}{3}\right)$$
$$= 4\sqrt{3} - \frac{4\pi}{3}$$

$a = 4, b = -4, c = 3$ (or $a = 4, b = 4, c = -3$)

c $\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (2\cos x + 2)\,\mathrm{d}x$

$$= \left[2\sin x + 2x\right]_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$$
$$= \left(-\sqrt{3} + \frac{10\pi}{3}\right) - \left(\sqrt{3} + \frac{2\pi}{3}\right)$$
$$= -2\sqrt{3} + \frac{8\pi}{3}$$

$\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (-2\cos x + 4)\,\mathrm{d}x$

$$= \left[-2\sin x + 4x\right]_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$$
$$= \left(\sqrt{3} + \frac{20\pi}{3}\right) - \left(-\sqrt{3} + \frac{4\pi}{3}\right)$$
$$= 2\sqrt{3} + \frac{16\pi}{3}$$

$$R_2 = \left(2\sqrt{3} + \frac{16\pi}{3}\right) - \left(-2\sqrt{3} + \frac{8\pi}{3}\right)$$
$$= 4\sqrt{3} + \frac{8\pi}{3}$$

$$R_2 : R_1 = 4\sqrt{3} + \frac{8\pi}{3} : 4\sqrt{3} - \frac{4\pi}{3}$$
$$= \left(3\sqrt{3} + 2\pi\right) : \left(3\sqrt{3} - \pi\right)$$

10 $y = \sin\theta$

Area under curve $= 2\int_0^{\pi} \sin\theta\,\mathrm{d}\theta$ because of the symmetry of the curve.

$$= 2\left[-\cos\theta\right]_0^{\pi} = 2 + 2 = 4$$

$y = \sin 2\theta$

Area under curve $= 4\int_0^{\frac{\pi}{2}} \sin 2\theta\,\mathrm{d}\theta$ because of the symmetry of the curve.

$$= 4\left[-\frac{1}{2}\cos 2\theta\right]_0^{\frac{\pi}{2}} = 2 + 2 = 4$$

11 a At A, $\cos x = \sin x$

$$\tan x = 1 \Rightarrow x = \frac{\pi}{4} \Rightarrow y = \frac{1}{\sqrt{2}}$$

$$A\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$$

b i $R_1 =$ Area under $y = \cos x$
− Area under $y = \sin x$ between
$x = 0$ and $x = \frac{\pi}{4}$

$$R_1 = \int_0^{\frac{\pi}{4}} \cos x \, dx \; - \; \int_0^{\frac{\pi}{4}} \sin x \, dx$$

$$= \left[\sin x + \cos x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1$$

ii $R_2 = 2 \times$ Area under $y = \sin x$
between $x = 0$ and $x = \frac{\pi}{4}$

$$R_2 = 2\int_0^{\frac{\pi}{4}} -\cos x \, dx$$

$$= 2\left(1 - \frac{1}{\sqrt{2}}\right) = 2 - \sqrt{2}$$

iii $R_3 = \int_{\frac{\pi}{4}}^{\pi} \sin x \, dx \; - \; \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \, dx$

$$= \left[-\cos x\right]_{\frac{\pi}{4}}^{\pi} - \left[\sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= 1 + \frac{1}{\sqrt{2}} - \left(1 - \frac{1}{\sqrt{2}}\right) = \sqrt{2}$$

c $R_1 : R_2 = \left(\sqrt{2} - 1\right) : \left(2 - \sqrt{2}\right)$

$$= \left(\sqrt{2} - 1\right)\left(2 + \sqrt{2}\right) : \left(2 - \sqrt{2}\right)\left(2 + \sqrt{2}\right)$$

$$= \sqrt{2} : 2$$

12 Area $= \int y \frac{dx}{dt} = \int_0^{\sqrt[3]{4}} t^2\left(3t^2\right) dt$

$$= \frac{3}{5}\left(\sqrt[3]{4}\right)^5 = \frac{3}{5}\left(2^{\frac{10}{3}}\right)$$

$$= \frac{3}{5}\left(2^3\right)\left(2^{\frac{1}{3}}\right) = \frac{24}{5}\sqrt[3]{2}$$

$$\Rightarrow k = \frac{24}{5}$$

13 Area $= \int y \frac{dx}{dt} = \int_0^{\frac{\pi}{2}} \sin 2t(\cos t) dt$

Using $\sin 2t = \cos t \sin t$:

$$\text{Area} = \int_0^{\frac{\pi}{2}} \sin t(\cos^2 t) dt$$

Let $u = \cos t \Rightarrow \frac{du}{dt} = -\sin t$

$$\Rightarrow dt = \frac{1}{-\sin t} du$$

$$\text{Area} = -2\int_0^{\frac{\pi}{2}} u^2 du = \left[-\frac{2u^3}{3}\right]_0^{\frac{\pi}{2}}$$

$$= \left[-\frac{2\cos^3 t}{3} + c\right]_0^{\frac{\pi}{2}}$$

$$= \left(\frac{2}{3}\right) - (0)$$

$$= \frac{2}{3}$$

14 a P is at point $t = 2$

$$x = (2+1)^2 = 9$$

$$y = \frac{1}{2}\left(2^3\right) + 3 = 7$$

$(9, 7)$

Equation of normal at P:

$$\frac{dy}{dx} = \frac{dy}{dt} \bigg/ \frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{3}{2}t^2, \quad \frac{dx}{dt} = 2t + 2$$

$$\frac{dy}{dx} = \frac{\frac{3}{2}t^2}{2t+2} = \frac{\frac{3}{2}(2)^2}{4+2} = 1$$

14 a Gradient of normal is negative reciprocal of derivative at P $\therefore m = -1$

$$y - y_1 = m\left(x - x_1\right)$$

$$y - 7 = -1\left(x - 9\right)$$

$$y + x = 16$$

b

$$\text{Area} = \int_0^2 \left(\frac{1}{2}t^3 + 3\right)\left(2t + 2\right)\mathrm{d}t + \int_9^{16}\left(16 - x\right)\mathrm{d}x$$

$$= \int_0^2 \left(t^4 + t^3 + 6t + 6\right)\mathrm{d}t + \int_9^{16}\left(16 - x\right)\mathrm{d}x$$

$$= \left[\frac{t^5}{5} + \frac{t^4}{4} + 3t^2 + 6t\right]_0^2 + \left[16x - \frac{x^2}{2}\right]_9^{16}$$

$$= 34.4 + 24.5$$

$$= 58.9$$

Challenge

Curves intersect at $\sin 2x = \cos x$

$$2\sin x\cos x = \cos x$$

$$\sin x = \frac{1}{2}, \cos x \neq 0$$

$$x = \frac{\pi}{6}$$

Shaded area

$$= \int_0^{\frac{\pi}{6}}\left(\cos x - \sin 2x\right)\,\mathrm{d}x + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\left(\sin 2x - \cos x\right)\,\mathrm{d}x$$

$$= \left[\sin x + \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{6}} + \left[-\frac{1}{2}\cos 2x - \sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= \left(\frac{1}{2} + \frac{1}{4}\right) - \left(0 + \frac{1}{2}\right) + \left(0 - \frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{4} - \frac{1}{2}\right)$$

$$= 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2} - 1}{\sqrt{2}}$$

$$= \frac{2 - \sqrt{2}}{2}$$

Integration 11I

1 a

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
y	1	1.1260	1.2559	1.4142	1.6529

b i $\text{Area} \approx \frac{1}{2}h\left(y_0 + 2(y_1 + \ldots) + y_n\right)$

$= \frac{1}{2} \times \frac{\pi}{6}\left(1 + 1.6529 + 2 \times 1.2559\right) = 1.352$

ii $\text{Area} \approx \frac{1}{2} \times \frac{\pi}{12}\left(1 + 1.6529 + 2(1.1260 + 1.2559 + 1.4142)\right) = 1.341$

2 a

θ	$-\frac{\pi}{5}$	$-\frac{\pi}{10}$	0	$\frac{\pi}{10}$	$\frac{\pi}{5}$
y	0	0.7071	1	0.7071	0

b $R = \frac{1}{2} \times \frac{\pi}{10}\left(0 + 0 + 2(0.7071 + 1 + 0.7071)\right) = 0.758$

c The shape of the graph is concave, so the trapezium lines will underestimate the area.

d $\int_{-\frac{\pi}{5}}^{\frac{\pi}{5}} \cos\frac{5\theta}{2}\,\mathrm{d}\theta = \left[\frac{2}{5}\sin\frac{5\theta}{2}\right]_{-\frac{\pi}{5}}^{\frac{\pi}{5}} = \frac{4}{5} = 0.8$

e $\text{Percentage error} = \frac{0.8 - 0.758}{0.8} \times 100\% = 5.25\%$

3 a

x	0	0.5	1	1.5	2
y	0.707	0.614	0.519	0.427	0.345

b Area using the trapezium rule:

$\approx \frac{1}{2}h\left(y_0 + 2(y_1 + \ldots) + y_n\right) = \frac{1}{4}\left(0.707 + 0.345 + 2(0.614 + 0.519 + 0.427)\right)$

= 1.04 to 2 decimal places

4 a

x	1	1.5	2	2.5	3
y	1	0.7973	1	1.4581	2.0986

b i $\text{Area} \approx \frac{1}{2} \times 1\left(1 + 2.0986 + 2\right) = 2.549$

ii $\text{Area} \approx \frac{1}{2} \times \frac{1}{2}\left(1 + 2.0986 + 2(0.7973 + 1 + 1.4581)\right) = 2.402$

c Increasing the number of values decreases the interval. This leads to an approximation more closely following the curve.

d $\int_1^3 \left((x-2)\ln x + 1\right) \mathrm{d}x = \int_1^3 (x-2)\ln x \; \mathrm{d}x + \int_1^3 \mathrm{d}x$

Let $u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = x - 2 \Rightarrow v = \frac{(x-2)^2}{2}$$

$$I = \left[\frac{(x-2)^2 \ln x}{2}\right]_1^3 - \int_1^3 \frac{(x-2)^2}{2x} \mathrm{d}x + 2$$

$$= \frac{1}{2}\ln 3 - \int_1^3 \frac{x^2 - 4x + 4}{2x} \mathrm{d}x + 2$$

$$= \frac{1}{2}\ln 3 - \int_1^3 \left(\frac{x}{2} - 2 + \frac{2}{x}\right) \mathrm{d}x + 2$$

$$= \frac{1}{2}\ln 3 - \left[\frac{x^2}{4} - 2x + 2\ln x\right]_1^3 + 2$$

$$= \frac{1}{2}\ln 3 + 2 - \left(\frac{9}{4} - 6 + 2\ln 3\right) + \left(\frac{1}{4} - 2\right) = -\frac{3}{2}\ln 3 + 4$$

5 a

x	0	0.5	1	1.5	2
y	0	0.6124	1	1.0607	0

b $\text{Area} \approx \frac{1}{2} \times 0.5\left(0 + 0 + 2(0.6124 + 1 + 1.0607)\right) = 1.337$

5 **c** $I=\int_0^2 x\sqrt{2-x}\ \mathrm{d}x$

Let $u=2-x\Rightarrow\dfrac{\mathrm{d}u}{\mathrm{d}x}=-1$

$$I=\int_2^0 -(2-u)u^{\frac{1}{2}}\ \mathrm{d}u=\int_2^0 u^{\frac{3}{2}}-2u^{\frac{1}{2}}\ \mathrm{d}u=\left[\frac{2}{5}u^{\frac{5}{2}}-\frac{4}{3}u^{\frac{3}{2}}\right]_2^0$$

$$=0-\left(\frac{2}{5}\sqrt{32}-\frac{4}{3}\sqrt{8}\right)=\frac{8}{3}\sqrt{2}-\frac{8}{5}\sqrt{2}=\frac{16}{15}\sqrt{2}=\frac{2^{\frac{9}{2}}}{15}$$

$p=\dfrac{16}{15},\ q=\dfrac{1}{2}$

d $\dfrac{16}{15}\sqrt{2}=1.509$

Percentage error $=\dfrac{1.509-1.337}{1.509}\times 100\%=11.4\%$

6 **a** $y=\dfrac{4x-5}{(x-3)(2x+1)}$

$y=0\Rightarrow 4x-5=0\Rightarrow x=\dfrac{5}{4}$

$A\left(\dfrac{5}{4},0\right)$

b

x	0	0.25	0.5	0.75	1	1.25
y	1.6667	0.9697	0.6	0.3556	0.1667	0

c Area $\approx\dfrac{1}{2}\times 0.25\big(1.6667+0+2(0.9697+0.6+0.3556+0.1667)\big)=0.7313$

d $\dfrac{4x-5}{(x-3)(2x+1)}=\dfrac{A}{x-3}+\dfrac{B}{2x+1}$

$4x-5=A(2x+1)+B(x-3)$

Let $x=3: 7=7A\Rightarrow A=1$

Let $x=-\dfrac{1}{2}: -7=-\dfrac{7}{2}B\Rightarrow B=2$

$y=\dfrac{1}{x-3}+\dfrac{2}{2x+1}$

$$I=\int_0^{\frac{5}{4}}\left(\frac{1}{x-3}+\frac{2}{2x+1}\right)\ \mathrm{d}x=\Big[\ln|x-3|+\ln|2x+1|\Big]_0^{\frac{5}{4}}$$

$$=\ln\frac{7}{4}+\ln\frac{7}{2}-\ln 3-\ln 1=\ln 7+\ln 7-(\ln 4+\ln 2+\ln 3)=\ln\frac{49}{24}$$

6 e $\ln\frac{49}{24} = 0.7137$

$$\text{Percentage error} = \frac{0.7137 - 0.7313}{0.7137} \times 100\% = 2.5\%$$

7 a

x	0	0.5	1	1.5	2	2.5	3
y	2.7183	4.1133	5.6522	7.3891	9.3565	11.5824	14.0940

b $\text{Area} \approx \frac{1}{2} \times 0.5\left(2.7183 + 14.0940 + 2(4.1133 + 5.6522 + 7.3891 + 9.3565 + 11.5824)\right) = 23.25$

c $I = \int_0^3 \mathrm{e}^{\sqrt{2x+1}}\, \mathrm{d}x$

Let $t = \sqrt{2x+1} \Rightarrow \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{\sqrt{2x+1}} = \frac{1}{t}$

$I = \int_1^{\sqrt{7}} t\mathrm{e}^t\, \mathrm{d}t$

$a = 1, b = \sqrt{7}, k = 1$

d $I = \int_1^{\sqrt{7}} t\mathrm{e}^t\, \mathrm{d}t$

Let $u = t \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}t} = 1$

$\frac{\mathrm{d}v}{\mathrm{d}t} = \mathrm{e}^t \Rightarrow v = \mathrm{e}^t$

$I = \left[t\mathrm{e}^t\right]_1^{\sqrt{7}} - \int_1^{\sqrt{7}} \mathrm{e}^t\, \mathrm{d}t = \sqrt{7}\mathrm{e}^{\sqrt{7}} - \mathrm{e} - \mathrm{e}^{\sqrt{7}} + \mathrm{e} = \left(\sqrt{7} - 1\right)\mathrm{e}^{\sqrt{7}} = 23.20$

Integration 11J

1 a $\dfrac{dy}{dx} = (1+y)(1-2x)$

$$\Rightarrow \int \frac{1}{1+y}\,dy = \int (1-2x)\,dx$$

$$\Rightarrow \ln|1+y| = x - x^2 + c$$

$$\Rightarrow 1+y = e^{(x-x^2+c)}$$

$$\Rightarrow 1+y = A\,e^{x-x^2}, \quad (A = e^c)$$

$$\Rightarrow y = A\,e^{x-x^2} - 1$$

b $\dfrac{dy}{dx} = y\tan x$

$$\Rightarrow \int \frac{1}{y}\,dy = \int \tan x\,dx$$

$$\Rightarrow \ln|y| = \ln|\sec x| + c$$

$$\Rightarrow \ln|y| = \ln|k\sec x|, \quad (c = \ln k)$$

$$\Rightarrow y = k\sec x$$

c $\cos^2 x\dfrac{dy}{dx} = y^2\sin^2 x$

$$\Rightarrow \int \frac{1}{y^2}\,dy = \int \frac{\sin^2 x}{\cos^2 x}\,dx$$

$$\Rightarrow \int \frac{1}{y^2}\,dy = \int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx$$

$$\Rightarrow -\frac{1}{y} = \tan x - x + c$$

$$\Rightarrow y = \frac{-1}{\tan x - x + c}$$

d $\dfrac{dy}{dx} = 2e^{x-y} = 2e^x e^{-y}$

$$\Rightarrow \int \frac{1}{e^{-y}}\,dy = \int 2e^x\,dx$$

$$\text{i.e. } \Rightarrow \int e^y\,dy = \int 2e^x\,dx$$

$$\Rightarrow e^y = 2e^x + c$$

$$\Rightarrow y = \ln|2e^x + c|$$

2 a $\dfrac{dy}{dx} = \sin x\cos^2 x$

$$\Rightarrow \int dy = \int \sin x\cos^2 x\,dx$$

$$\Rightarrow y = -\frac{\cos^3 x}{3} + c$$

$$y = 0, x = \frac{\pi}{3} \Rightarrow 0 = -\frac{\left(\frac{1}{8}\right)}{3} + c \Rightarrow c = \frac{1}{24}$$

$$\therefore\ y = \frac{1}{24} - \frac{1}{3}\cos^3 x$$

b $\dfrac{dy}{dx} = \sec^2 x\sec^2 y$

$$\Rightarrow \int \frac{1}{\sec^2 y}\,dy = \int \sec^2 x\,dx$$

$$\Rightarrow \int \cos^2 y\,dy = \int \sec^2 x\,dx$$

$$\Rightarrow \int \left(\frac{1}{2} + \frac{1}{2}\cos 2y\right)dy = \int \sec^2 x\,dx$$

$$\Rightarrow \frac{1}{2}y + \frac{1}{4}\sin 2y = \tan x + c$$

$$\text{or } \sin 2y + 2y = 4\tan x + k$$

$$y = 0, x = \frac{\pi}{4} \Rightarrow 0 = 4 + k \Rightarrow k = -4$$

$$\therefore \sin 2y + 2y = 4\tan x - 4$$

c $\dfrac{dy}{dx} = 2\cos^2 y\cos^2 x$

$$\Rightarrow \int \frac{1}{\cos^2 y}\,dy = \int 2\cos^2 x\,dx$$

$$\Rightarrow \int \sec^2 y\,dy = \int (1 + \cos 2x)\,dx$$

$$\Rightarrow \tan y = x + \frac{1}{2}\sin 2x + c$$

$$x = 0, \quad y = \frac{\pi}{4} \Rightarrow 1 = 0 + c$$

$$\therefore\ \tan y = x + \frac{1}{2}\sin 2x + 1$$

2 d $$\sin y\cos x\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\cos y}{\cos x}$$

$$\tan y\frac{\mathrm{d}y}{\mathrm{d}x}=\sec^2 x$$
$$\int \tan y\ \mathrm{d}y=\int \sec^2 x\ \mathrm{d}x$$
$$-\ln|\cos y|=\tan x+c$$

$$x=0, y=0\Rightarrow 0=0+c\Rightarrow c=0$$

$$-\ln|\cos y|=\tan x$$
$$\cos y=\mathrm{e}^{-\tan x}$$
$$y=\arccos\left(\mathrm{e}^{-\tan x}\right)$$

3 a $$x^2\frac{\mathrm{d}y}{\mathrm{d}x}=y+xy$$
$$x^2\frac{\mathrm{d}y}{\mathrm{d}x}=y(1+x)$$
$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1+x}{x^2}=\frac{1}{x^2}+\frac{1}{x}$$
$$\int\frac{1}{y}\ \mathrm{d}y=\int\left(\frac{1}{x^2}+\frac{1}{x}\right)\mathrm{d}x$$
$$\ln y=-\frac{1}{x}+\ln x+\ln A$$
$$y=\mathrm{e}^{-\frac{1}{x}+\ln x+\ln A}=\mathrm{e}^{-\frac{1}{x}}\times\mathrm{e}^{\ln x}\times\mathrm{e}^{\ln A}$$
$$y=Ax\mathrm{e}^{-\frac{1}{x}}$$

b $$y=\mathrm{e}^4, x=-1\Rightarrow \mathrm{e}^4=-A\mathrm{e}$$
$$A=-\mathrm{e}^3$$
$$y=-\mathrm{e}^3x\mathrm{e}^{-\frac{1}{x}}=-x\mathrm{e}^{\left(\frac{3x-1}{x}\right)}$$

4 $$(2y+2yx)\frac{\mathrm{d}y}{\mathrm{d}x}=1-y^2$$
$$2y(1+x)\frac{\mathrm{d}y}{\mathrm{d}x}=1-y^2$$
$$\int\frac{2y}{1-y^2}\ \mathrm{d}y=\int\frac{1}{1+x}\ \mathrm{d}x$$
$$\ln k-\ln\left|1-y^2\right|=\ln|1+x|$$
$$\ln\left|\frac{k}{1-y^2}\right|=\ln|1+x|$$

$$\frac{k}{1-y^2}=1+x$$
$$x=0, y=0\Rightarrow k=1$$
$$\frac{1}{1-y^2}=1+x$$
$$1-y^2=\frac{1}{x+1}$$
$$y^2=1-\frac{1}{x+1}=\frac{x}{x+1}$$

$$y=\sqrt{\frac{x}{x+1}}$$

5 $$\mathrm{e}^{x+y}\frac{\mathrm{d}y}{\mathrm{d}x}=2x+x\mathrm{e}^y$$
$$\mathrm{e}^x\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x}=x(2+\mathrm{e}^y)$$
$$\frac{\mathrm{e}^y}{2+\mathrm{e}^y}\frac{\mathrm{d}y}{\mathrm{d}x}=x\mathrm{e}^{-x}$$
$$\int\frac{\mathrm{e}^y}{2+\mathrm{e}^y}\ \mathrm{d}y=\int x\mathrm{e}^{-x}\ \mathrm{d}x$$
$$\ln\left|2+\mathrm{e}^y\right|=\int x\mathrm{e}^{-x}\ \mathrm{d}x$$
Let $u=x\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=1$
$$\frac{\mathrm{d}v}{\mathrm{d}x}=\mathrm{e}^{-x}\Rightarrow v=-\mathrm{e}^{-x}$$

So $\ln\left|2+\mathrm{e}^y\right|=-x\mathrm{e}^{-x}+\int\mathrm{e}^{-x}\ \mathrm{d}x$
$$\ln\left|2+\mathrm{e}^y\right|=-x\mathrm{e}^{-x}-\mathrm{e}^{-x}+c$$

6 $(1-x^2)\frac{dy}{dx} = xy + y$

$$(1-x^2)\frac{dy}{dx} = y(x+1)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{x+1}{(1-x^2)} = \frac{x+1}{(1-x)(1+x)} = \frac{1}{1-x}$$

$$\int \frac{1}{y}\ dy = \int \frac{1}{1-x} dx$$

$$\ln y = \ln k - \ln|1-x| = \ln\left|\frac{k}{1-x}\right|$$

$$y = \frac{k}{1-x}$$

$$y = 6, x = 0.5 \Rightarrow 6 = 2k \Rightarrow k = 3$$

$$y = \frac{3}{1-x}$$

7 $(1+x^2)\frac{dy}{dx} = x - xy^2$

$$(1+x^2)\frac{dy}{dx} = x(1-y^2)$$

$$\int \frac{1}{1-y^2}\ dy = \int \frac{x}{1+x^2}\ dx$$

$$\frac{1}{2}\int\left(\frac{1}{1+y} + \frac{1}{1-y}\right)\ dy = \int \frac{x}{1+x^2}\ dx$$

$$\frac{1}{2}\left(\ln|1+y| - \ln|1-y|\right) = \frac{1}{2}\ln\left|1+x^2\right| + c$$

$$x = 0, y = 2 \Rightarrow c = \frac{1}{2}\ln 3$$

$$\left(\ln|1+y| - \ln|1-y|\right) = \ln\left|3(1+x^2)\right|$$

$$\frac{1+y}{y-1} = 3 + 3x^2$$

$$1 + y = 3y + 3x^2y - 3 - 3x^2$$

$$1 + 3 + 3x^2 = 3y + 3x^2y - y$$

$$3(x^2+1) + 1 = y(3(x^2+1) - 1)$$

$$y = \frac{3(1+x^2)+1}{3(1+x^2)-1}$$

8 $\frac{dy}{dx} = xe^{-y}$

$$\int e^y\ dy = \int x\ dx$$

$$e^y = \frac{x^2}{2} + c$$

$$x = 4, y = \ln 2 \Rightarrow 2 = 8 + c \Rightarrow c = -6$$

$$e^y = \frac{x^2 - 12}{2}$$

$$y = \ln\left|\frac{x^2-12}{2}\right|$$

9 $\frac{dy}{dx} = \cos^2 y + \cos 2x \cos^2 y$

$$\frac{dy}{dx} = \cos^2 y(1 + \cos 2x)$$

$$\int \sec^2 y\ dy = \int \left(1 + \cos 2x\right)\ dx$$

$$\tan y = x + \frac{1}{2}\sin 2x + c$$

$$x = \frac{\pi}{4}, y = \frac{\pi}{4} \Rightarrow 1 = \frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2} + c$$

$$c = \frac{1}{2} - \frac{\pi}{4} = \frac{2-\pi}{4}$$

$$\tan y = x + \frac{1}{2}\sin 2x + \frac{2-\pi}{4}$$

10 $\frac{dy}{dx} = xy\sin x$

$$\int \frac{1}{y}\ dy = \int x\sin x\ dx$$

$$\ln|y| = -x\cos x + \int \cos x\ dx$$

$$\ln|y| = -x\cos x + \sin x + c$$

$$x = \frac{\pi}{2}, y = 1 \Rightarrow c = -1$$

$$\ln|y| = -x\cos x + \sin x - 1$$

11 a $I = \int \frac{3x+4}{x}\ dx$

$$I = \int 3 + \frac{4}{x}\ dx = 3x + 4\ln x + c$$

11 b $$\frac{dy}{dx}=\frac{3x\sqrt{y}+4\sqrt{y}}{x}=\sqrt{y}\,\frac{3x+4}{x}$$

$$\int\frac{1}{\sqrt{y}}\,dy=3x+4\ln x+c \text{ (from } \mathbf{a}\text{)}$$

$$2\sqrt{y}=3x+4\ln x+c$$

$$x=1, y=16\Rightarrow 8=3+c\Rightarrow c=5$$

$$\sqrt{y}=\frac{3}{2}x+2\ln x+\frac{5}{2}$$

$$y=\left(\frac{3}{2}x+2\ln x+\frac{5}{2}\right)^2$$

12 a $$\frac{8x-18}{(3x-8)(x-2)}=\frac{A}{3x-8}+\frac{B}{x-2}$$

$$8x-18=A(x-2)+B(3x-8)$$

$$x=2: -2=-2B\Rightarrow B=1$$

$$x=\frac{8}{3}: \frac{64}{3}-18=\frac{2}{3}A\Rightarrow A=5$$

$$\frac{8x-18}{(3x-8)(x-2)}=\frac{5}{3x-8}+\frac{1}{x-2}$$

b $$(x-2)(3x-8)\frac{dy}{dx}=(8x-18)y$$

$$\int\frac{1}{y}\,dy=\int\frac{8x-18}{(3x-8)(x-2)}\,dx$$

$$\ln|y|=\int\left(\frac{5}{3x-8}+\frac{1}{x-2}\right)dx$$

$$\ln|y|=\frac{5}{3}\ln|3x-8|+\ln|x-2|+c$$

c $$x=3, y=8\Rightarrow c=\ln 8$$

$$\ln|y|=\ln|3x-8|^{\frac{5}{3}}+\ln|8x-2|$$

$$\ln|y|=\ln\left|(3x-8)^{\frac{5}{3}}(8x-2)\right|$$

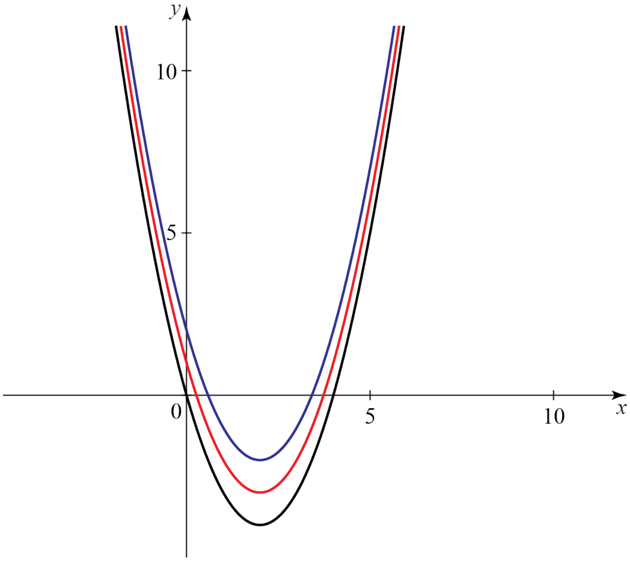
$$y=(3x-8)^{\frac{5}{3}}(8x-2)$$

$$y=8(x-2)(3x-8)^{\frac{5}{3}}$$

13 a $$\frac{dy}{dx}=2x-4$$

$$y=x^2-4x+c$$

b



$$y=x^2-4x,\quad y=x^2-4x+1,\quad y=x^2-4x+2$$

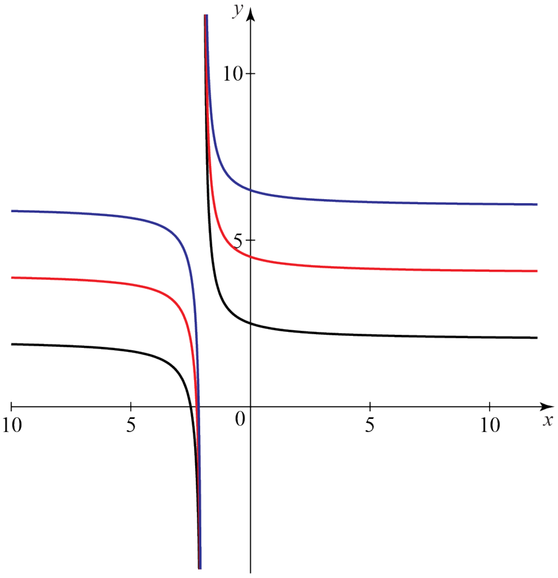
Any graphs of the form $y=x^2-4x+c$, where c is any real number.

14 a $$\frac{dy}{dx}=-\frac{1}{(x+2)^2}$$

$$y=-\int\frac{1}{(x+2)^2}\,dx$$

$$y=\frac{1}{x+2}+c$$

b



$$y=\frac{1}{x+2}+2,\quad y=\frac{1}{x+2}+4,\quad y=\frac{1}{x+2}+6$$

Any graphs of the form $y=\frac{1}{x+2}+c$, where c is any real number.

14 c $3.1 = \frac{1}{10} + c \Rightarrow c = 3$

$$y = \frac{1}{x+2} + 3$$

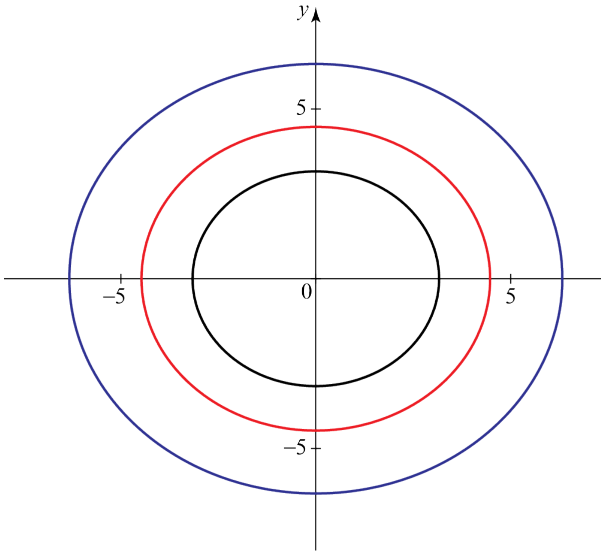
15 a $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$

$$\int y \, \mathrm{d}y = \int x \, \mathrm{d}x$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + k$$

$$x^2 + y^2 = c$$

15 b



$x^2 + y^2 = 10, \quad x^2 + y^2 = 20, \quad x^2 + y^2 = 40$

Circles with centre (0, 0) and radius $\sqrt{c}$ where c is any positive real number.

c $x^2 + y^2 = 49$

Integration 11K

1 a $$\frac{\mathrm{d}P}{\mathrm{d}t}=kP$$

$$\int\frac{1}{P}\ \mathrm{d}P=kt+c$$

$$\ln P=kt+c$$

$$t=0, P=200\Rightarrow c=\ln 200$$

$$\ln P-\ln 200=kt$$

$$\ln\left(\frac{P}{200}\right)=kt$$

$$\frac{P}{200}=\mathrm{e}^{kt}$$

$$P=200\mathrm{e}^{kt}$$

b $k=3$: $\quad 4000=200\mathrm{e}^{3t}$

$$\mathrm{e}^{3t}=20$$

$$t=\frac{1}{3}\ln 20\approx 1 \text{ year}$$

c The population could not increase in size in this way forever due to limitations such as available food or space.

2 $$\frac{\mathrm{d}M}{\mathrm{d}t}=M-M^2$$

$$\Rightarrow\int\frac{1}{M(1-M)}\mathrm{d}M=\int 1\mathrm{d}t$$

but $$\frac{1}{M(1-M)}\equiv\frac{A}{M}+\frac{B}{1-M}$$

$$\therefore 1\equiv A(1-M)+BM$$

$$M=0:1=1A, A=1$$

$$M=1:1=1B, B=1$$

$$\Rightarrow\int\left(\frac{1}{M}+\frac{1}{1-M}\right)\mathrm{d}M=\int 1\mathrm{d}t$$

$$\Rightarrow 1\mathrm{n}|M|-1\mathrm{n}|1-M|=t+c$$

$$\Rightarrow 1\mathrm{n}\left|\frac{M}{1-M}\right|=t+c$$

$$\Rightarrow\frac{M}{1-M}=A\mathrm{e}^t$$

a $$t=0, M=0.5\Rightarrow\frac{0.5}{0.5}=A^0\Rightarrow A=1$$

$$\therefore M=\mathrm{e}^t-\mathrm{e}^tM\Rightarrow M=\frac{\mathrm{e}^t}{1+\mathrm{e}^t}$$

b $$t=1\mathrm{n}2\Rightarrow M=\frac{\mathrm{e}^{1\mathrm{n}2}}{1+\mathrm{e}^{1\mathrm{n}2}}=\frac{2}{1+2}=\frac{2}{3}$$

c $$t\to\infty\Rightarrow M=\frac{1}{\mathrm{e}^{-t}+1}\to\frac{1}{1}=1$$

3 a $$\frac{\mathrm{d}x}{\mathrm{d}t}\propto\frac{1}{x^2}\Rightarrow\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{k}{x^2}$$

$$\int x^2\ \mathrm{d}x=kt$$

$$\frac{x^3}{3}=kt+c$$

$$t=0, x=1\Rightarrow c=\frac{1}{3}$$

$$x^3=3kt+1$$

$$t=20, x=2\Rightarrow 8=60k+1\Rightarrow k=\frac{7}{60}$$

$$x^3=\frac{7}{20}t+1$$

$$x=\sqrt[3]{\frac{7}{20}t+1}$$

b $$x=3\Rightarrow 27=\frac{7}{20}t+1$$

$$t=\frac{520}{7}=74.3\text{ days}$$

So time taken to go from 2 cm to 3 cm is 74.3 – 20 = 54.3 days.

4 a $$\frac{\mathrm{d}T}{\mathrm{d}t}\propto-(T-25)$$

$$\frac{\mathrm{d}T}{\mathrm{d}t}=-k(T-25)$$

The difference in temperature is $T-25$. The tea is cooling, so there should be a negative sign. k has to be positive or the tea would be warming.

4 b $\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T-25)$

$$\int \frac{1}{T-25}\ \mathrm{d}T = -kt + c$$

$$\ln|T-25| = -kt + c$$

$$t = 0, T = 85 \Rightarrow c = \ln 60$$

$$\frac{T-25}{60} = \mathrm{e}^{-kt}$$

$$t = 10, T = 55 \Rightarrow \frac{30}{60} = \mathrm{e}^{-10k}$$

$$\ln\frac{1}{2} = -10k \Rightarrow k = 0.0693$$

$$t = 15 \Rightarrow \frac{T-25}{60} = \mathrm{e}^{-0.0693\times 15}$$

$T = 60\times \mathrm{e}^{-0.0693\times 15} + 25 = 46.2\ ^\circ\mathrm{C}$ to 1 d.p.

5 a $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{A^{\frac{3}{2}}}{10t^2}$

$$\int A^{-\frac{3}{2}}\ \mathrm{d}A = \int \frac{1}{10t^2}\ \mathrm{d}t$$

$$-\frac{2}{\sqrt{A}} = -\frac{1}{10t} + c$$

$$t = 1, A = 1 \Rightarrow -2 = -\frac{1}{10} + c$$

$$c = -\frac{19}{10}$$

$$-\frac{2}{\sqrt{A}} = -\frac{1}{10t} - \frac{19}{10}$$

$$\frac{2}{\sqrt{A}} = \frac{1}{10t} + \frac{19}{10} = \frac{1+19t}{10t}$$

$$\sqrt{A} = \frac{20t}{1+19t}$$

$$A = \left(\frac{20t}{1+19t}\right)^2$$

b As $t \to \infty, A \to \left(\frac{20}{19}\right)^2 = \frac{400}{361}$ from below.

6 a Volume $V = 6000h \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}h} = 6000$

$\frac{\mathrm{d}V}{\mathrm{d}t} = 12\,000 - 500h$ as the tub is filling at the rate of $12\,000\ \mathrm{cm}^3/\mathrm{min}$ and losing water at the rate of $500h\ \mathrm{cm}^3/\mathrm{min}$.

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{6000}(12\,000 - 500h)$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{60}(120 - 5h)$$

$$60\frac{\mathrm{d}h}{\mathrm{d}t} = 120 - 5h$$

b $60\frac{dh}{dt} = 120 - 5h$

$$\int \frac{60}{120-5h}\ \mathrm{d}h = t + c$$

$$-12\ln(120-5h) = t + c$$

$$t = 0, h = 6 \Rightarrow c = -12\ln 90$$

$$-12\ln(120-5h) = t - 12\ln 90$$

$h = 10$:

$t = 12\ln 90 - 12\ln 70$

$t = 12\ln\frac{9}{7}$

7 a $\frac{1}{P(10\,000-P)} = \frac{A}{P} + \frac{B}{10\,000-P}$

$$1 = A(10\,000-P) + BP$$

$$P = 0 \Rightarrow A = \frac{1}{10\,000}$$

$$P = 10\,000 \Rightarrow B = \frac{1}{10\,000}$$

$$\frac{1}{P(10\,000-P)} = \frac{\frac{1}{10\,000}}{P} + \frac{\frac{1}{10\,000}}{10\,000-P}$$

7 **b** $\dfrac{dP}{dt} = \dfrac{1}{200}P(10\,000 - P)$

$$\frac{1}{10\,000}\int\left(\frac{1}{P} + \frac{1}{10\,000 - P}\right)dP = \frac{1}{200}t + c$$

$$\frac{1}{10\,000}\left(\ln|P| - \ln|10\,000 - P|\right) = \frac{1}{200}t + c$$

$$\ln P - \ln|10\,000 - P| = 50t + d$$

$$t = 0, P = 2500 \Rightarrow d = \ln\left(\frac{2500}{7500}\right) = \ln\frac{1}{3}$$

$$\ln P - \ln|10\,000 - P| - \ln\frac{1}{3} = 50t$$

$$\ln\left|\frac{3P}{10\,000 - P}\right| = 50t$$

$$\frac{3P}{10\,000 - P} = e^{50t}$$

$$3P = (10\,000 - P)e^{50t}$$

$$3Pe^{-50t} + P = 10\,000$$

$$P(3e^{-50t} + 1) = 10\,000$$

$$P = \frac{10\,000}{1 + 3e^{-50t}}$$

$$a = 10\,000, b = 1, c = 3$$

c Maximum number of deer is when

$$\frac{dP}{dt} = 0 \Rightarrow P = 0 \text{ or } 10\,000$$

So maximum population is 10 000.

8 **a** $\dfrac{dV}{dt} = 40 - \dfrac{1}{4}V$

$$-4\frac{dV}{dt} = V - 160$$

b $\displaystyle\int -\frac{4}{V - 160}\,dV = t + c$

$$-4\ln|V - 160| = t + c$$

$$t = 0, V = 5000 \Rightarrow c = -4\ln 4840$$

$$4\ln 4840 - 4\ln|V - 160| = t$$

$$\ln\left|\frac{4840}{V - 60}\right| = \frac{t}{4}$$

$$\frac{4840}{V - 60} = e^{\frac{t}{4}}$$

$$\frac{V - 160}{4840} = e^{-\frac{t}{4}}$$

$$V = 160 + 4840e^{-\frac{t}{4}}$$

$$a = 160, b = 4840$$

c $t \to \infty \Rightarrow V \to 160$, as $e^{-\frac{t}{4}} \to 0$

9 **a** $\dfrac{dR}{dt} = -kR$

$$\ln|R| = -kt + c$$

$$t = 0, R = R_0 \Rightarrow c = \ln R_0$$

$$\ln R - \ln R_0 = -kt$$

$$\frac{R}{R_0} = e^{-kt}$$

$$R = R_0e^{-kt}$$

b $t = 5730, R = \dfrac{R_0}{2} \Rightarrow \dfrac{R_0}{2} = R_0e^{-5730k}$

$$e^{5730k} = 2$$

$$5730k = \ln 2$$

$$k = \frac{1}{5730}\ln 2$$

c $R = \dfrac{R_0}{10} \Rightarrow \dfrac{R_0}{10} = R_0e^{-kt}$

$$e^{kt} = 10$$

$$t = \frac{1}{k}\ln 10 = \frac{5730\ln 10}{\ln 2} = 19\,035$$

Integration 11L

1 **a** $$\lim_{\delta x\to 0}\sum_{x=3}^{12} x^2\ \delta x = \int_3^{12} x^2 \mathrm{d}x$$

This is the limit as the width of the strip tends towards 0.

$$\int_3^{12} x^2 \mathrm{d}x = \left[\frac{x^3}{3}\right]_3^{12}$$
$$= 576 - 9$$
$$= 567$$

b $$\lim_{\delta x\to 0}\sum_{x=9}^{25} \sqrt{x}\ \delta x = \int_9^{25} \sqrt{x}\, \mathrm{d}x$$

This is the limit as the width of the strip tends towards 0.

$$\int_9^{25} \sqrt{x}\, \mathrm{d}x = \int_9^{25} x^{\frac{1}{2}}\, \mathrm{d}x$$
$$= \left[\frac{2x^{\frac{3}{2}}}{3}\right]_9^{25}$$
$$= \frac{250}{3} - \frac{54}{3}$$
$$= \frac{196}{3}$$

c $$\lim_{\delta x\to 0}\sum_{x=5}^{10} \left(x\sqrt{x-1}\right)\delta x = \int_5^{10} \left(x\sqrt{x-1}\right)\mathrm{d}x$$

This is the limit as the width of the strip tends towards 0.

Let $I = \int_5^{10} \left(x\sqrt{x-1}\right)\mathrm{d}x$

Let $u = x - 1$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 1$$

So replace dx with du
and replace $\sqrt{x-1}$ with u
and replace x with u + 1

Change the limits:

x	u
10	9
5	4

c $$I = \int_4^9 \left((u+1)u^{\frac{1}{2}}\right)\mathrm{d}u$$
$$= \int_4^9 \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)\mathrm{d}u$$
$$= \left[\frac{2u^{\frac{5}{2}}}{5} + \frac{2u^{\frac{3}{2}}}{3}\right]_4^9$$
$$= \frac{2\times 9^{\frac{5}{2}}}{5} + \frac{2\times 9^{\frac{3}{2}}}{3} - \frac{2\times 4^{\frac{5}{2}}}{5} - \frac{2\times 4^{\frac{3}{2}}}{3}$$
$$= \frac{486}{5} + \frac{54}{3} - \frac{64}{5} - \frac{16}{3}$$
$$= \frac{1456}{15}$$

2 $$\lim_{\delta x\to 0}\sum_{x=2}^{3} \ln x\ \delta x = \int_2^3 \ln x\, \mathrm{d}x$$

This is the limit as the width of the strip tends towards 0.

Let $I = \int_2^3 \ln x\, \mathrm{d}x$

Let $u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = 1 \quad \Rightarrow v = x$$

$$I = \left[x\ln x\right]_2^3 - \int_2^3 x\times\frac{1}{x}\mathrm{d}x$$
$$= 3\ln 3 - 2\ln 2 - \int_2^3 1\mathrm{d}x$$
$$= \ln 3^3 - \ln 2^2 - \left[x\right]_2^3$$
$$= \ln\left(\frac{27}{4}\right) - 3 + 2$$
$$= -1 + \ln\left(\frac{27}{4}\right)$$

3 $\lim\limits_{\delta x \to 0} \sum\limits_{x=2}^{5} \sqrt[3]{x}\ \delta x = \int_2^5 \sqrt[3]{x}\ \mathrm{d}x$

This is the limit as the width of the strip tends towards 0.

$$\int_2^5 \sqrt[3]{x}\ \mathrm{d}x = \int_2^5 x^{\frac{1}{3}}\ \mathrm{d}x$$

$$= \left[\frac{3x^{\frac{4}{3}}}{4}\right]_2^5$$

$$= 6.41240... - 1.88988...$$

$$= 4.523 \text{ (4 s.f.)}$$

Integration, Mixed exercise 11

1 a $I=\int(2x-3)^7\ \mathrm{d}x$

Consider $y=(2x-3)^8$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=16(2x-3)^7$$

$$I=\frac{(2x-3)^8}{16}+c$$

b $I=\int x\sqrt{4x-1}\ \mathrm{d}x$

Let $u=4x-1\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=4$

$$I=\int\frac{u+1}{16}\sqrt{u}\ \mathrm{d}u$$

$$=\frac{2u^{\frac{5}{2}}}{80}+\frac{2u^{\frac{3}{2}}}{48}+c$$

$$=\frac{(4x-1)^{\frac{5}{2}}}{40}+\frac{(4x-1)^{\frac{3}{2}}}{24}+c$$

c $I=\int\sin^2 x\cos x\ \mathrm{d}x$

Consider $y=\sin^3 x\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=3\sin^2 x\cos x$

$$I=\frac{1}{3}\sin^3 x+c$$

d $I=\int x\ln x\ \mathrm{d}x$

Let $u=\ln x\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x}$

$$\frac{\mathrm{d}v}{\mathrm{d}x}=x\Rightarrow v=\frac{x^2}{2}$$

$$I=\frac{x^2}{2}\ln x-\int\frac{x^2}{2}\times\frac{1}{x}\ \mathrm{d}x$$

$$=\frac{x^2}{2}\ln x-\frac{x^2}{4}+c$$

e $I=\int\frac{4\sin x\cos x}{4-8\sin^2 x}\ \mathrm{d}x$

$$I=\int\frac{2\sin 2x}{4(1-2\sin^2 x)}\ \mathrm{d}x$$

$$I=\int\frac{2\sin 2x}{4\cos 2x}\ \mathrm{d}x$$

$$=-\frac{1}{4}\ln|\cos 2x|+c$$

f $I=\int\frac{1}{3-4x}\ \mathrm{d}x$

$$=-\frac{1}{4}\ln|3-4x|+c$$

2 a $I=\int_{-3}^{0}x(x^2+3)^5\ \mathrm{d}x$

Consider $y=(x^2+3)^6$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=12x(x^2+3)^5$$

$$I=\left[\frac{1}{12}(x^2+3)^6\right]_{-3}^{0}$$

$$=\frac{1}{12}(729-2985984)$$

$$=-\frac{995085}{4}$$

b $I=\int_0^{\frac{\pi}{4}}x\sec^2 x\ \mathrm{d}x$

Let $u=x\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=1$

$$\frac{\mathrm{d}v}{\mathrm{d}x}=\sec^2 x\Rightarrow v=\tan x$$

$$I=[x\tan x]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\tan x\ \mathrm{d}x$$

$$=\frac{\pi}{4}+[\ln|\cos x|]_0^{\frac{\pi}{4}}=\frac{\pi}{4}+\ln\frac{1}{\sqrt{2}}$$

$$=\frac{\pi}{4}-\frac{1}{2}\ln 2$$

c $I=\int_1^4\left(16x^{\frac{3}{2}}-\frac{2}{x}\right)\mathrm{d}x$

$$=\left[\frac{32}{5}x^{\frac{5}{2}}-2\ln|x|\right]_1^4$$

$$=\frac{1024}{5}-2\ln 4-\frac{32}{5}$$

$$=\frac{992}{5}-2\ln 4$$

2 d $I = \int_{\frac{\pi}{12}}^{\frac{\pi}{3}} (\cos x + \sin x)(\cos x - \sin x)\ \mathrm{d}x$

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{3}} (\cos^2 x - \sin^2 x)\ \mathrm{d}x$$

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{3}} \cos 2x\ \mathrm{d}x$$

$$= \left[\frac{1}{2}\sin 2x\right]_{\frac{\pi}{12}}^{\frac{\pi}{3}}$$

$$= \frac{\sqrt{3}}{4} - \frac{1}{4}$$

$$= \frac{\sqrt{3}-1}{4}$$

e $I = \int_1^4 \frac{4}{16x^2 + 8x - 3}\ \mathrm{d}x$

$$\frac{4}{16x^2 + 8x - 3} = \frac{4}{(4x+3)(4x-1)}$$

$$\frac{4}{(4x+3)(4x-1)} = \frac{A}{4x+3} + \frac{B}{4x-1}$$

$$4 = A(4x-1) + B(4x+3)$$

$$x = \frac{1}{4} \Rightarrow 4 = 4B \Rightarrow B = 1$$

$$x = -\frac{3}{4} \Rightarrow 4 = -4A \Rightarrow A = -1$$

$$I = \int_1^4 \frac{1}{4x-1} - \frac{1}{4x+3}\ \mathrm{d}x$$

$$= \frac{1}{4}\left[\ln|4x-1| - \ln|4x+3|\right]_1^4$$

$$= \frac{1}{4}(\ln 15 - \ln 19 - \ln 3 + \ln 7)$$

$$= \frac{1}{4}\ln\frac{105}{57}$$

$$= \frac{1}{4}\ln\frac{35}{19}$$

f $I = \int_0^{\ln 2} \frac{1}{1+\mathrm{e}^x}\ \mathrm{d}x$

Let $u = 1 + \mathrm{e}^x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x = u - 1$

$$I = \int_2^3 \frac{1}{(u-1)u}\ \mathrm{d}u$$

$$I = \int_2^3 \left(\frac{1}{(u-1)} - \frac{1}{u}\right)\ \mathrm{d}u$$

$$= \left[\ln|u-1| - \ln|u|\right]_2^3$$

$$= \ln 2 - \ln 3 - \ln 1 + \ln 2$$

$$= \ln\frac{4}{3}$$

3 a $I = \int_1^{\mathrm{e}} \frac{1}{x^2}\ln x\,\mathrm{d}x$

$$u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x^2} \Rightarrow v = -\frac{1}{x}$$

$$\therefore I = \left[-\frac{1}{x}\ln x\right]_1^{\mathrm{e}} - \int_1^{\mathrm{e}} \left(-\frac{1}{x^2}\right)\mathrm{d}x$$

$$= \left(-\frac{1}{\mathrm{e}}\right) - (0) + \left[-\frac{1}{x}\right]_1^{\mathrm{e}}$$

$$= -\frac{1}{\mathrm{e}} + \left(-\frac{1}{\mathrm{e}}\right) - (-1)$$

$$= 1 - \frac{2}{\mathrm{e}}$$

3 b

$$\frac{1}{(x+1)(2x-1)} \equiv \frac{A}{x+1} + \frac{B}{2x-1}$$

$$\Rightarrow 1 \equiv A(2x-1) + B(x+1)$$

$$x - \frac{1}{2} \Rightarrow 1 = \frac{3}{2}B \Rightarrow B = \frac{2}{3}$$

$$x = -1 \Rightarrow 1 = -3A \Rightarrow A = -\frac{1}{3}$$

$$\therefore \int_1^p \frac{1}{(x+1)(2x-1)}\,\mathrm{d}x = \int_1^p \left(\frac{\frac{2}{3}}{2x-1} + \frac{-\frac{1}{3}}{x+1}\right)\mathrm{d}x$$

$$= \left[\frac{1}{3}1\mathrm{n}\,|2x-1| - \frac{1}{3}1\mathrm{n}\,|x+1|\right]_1^p$$

$$= \left[\frac{1}{3}1\mathrm{n}\left|\frac{2x-1}{x+1}\right|\right]_1^p$$

$$= \frac{1}{3}1\mathrm{n}\left(\frac{2p-1}{p+1}\right) - \left(\frac{1}{3}1\mathrm{n}\,\frac{1}{2}\right)$$

$$= \frac{1}{3}1\mathrm{n}\left(\frac{4p-2}{p+1}\right)$$

4

$$\int_{\frac{1}{2}}^{b}\left(\frac{2}{x^3} - \frac{1}{x^2}\right)\,\mathrm{d}x = \frac{9}{4}$$

$$\left[-\frac{1}{x^2} + \frac{1}{x}\right]_{\frac{1}{2}}^{b} = \frac{9}{4}$$

$$-\frac{1}{b^2} + \frac{1}{b} + 4 - 2 = \frac{9}{4}$$

$$\frac{b-1}{b^2} = \frac{1}{4}$$

$$b^2 - 4b + 4 = 0$$

$$(b-2)^2 = 0$$

$$b = 2$$

5

$$I = \int_0^\theta \cos x \sin^3 x\ \mathrm{d}x = \frac{9}{64}$$

$$\left[\frac{\sin^4 x}{4}\right]_0^\theta = \frac{9}{64}$$

$$\sin^4\theta = \frac{9}{16}$$

$$\sin\theta = \pm\frac{\sqrt{3}}{2}$$

$$\theta = \frac{\pi}{3}$$

6 a

$$t^2 = x+1 \Rightarrow 2t\,\mathrm{d}t = \mathrm{d}x$$

$$\therefore I = \int \frac{x}{\sqrt{x+1}}\mathrm{d}x$$

$$= \int \frac{t^2-1}{t} \times 2t\,\mathrm{d}t$$

$$= \int (2t^2-2)\mathrm{d}t$$

$$= \frac{2}{3}t^3 - 2t + c$$

$$= \frac{2}{3}(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + c$$

$$= \frac{2}{3}\sqrt{x+1}(x-2) + c$$

b

$$\int_0^3 \frac{x}{\sqrt{x+1}}\mathrm{d}x = \left[\frac{2}{3}(x-2)\sqrt{x+1}\right]_0^3$$

$$= \left(\frac{2}{3}\times 2\right) - \left(-\frac{4}{3}\right) = \frac{8}{3}$$

7 a

$$I = \int x \sin 8x\ \mathrm{d}x$$

Let $u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin 8x \Rightarrow v = -\frac{1}{8}\cos 8x$$

$$I = -\frac{1}{8}x\cos 8x + \frac{1}{8}\int \cos 8x\ \mathrm{d}x$$

$$= -\frac{1}{8}x\cos 8x + \frac{1}{64}\sin 8x + c$$

b

$$I = \int x^2 \cos 8x\ \mathrm{d}x$$

Let $u = x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \cos 8x \Rightarrow v = \frac{1}{8}\sin 8x$$

$$I = \frac{1}{8}x^2\sin 8x - \frac{2}{8}\int x\sin 8x\ \mathrm{d}x$$

$$= \frac{1}{8}x^2\sin 8x - \frac{2}{8}\left(-\frac{1}{8}x\cos 8x + \frac{1}{64}\sin 8x\right) + c$$

$$= \frac{1}{8}x^2\sin 8x + \frac{1}{32}x\cos 8x - \frac{1}{256}\sin 8x + c$$

8 a $$\mathrm{f}(x) \equiv \frac{5x^2-8x+1}{2x(x-1)^2} \equiv \frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

$$\Rightarrow 5x^2-8x+1 \equiv 2A(x-1)^2$$
$$+2Bx(x-1)+2Cx$$

$$x=0 \Rightarrow 1=2A \Rightarrow A=\frac{1}{2}$$

$$x=1 \Rightarrow -2=2C \Rightarrow C=-1$$

Coefficients of x^2: $5=2A+2B \Rightarrow B=2$

b $$\int \mathrm{f}(x)\mathrm{d}x = \int\left(\frac{\frac{1}{2}}{x}+\frac{2}{x-1}-\frac{1}{(x-1)^2}\right)\mathrm{d}x$$

$$=\frac{1}{2}\ln|x|+2\ln|x-1|+\frac{1}{x-1}+c$$

c $$\int_4^9 \mathrm{f}(x)\mathrm{d}x = \left[\frac{1}{2}\ln|x|+2\ln|x-1|+\frac{1}{x-1}\right]_4^9$$

$$=\left[\mathrm{1n}\left|\sqrt{x(x-1)^2}\right|+\frac{1}{x-1}\right]_4^9$$

$$=\left[\mathrm{1n}(3\times 64)+\frac{1}{8}\right]-\left[\mathrm{1n}(2\times 9)+\frac{1}{3}\right]$$

$$=\mathrm{1n}\left(\frac{3\times 64}{2\times 9}\right)+\frac{1}{8}-\frac{1}{3}$$

$$=\mathrm{1n}\frac{32}{3}-\frac{5}{24}$$

9 a $$y=x^{\frac{3}{2}}+48x^{-1} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3}{2}x^{\frac{1}{2}}-48x^{-2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow \frac{3}{2}x^{\frac{1}{2}}=\frac{48}{x^2}$$

$$\Rightarrow x^{\frac{5}{2}}=\frac{2}{3}\times 48=32$$

$$\Rightarrow x=4,\ y=2^3+12=20$$

$$\Rightarrow x=4, y=20$$

b $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{3}{4}x^{-\frac{1}{2}}+96x^{-3}>\text{for all } x>0$$

$\therefore$ 20 is a minimum value of y

c $$\text{Area } = \int_1^4\left(x^{\frac{3}{2}}+\frac{48}{x}\right)\mathrm{d}x$$

$$=\left[\frac{2}{5}x^{\frac{5}{2}}+48\ln|x|\right]_1^4$$

$$=\left(\frac{2}{5}\times 32+48\ln 4\right)-\left(\frac{2}{5}+0\right)$$

$$=\frac{62}{5}+48\ln 4$$

10 a $$I=\int x^2\ln 2x\ \mathrm{d}x$$

Let $u=\ln 2x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x}$

$$\frac{\mathrm{d}v}{\mathrm{d}x}=x^2 \Rightarrow v=\frac{x^3}{3}$$

$$I=\frac{1}{3}x^3\ln 2x-\int\frac{x^2}{3}\ \mathrm{d}x$$

$$=\frac{1}{3}x^3\ln 2x-\frac{1}{9}x^3+c$$

b $$\int_{\frac{1}{2}}^3 x^2\ln 2x\ \mathrm{d}x=\left[\frac{1}{3}x^3\ln 2x-\frac{1}{9}x^3\right]_{\frac{1}{2}}^3$$

$$=9\ln 6-3-0+\frac{1}{72}$$

$$=9\ln 6-\frac{215}{9}$$

11 a $$y=(1+\sin 2x)^2$$

$$=1+2\sin 2x+\sin^2 2x$$

$$=1+2\sin 2x+\frac{1}{2}-\frac{1}{2}\cos 4x$$

$$=\frac{1}{2}(3+4\sin 2x-\cos 4x)$$

11 b Area of $R = \int_0^{\frac{3\pi}{4}} (1+\sin 2x)^2 \; \mathrm{d}x$

$$= \frac{1}{2}\int_0^{\frac{3\pi}{4}} (3+4\sin 2x - \cos 4x) \; \mathrm{d}x$$

$$= \frac{1}{2}\left[3x - 2\cos 2x - \frac{1}{4}\sin 4x\right]_0^{\frac{3\pi}{4}}$$

$$= \left(\frac{9\pi}{8} - 0 - 0\right) - (0 - 1 - 0)$$

$$= \frac{9\pi}{8} + 1$$

c $\frac{\mathrm{d}y}{\mathrm{d}x} = 4\cos 2x + 2\sin 4x$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 4\cos 2x + 2\sin 4x = 0$$

$$4\cos 2x + 4\sin 2x\cos 2x = 0$$

$$4\cos 2x(1+\sin 2x) = 0$$

$$2x = \frac{\pi}{2} \quad \text{for } x < \frac{3\pi}{4}$$

$$x = \frac{\pi}{4}, y = \left(1+\sin\frac{\pi}{2}\right)^2 = 4$$

Coordinates of $A \left(\frac{\pi}{4}, 4\right)$

12 a $I = \int x\mathrm{e}^{-x}\mathrm{d}x$

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{-x} \Rightarrow v = -\mathrm{e}^{-x}$$

$$\therefore I = -x\mathrm{e}^{-x} - \int(-\mathrm{e}^{-x})\mathrm{d}x$$

i.e. $I = -x\mathrm{e}^{-x} - \mathrm{e}^{-x} + c$

b $\mathrm{e}^x \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{\sin 2y}$

$$\Rightarrow \int \sin 2y \, \mathrm{d}y = \int x\mathrm{e}^{-x}\mathrm{d}x$$

$$\Rightarrow -\frac{1}{2}\cos 2y = -x\mathrm{e}^{-x} - \mathrm{e}^{-x} + c$$

$$x = 0, y = \frac{\pi}{4} \Rightarrow 0 = 0 - 1 + c \Rightarrow c = 1$$

$$\therefore \frac{1}{2}\cos 2y = x\mathrm{e}^{-x} + \mathrm{e}^{-x} - 1$$

or $\cos 2y = 2(x\mathrm{e}^{-x} + \mathrm{e}^{-x} - 1)$

13 a $I = \int x\sin 2x \; \mathrm{d}x$

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin 2x \Rightarrow v = \frac{-1}{2}\cos 2x$$

$$\therefore I = -\frac{1}{2}x\cos 2x - \int \frac{-1}{2}\cos 2x \; \mathrm{d}x$$

$$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + c$$

b $\frac{\mathrm{d}y}{\mathrm{d}x} = x\sin 2x\cos^2 y$

$$\Rightarrow \int \sec^2 y \, \mathrm{d}y = \int x\sin 2x \, \mathrm{d}x$$

$$\Rightarrow \tan y = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + c$$

$$y = 0, \; x = \frac{\pi}{4} \Rightarrow 0 = 0 + \frac{1}{4} + c \Rightarrow c = -\frac{1}{4}$$

$$\therefore \tan y = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x - \frac{1}{4}$$

14 a $\frac{\mathrm{d}y}{\mathrm{d}x} = xy^2$

$$\Rightarrow \int \frac{1}{y^2}\mathrm{d}y = \int x\,\mathrm{d}x$$

$$\Rightarrow -\frac{1}{y} = \frac{x^2}{2} + c$$

or $y = \frac{-2}{x^2 + k} \qquad (k = 2c)$

b $y = 1, x = 1 \Rightarrow 1 = \frac{-2}{1+k} \Rightarrow k = -3$

$$\therefore y = \frac{2}{3 - x^2}$$

for $x^2 \neq 3$ and $y > 0$, i.e. $-\sqrt{3} < x < \sqrt{3}$

c When $x = 1, y = 1, \frac{\mathrm{d}y}{\mathrm{d}x}$ is 1

14 d Equation of tangent is:

$$y-1=1(x-1)$$
$$y=x$$

This meets the curve again when:

$$x=\frac{2}{3-x^2}$$
$$3x-x^3=2$$
$$x^3-3x+2=0$$
$$(x-1)(x-1)(x+2)=0$$

Other point is when $x=-2, y=-2$
i.e. (−2, −2)

15 a $I=\int\frac{4x}{(1+2x)^2}\,\mathrm{d}x$

$$u=1+2x$$
$$\Rightarrow\frac{\mathrm{d}u}{2}=\mathrm{d}x \text{ and } 4x=2(u-1)$$
$$\therefore I=\int\frac{2(u-1)}{u^2}\times\frac{\mathrm{d}u}{2}$$
$$=\int\left(\frac{1}{u}-u^{-2}\right)\mathrm{d}u$$
$$=\ln|u|+\frac{1}{u}+c$$
$$=\ln|1+2x|+\frac{1}{1+2x}+c$$

b $(1+2x)^2\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x}{\sin^2 y}$

$$\Rightarrow\int\sin^2 y\,\mathrm{d}y=\int\frac{x}{(1+2x)}\mathrm{d}x$$
$$\Rightarrow\int 4\sin^2 y\,\mathrm{d}y=\int\frac{4x}{(1+2x)^2}\mathrm{d}x$$
$$\Rightarrow\int(2-2\cos 2y)\,\mathrm{d}y=I$$
$$\Rightarrow 2y-\sin 2y=\ln|1+2x|+\frac{1}{1+2x}+c$$
$$x=0, y=\frac{\pi}{4}\Rightarrow\frac{\pi}{2}-1=\ln 1+1+c$$
$$\Rightarrow c=\frac{\pi}{2}-2$$
$$\therefore 2y-\sin 2y=\ln|1+2x|+\frac{1}{1+2x}+\frac{\pi}{2}-2$$

16 a $\int xe^{2x}\mathrm{d}x$

$$u=x\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=1$$
$$\frac{\mathrm{d}v}{\mathrm{d}x}=e^{2x}\Rightarrow v=\frac{1}{2}e^{2x}$$
$$\therefore\int xe^{2x}\mathrm{d}x=\frac{1}{2}xe^{2x}$$
$$-\int\frac{1}{2}e^{2x}\mathrm{d}x=\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+c$$

$$A_1=-\left[\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}\right]_{-\frac{1}{2}}^{0}$$
$$=-\left(\left(0-\frac{1}{4}\right)-\left(-\frac{1}{4}e^{-1}-\frac{1}{4}e^{-1}\right)\right)$$
$$=\frac{1}{4}(1-2e^{-1})$$

$$A_2=\left[\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}\right]_{0}^{\frac{1}{2}}$$
$$=\left(\frac{1}{4}e^{1}-\frac{1}{4}e^{1}\right)-\left(0-\frac{1}{4}\right)$$
$$=\frac{1}{4}$$

16 b $$\frac{A_1}{A_2}=\frac{\frac{1}{4}\left(1-2\mathrm{e}^{-1}\right)}{\frac{1}{4}}=1-2\mathrm{e}^{-1}=\frac{\mathrm{e}-2}{\mathrm{e}}$$

$$\therefore A_1 : A_2=(\mathrm{e}-2):\mathrm{e}$$

17 a $$I=\int x^2\mathrm{e}^{-x}\,\mathrm{d}x$$

Let $u=x^2\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=2x$

$$\frac{\mathrm{d}v}{\mathrm{d}x}=\mathrm{e}^{-x}\Rightarrow v=-\mathrm{e}^{-x}$$

$$I=-x^2\mathrm{e}^{-x}+2\int x\mathrm{e}^{-x}\,\mathrm{d}x$$

Again, let $u=x\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=1$

$$\frac{\mathrm{d}v}{\mathrm{d}x}=\mathrm{e}^{-x}\Rightarrow v=-\mathrm{e}^{-x}$$

$$I=-x^2\mathrm{e}^{-x}-2x\mathrm{e}^{-x}+2\int\mathrm{e}^{-x}\,\mathrm{d}x$$

$$=-x^2\mathrm{e}^{-x}-2x\mathrm{e}^{-x}-2\mathrm{e}^{-x}+c$$

$$=-\mathrm{e}^{-x}\left(x^2+2x+2\right)+c$$

b $$\frac{\mathrm{d}y}{\mathrm{d}x}=x^2\mathrm{e}^{3y-x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=x^2\mathrm{e}^{3y}\mathrm{e}^{-x}$$

$$\int\mathrm{e}^{-3y}\,\mathrm{d}x=\int x^2\mathrm{e}^{-x}\,\mathrm{d}x$$

$$-\frac{1}{3}\mathrm{e}^{-3y}=-\mathrm{e}^{-x}\left(x^2+2x+2\right)+c$$

$$x=0,y=0\Rightarrow-\frac{1}{3}=-2+c\Rightarrow c=\frac{5}{3}$$

$$\mathrm{e}^{-3y}=3\mathrm{e}^{-x}\left(x^2+2x+2\right)-5$$

$$3y=-\ln\left(3\mathrm{e}^{-x}\left(x^2+2x+2\right)-5\right)$$

$$y=-\frac{1}{3}\ln\left(3\mathrm{e}^{-x}\left(x^2+2x+2\right)-5\right)$$

18 a $$y=\mathrm{e}^{3x}+1$$

$$y=8\Rightarrow\mathrm{e}^{3h}=7$$

$$3h=\ln 7$$

$$h=\frac{1}{3}\ln 7$$

b Area of *R* is given by

$$\int_0^{\frac{1}{3}\ln 7}\left(\mathrm{e}^{3x}+1\right)\mathrm{d}x=\left[\frac{1}{3}\mathrm{e}^{3x}+x\right]_0^{\frac{1}{3}\ln 7}$$

$$=\left(\frac{1}{3}\mathrm{e}^{\ln 7}+\frac{1}{3}\ln 7\right)-\left(\frac{1}{3}+0\right)$$

$$=2+\frac{1}{3}\ln 7$$

19 a $$\frac{x^2}{x^2-1}\equiv A+\frac{B}{x-1}+\frac{C}{x+1}$$

$$\Rightarrow x^2\equiv A(x-1)(x+1)+B(x+1)+C(x-1)$$

$$x=1\Rightarrow 1=2B\Rightarrow B=\frac{1}{2}$$

$$x=-1\Rightarrow 1=-2C\Rightarrow C=-\frac{1}{2}.$$

Coefficients of x^2: $1=A\Rightarrow A=1$

b $$\frac{\mathrm{d}x}{\mathrm{d}t}=2\frac{(x^2-1)}{x^2}$$

$$\Rightarrow\int\frac{x^2}{x^2-1}\mathrm{d}x=\int 2\,\mathrm{d}t$$

$$\Rightarrow\int\left(1+\frac{\left(\frac{1}{2}\right)}{x-1}-\frac{\left(\frac{1}{2}\right)}{x+1}\right)\mathrm{d}x=2t$$

$$\Rightarrow x+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|=2t+c$$

$$x=2,t=1\Rightarrow 2+\frac{1}{2}\ln\frac{1}{3}=2+c\Rightarrow c=\frac{1}{2}\ln\frac{1}{3}$$

$$\therefore x+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|=2t+\frac{1}{2}\ln\frac{1}{3}$$

20 a $$y=\mathrm{e}^{2x}-\mathrm{e}^{-x}$$

x	0	0.25	0.5	0.75	1
y	0	0.86992	2.11175	4.00932	7.02118

b Area $\approx\frac{1}{2}h\left(y_0+2(y_1+\ldots)+y_n\right)$

$$=\frac{1}{8}(7.02118+2\times 6.99099)$$

$$=2.6254$$

20 c The curve is convex, so it is an overestimate.

d $\int_0^1 \left(e^{2x} - e^{-x}\right)\,dx = \left[\frac{1}{2}e^{2x} + e^{-x}\right]_0^1$

$= \frac{1}{2}e^2 + e^{-1} - \frac{1}{2} - 1$

$= \frac{1}{2}e^2 + \frac{1}{e} - \frac{3}{2}$

$= \frac{e^3 - 3e + 2}{2e}$

$P = -3,\ Q = 2$

e $\frac{e^3 - 3e + 2}{2e} = 2.5624$

Percentage error

$= \frac{2.5624 - 2.6254}{2.5624} \times 100\% \approx 2.5\%$

21 a $\frac{dv}{dt} = -kV$

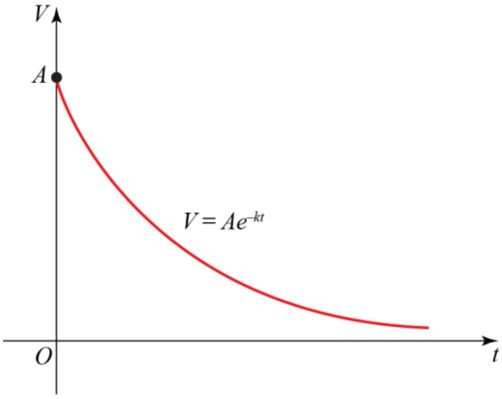
$\Rightarrow \int \frac{1}{V}\,dV = \int -k\,dt$

$\Rightarrow 1n|V| = -kt + C$

$\Rightarrow V = A_1e^{-kt}$

$t = 0, V = A \Rightarrow V = Ae^{-kt} \qquad (A_1 = A)$

b



c $t = T, V = \frac{1}{2}A \Rightarrow \frac{1}{2}A = Ae^{-kT}$

$\Rightarrow -1n\,2 = -kT$

$\Rightarrow kT = 1n\,2$

22 a $\frac{dy}{dx} = \frac{x}{k - y}$

$\int (k - y)\,dy = \int x\,dx$

$-\frac{(k - y)^2}{2} + c = \frac{x^2}{2}$

$x^2 + (y - k)^2 = c$

b Concentric circles with centre (0, 2).

23 a

x	1	1.5	2	2.5
y	1	0.6825	0.5545	0.6454

x	3	3.5	4
y	0.9775	1.5693	2.4361

b Area $= \frac{1}{4}(3.4361 + 2 \times 4.4262)$

$= 3.074$

c If smaller intervals are used and consequently more values, the lines would follow the curve more closely.

23 d $\text{Area} = \int_1^4 \frac{1}{5}x^2 \ln x - x + 2$

Now let $I = \int x^2 \ln x \, dx$

Let $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x^2 \Rightarrow v = \frac{x^3}{3}$

$I = \frac{1}{3}x^3 \ln x - \int \frac{x^2}{3} \, dx$

$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + c$

$\text{Area} = \left[\frac{1}{15}x^3 \ln x - \frac{1}{45}x^3 - \frac{x^2}{2} + 2x \right]_1^4$

$= \frac{64}{15}\ln 4 - \frac{64}{15} - 8 + 8 - \left(0 - \frac{1}{45} - \frac{1}{2} + 2 \right)$

$= -\frac{29}{10} + \frac{64}{15}\ln 4$

e $-\frac{29}{10} + \frac{64}{15}\ln 4 = 3.015$

Percentage error

$= \frac{3.015 - 3.074}{3.015} \times 100\% \approx 2.0\%$

24 a $u = 1 + 2x^2 \Rightarrow du = 4x\,dx \Rightarrow x\,dx = \frac{du}{4}$

So $\int x(1+2x^2)^5 dx = \int \frac{u^5}{4} du = \frac{u^6}{24}$

$+ c_1 = \frac{(1+2x^2)^6}{24} + c_1$

b $\frac{dy}{dx} = x(1+2x^2)^5 \cos^2 2y$

$\Rightarrow \int \sec^2 2y \, dy = \int x(1+2x^2)^5 dx$

$\Rightarrow \frac{1}{2}\tan 2y = \frac{(1+2x^2)^6}{24} + c_2$

$y = \frac{\pi}{8}, x = 0 \Rightarrow \frac{1}{2} = \frac{1}{24} + c_2 \Rightarrow c_2 = \frac{11}{24}$

$\therefore \tan 2y = \frac{(1+2x^2)^6}{12} + \frac{11}{12}$

25 $I = \int \frac{1}{1+x^2} \, dx$

Let $x = \tan u \Rightarrow \frac{dx}{du} = \sec^2 u$

$I = \int \frac{1}{1+\tan^2 u} \sec^2 u \, du$

But $1 + \tan^2 u = \sec^2 u$

So $I = \int du = u + c$

$= \arctan x + c$

26 $x(x+2)\frac{dy}{dx} = y$

$\Rightarrow \int \frac{1}{y} dy = \int \frac{1}{x(x+2)} dx$

$\frac{1}{x(x+2)} \equiv \frac{A}{x} + \frac{B}{x+2}$

$\Rightarrow 1 \equiv A(x+2) + Bx$

$x = 0 \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$

$x = -2 \Rightarrow 1 = -2B \Rightarrow B = -\frac{1}{2}$

So $\ln y = \int \left(\frac{\left(\frac{1}{2}\right)}{x} - \frac{\left(\frac{1}{2}\right)}{x+2} \right) dx$

$= \frac{1}{2}\ln|x| - \frac{1}{2}\ln|x+2| + c$

$\therefore y = \sqrt{\frac{kx}{x+2}} \quad \left(c = \frac{1}{2}\ln k \right)$

$x = 2, y = 2 \Rightarrow 2 = \sqrt{\frac{2k}{4}} \Rightarrow 4 \times 2 = k$

$\therefore y = \sqrt{\frac{8x}{x+2}}$ or $y^2 = \frac{8x}{x+2}$

27 a $A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$

$\frac{dr}{dt} = \frac{dr}{dA} \times \frac{dA}{dt} = \frac{1}{2\pi r} \times k \sin\left(\frac{t}{3\pi}\right)$

$\frac{dr}{dt} = \frac{k}{2\pi r} \sin\left(\frac{t}{3\pi}\right)$

27 b $\int 2\pi r \ \mathrm{d}r = \int k\sin\left(\frac{t}{3\pi}\right)\mathrm{d}t$

$$\pi r^2 = -3\pi k\cos\left(\frac{t}{3\pi}\right) + c$$

$$r^2 = -3k\cos\left(\frac{t}{3\pi}\right) + c$$

$$r = 1, t = 0 \Rightarrow 1 = -3k + c \Rightarrow c = 3k + 1$$

$$r = 2, t = \pi^2 \Rightarrow 4 = -\frac{3k}{2} + 3k + 1$$

So $r^2 = -6\cos\left(\frac{t}{3\pi}\right) + 6 + 1$

$$r^2 = -6\cos\left(\frac{t}{3\pi}\right) + 7$$

c $r = 1.5 \Rightarrow 2.25 = -6\cos\left(\frac{t}{3\pi}\right) + 7$

$$6\cos\left(\frac{t}{3\pi}\right) = 4.75$$

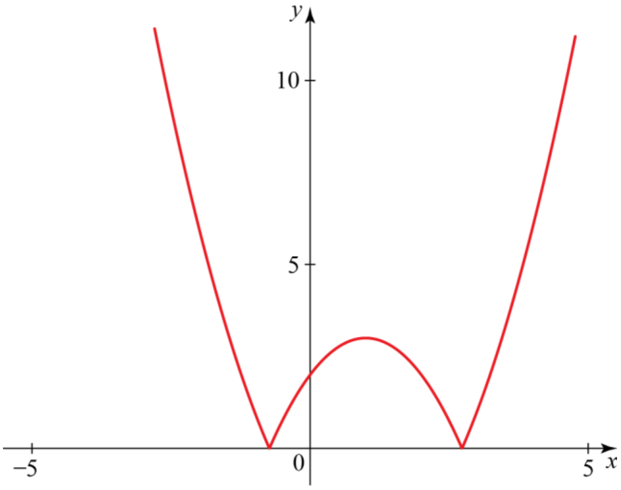
$$\cos\left(\frac{t}{3\pi}\right) \approx 0.7917$$

$$\frac{t}{3\pi} = 0.6527$$

$t = 6.19$ days

Challenge

a



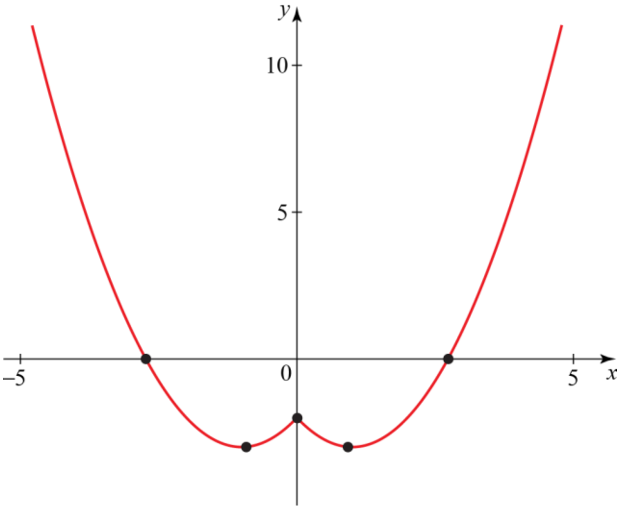
$$\int_{-3}^{3}|\mathrm{f}(x)|\ \mathrm{d}x = \int_{-3}^{3}\mathrm{f}(x)\ \mathrm{d}x + 2\times\left|\int_{-1}^{2}\mathrm{f}(x)\ \mathrm{d}x\right|$$

$$\int_{-3}^{3}\mathrm{f}(x)\ \mathrm{d}x = \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-3}^{3} = 6$$

$$\int_{-1}^{2}\mathrm{f}(x)\ \mathrm{d}x = \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-1}^{2} = \frac{9}{2}$$

$$\int_{-3}^{3}|\mathrm{f}(x)|\ \mathrm{d}x = 6 + 2\times\frac{9}{2} = 15$$

b



$$\int_{-3}^{3}\mathrm{f}(|x|)\ \mathrm{d}x = 2\times\int_{0}^{3}\mathrm{f}(x)\ \mathrm{d}x$$

$$\int_{0}^{3}\mathrm{f}(x)\ \mathrm{d}x = \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{0}^{3} = -\frac{3}{2}$$

$$\int_{-3}^{3}\mathrm{f}(|x|)\ \mathrm{d}x = 2\times\left(-\frac{3}{2}\right) = -3$$

Vectors 12A

1 $OP = \sqrt{2^2 + 8^2 + (-4)^2}$

$= \sqrt{4 + 64 + 16} = \sqrt{84}$

$= 2\sqrt{21} \approx 9.17$ (3 s.f.)

2 $OP = \sqrt{7^2 + 7^2 + 7^2}$

$= \sqrt{49 + 49 + 49} = \sqrt{147}$

$= 7\sqrt{3} \approx 12.1$ (3 s.f.)

3 **a** $AB = \sqrt{(3-1)^2 + (0-(-1))^2 + (5-8)^2}$

$= \sqrt{2^2 + 1^2 + (-3)^2}$

$= \sqrt{14} \approx 3.74$ (3 s.f.)

b $AB = \sqrt{(8-(-3))^2 + (11-1)^2 + (8-6)^2}$

$= \sqrt{11^2 + 10^2 + 2^2}$

$= \sqrt{225} = 15$

c $AB = \sqrt{(3-3)^2 + (5-10)^2 + (-2-3)^2}$

$= \sqrt{0^2 + (-5)^2 + (-5)^2}$

$= \sqrt{50} = 5\sqrt{2} \approx 7.07$ (3 s.f.)

d $AB = \sqrt{(-1-4)^2 + (-2-(-1))^2 + (5-3)^2}$

$= \sqrt{(-5)^2 + (-1)^2 + 2^2}$

$= \sqrt{30} \approx 5.48$ (3 s.f.)

4 $AB = \sqrt{(7-k)^2 + (-1-0)^2 + (2-4)^2} = 3$

$\sqrt{(49 - 14k + k^2) + 1 + 4} = 3$

$49 - 14k + k^2 + 1 + 4 = 9$

$k^2 - 14k + 45 = 0$

$(k-5)(k-9) = 0$

$k = 5$ or $k = 9$

5 $AB = \sqrt{(5-1)^2 + (3-k)^2 + (-8-(-3))^2}$

$= 3\sqrt{10}$

$\sqrt{16 + (9 - 6k + k^2) + 25} = 3\sqrt{10}$

$16 + 9 - 6k + k^2 + 25 = 9 \times 10$

$k^2 - 6k - 40 = 0$

$(k+4)(k-10) = 0$

$k = -4$ or $k = 10$

Challenge

a Coordinates of other points in the plane $x = 1$ will be (1, −3, 4) and (1, −3, −2).

Coordinates of other points in the plane $x = 7$ will be (7, 3, 4), (7, 3, −2) and (7, −3, −2).

b Shortest route for the ant will be from *A* to half way along one of the opposite edges and then across the next face to *C*.

Distance $= 2 \times \sqrt{6^2 + 3^2} = 2 \times \sqrt{45}$

$= 2 \times 3\sqrt{5} = 6\sqrt{5}$

Vectors 12B

1 a $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}$

i $\mathbf{a} - \mathbf{b} = \begin{pmatrix} -3 \\ 5 \\ -9 \end{pmatrix}$

ii $-\mathbf{a} + 3\mathbf{b} = \begin{pmatrix} -1+12 \\ -2-9 \\ 4+15 \end{pmatrix} = \begin{pmatrix} 11 \\ -11 \\ 19 \end{pmatrix}$

b **a** – **b** is parallel since
$-2(\mathbf{a} - \mathbf{b}) = 6\mathbf{i} - 10\mathbf{j} + 18\mathbf{k}$.

–**a** + 3**b** is not parallel as it is not a multiple of $6\mathbf{i} - 10\mathbf{j} + 18\mathbf{k}$.

2 $3\mathbf{a} + 2\mathbf{b} = 3\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + 2\begin{pmatrix} -3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$

$3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} = \frac{1}{2}(6\mathbf{i} + 4\mathbf{j} + 10\mathbf{k})$

So the vectors are parallel.

3 $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

$\mathbf{a} + 2\mathbf{b} = (1 + 2p)\mathbf{i} + (2 + 2q)\mathbf{j} + (-4 + 2r)\mathbf{k}$

$(1 + 2p)\mathbf{i} + (2 + 2q)\mathbf{j} + (-4 + 2r)\mathbf{k} = 5\mathbf{i} + 4\mathbf{j}$

$1 + 2p = 5 \Rightarrow p = 2$
$2 + 2q = 4 \Rightarrow q = 1$
$-4 + 2r = 0 \Rightarrow r = 2$

4 a $|3\mathbf{i} + 5\mathbf{j} + \mathbf{k}| = \sqrt{3^2 + 5^2 + 1^2}$
$= \sqrt{9 + 25 + 1} = \sqrt{35}$

b $|4\mathbf{i} - 2\mathbf{k}| = \sqrt{4^2 + 0^2 + (-2)^2}$
$= \sqrt{16 + 4} = \sqrt{20} = \sqrt{4}\sqrt{5} = 2\sqrt{5}$

c $|\mathbf{i} + \mathbf{j} - \mathbf{k}| = \sqrt{1^2 + 1^2 + (-1)^2}$
$= \sqrt{1 + 1 + 1} = \sqrt{3}$

d $|5\mathbf{i} - 9\mathbf{j} - 8\mathbf{k}| = \sqrt{5^2 + (-9)^2 + (-8)^2}$
$= \sqrt{25 + 81 + 64} = \sqrt{170}$

e $|\mathbf{i} + 5\mathbf{j} - 7\mathbf{k}| = \sqrt{1^2 + 5^2 + (-7)^2}$
$= \sqrt{1 + 25 + 49} = \sqrt{75}$
$= \sqrt{25}\sqrt{3} = 5\sqrt{3}$

5 a $\mathbf{p} + \mathbf{q} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ -1 \end{pmatrix}$

b $\mathbf{q} - \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} - \begin{pmatrix} 7 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \\ -5 \end{pmatrix}$

c $\mathbf{p} + \mathbf{q} + \mathbf{r} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 7 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 14 \\ -3 \\ 1 \end{pmatrix}$

d $3\mathbf{p} - \mathbf{r} = 3\begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 7 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 15 \\ 0 \\ 6 \end{pmatrix} - \begin{pmatrix} 7 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 4 \end{pmatrix}$

e $\mathbf{p} - 2\mathbf{q} + \mathbf{r} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 7 \\ -4 \\ 2 \end{pmatrix}$
$= \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ -6 \end{pmatrix} + \begin{pmatrix} 7 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ -6 \\ 10 \end{pmatrix}$

6 $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, so $\mathbf{b} = \overrightarrow{AB} + \mathbf{a}$

$\mathbf{b} = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ -7 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ 2 \end{pmatrix}$

Position vector of B is $7\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

7 $|\mathbf{a}| = \sqrt{t^2+2^2+3^2} = 7$

$$\begin{aligned}
&\sqrt{t^2+4+9} = 7\\
&t^2+4+9 = 49\\
&t^2 = 36\\
&t = 6 \text{ or } t = -6
\end{aligned}$$

8 $|\mathbf{a}| = \sqrt{(5t)^2+(2t)^2+t^2} = 3\sqrt{10}$

$$\begin{aligned}
&\sqrt{25t^2+4t^2+t^2} = 3\sqrt{10}\\
&\sqrt{30t^2} = 3\sqrt{10}\\
&30t^2 = 9\times 10\\
&t^2 = 3\\
&t = \sqrt{3} \text{ or } t = -\sqrt{3}
\end{aligned}$$

9 a i Position vector of A is $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$
Position vector of B is $3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$
Position vector of C is $-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

ii
$$\begin{aligned}
\overrightarrow{AC} &= \overrightarrow{OC} - \overrightarrow{OA}\\
&= (-\mathbf{i}+2\mathbf{j}+2\mathbf{k}) - (2\mathbf{i}+\mathbf{j}+4\mathbf{k})\\
&= -3\mathbf{i}+\mathbf{j}-2\mathbf{k}
\end{aligned}$$

b i $\left|\overrightarrow{AC}\right| = \sqrt{3^2+1^2+2^2} = \sqrt{14}$

ii $\left|\overrightarrow{OC}\right| = \sqrt{1^2+2^2+2^2} = \sqrt{9} = 3$

10 a
$$\begin{aligned}
\overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP}\\
&= -\mathbf{i}+3\mathbf{j}-5\mathbf{k} - (3\mathbf{i}+7\mathbf{k})\\
&= -4\mathbf{i}+3\mathbf{j}-12\mathbf{k}
\end{aligned}$$

b Distance between P and Q is
$$\left|\overrightarrow{PQ}\right| = \sqrt{4^2+3^2+12^2} = \sqrt{169} = 13$$

c Unit vector in the direction of $\overrightarrow{PQ}$ is
$$\frac{1}{13}(-4\mathbf{i}+3\mathbf{j}-12\mathbf{k}) = -\frac{4}{13}\mathbf{i}+\frac{3}{13}\mathbf{j}-\frac{12}{13}\mathbf{k}$$

11 a
$$\begin{aligned}
\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA}\\
&= -2\mathbf{i}+3\mathbf{j}+\mathbf{k} - (4\mathbf{i}-\mathbf{j}-2\mathbf{k})\\
&= -6\mathbf{i}+4\mathbf{j}+3\mathbf{k}
\end{aligned}$$

b Distance between A and B is
$$\left|\overrightarrow{AB}\right| = \sqrt{6^2+4^2+3^2} = \sqrt{61}$$

c Unit vector in the direction of $\overrightarrow{AB}$ is
$$\begin{aligned}
&\frac{1}{\sqrt{61}}(-6\mathbf{i}+4\mathbf{j}+3\mathbf{k})\\
&= -\frac{6}{\sqrt{61}}\mathbf{i}+\frac{4}{\sqrt{61}}\mathbf{j}+\frac{3}{\sqrt{61}}\mathbf{k}
\end{aligned}$$

12 a $|\mathbf{p}| = \sqrt{3^2+4^2+2^2} = \sqrt{29}$
$$\begin{aligned}
\hat{\mathbf{p}} &= \frac{1}{|\mathbf{p}|}\mathbf{p} = \frac{1}{\sqrt{29}}(3\mathbf{i}-4\mathbf{j}-2\mathbf{k})\\
&= \frac{3}{\sqrt{29}}\mathbf{i}-\frac{4}{\sqrt{29}}\mathbf{j}-\frac{2}{\sqrt{29}}\mathbf{k}
\end{aligned}$$

b $|\mathbf{q}| = \sqrt{2+4^2+7} = \sqrt{25} = 5$
$$\begin{aligned}
\hat{\mathbf{q}} &= \frac{1}{|\mathbf{q}|}\mathbf{q} = \frac{1}{5}\left(\sqrt{2}\mathbf{i}-4\mathbf{j}-\sqrt{7}\mathbf{k}\right)\\
&= \frac{\sqrt{2}}{5}\mathbf{i}-\frac{4}{5}\mathbf{j}-\frac{\sqrt{7}}{5}\mathbf{k}
\end{aligned}$$

c $|\mathbf{r}| = \sqrt{5+8+3} = \sqrt{16} = 4$
$$\begin{aligned}
\hat{\mathbf{r}} &= \frac{1}{|\mathbf{r}|}\mathbf{r} = \frac{1}{4}\left(\sqrt{5}\mathbf{i}-2\sqrt{2}\mathbf{j}-\sqrt{3}\mathbf{k}\right)\\
&= \frac{\sqrt{5}}{4}\mathbf{i}-\frac{2\sqrt{2}}{4}\mathbf{j}-\frac{\sqrt{3}}{4}\mathbf{k}
\end{aligned}$$

13 a
$$\begin{aligned}
\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA}\\
&= 8\mathbf{i}-3\mathbf{j}+3\mathbf{k} - (8\mathbf{i}-7\mathbf{j}+4\mathbf{k})\\
&= 4\mathbf{j}-\mathbf{k}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{AC} &= \overrightarrow{OC} - \overrightarrow{OA}\\
&= 12\mathbf{i}-6\mathbf{j}+3\mathbf{k} - (8\mathbf{i}-7\mathbf{j}+4\mathbf{k})\\
&= 4\mathbf{i}+\mathbf{j}-\mathbf{k}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{BC} &= \overrightarrow{OC} - \overrightarrow{OB}\\
&= 12\mathbf{i}-6\mathbf{j}+3\mathbf{k} - (8\mathbf{i}-3\mathbf{j}+3\mathbf{k})\\
&= 4\mathbf{i}-3\mathbf{j}
\end{aligned}$$

13 b $\left|\overrightarrow{AB}\right| = \sqrt{4^2+1} = \sqrt{17}$

$\left|\overrightarrow{AC}\right| = \sqrt{4^2+1+1} = \sqrt{18} = 3\sqrt{2}$

$\left|\overrightarrow{BC}\right| = \sqrt{4^2+3^2} = \sqrt{25} = 5$

c As the sides are all different lengths, the triangle is scalene.

14 a $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \mathbf{i} - 2\mathbf{j} + 5\mathbf{k} - (3\mathbf{i} + 4\mathbf{j} + 8\mathbf{k})$

$= -2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= 7\mathbf{i} - 5\mathbf{j} + 7\mathbf{k} - (3\mathbf{i} + 4\mathbf{j} + 8\mathbf{k})$

$= 4\mathbf{i} - 9\mathbf{j} - \mathbf{k}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= 7\mathbf{i} - 5\mathbf{j} + 7\mathbf{k} - (\mathbf{i} - 2\mathbf{j} + 5\mathbf{k})$

$= 6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

b $\left|\overrightarrow{AB}\right| = \sqrt{2^2+6^2+3^2} = \sqrt{49} = 7$

$\left|\overrightarrow{AC}\right| = \sqrt{4^2+9^2+1} = \sqrt{98} = 7\sqrt{2}$

$\left|\overrightarrow{BC}\right| = \sqrt{6^2+3^2+2^2} = \sqrt{49} = 7$

c Triangle *ABC* is isosceles.

Since angle $ABC = 90°$, angle $BAC = 45°$

15 a $\left|-\mathbf{i} + 7\mathbf{j} + \mathbf{k}\right| = \sqrt{51}$

i $\cos\theta_x = \frac{-1}{\sqrt{51}} \Rightarrow \theta_x = 98.0°$

ii $\cos\theta_y = \frac{7}{\sqrt{51}} \Rightarrow \theta_y = 11.4°$

iii $\cos\theta_z = \frac{1}{\sqrt{51}} \Rightarrow \theta_z = 82.0°$

b $\left|3\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}\right| = \sqrt{74}$

i $\cos\theta_x = \frac{3}{\sqrt{74}} \Rightarrow \theta_x = 69.6°$

ii $\cos\theta_y = \frac{4}{\sqrt{74}} \Rightarrow \theta_y = 62.3°$

iii $\cos\theta_z = \frac{7}{\sqrt{74}} \Rightarrow \theta_z = 35.5°$

c $\left|2\mathbf{i} - 3\mathbf{k}\right| = \sqrt{13}$

i $\cos\theta_x = \frac{2}{\sqrt{13}} \Rightarrow \theta_x = 56.3°$

ii $\cos\theta_y = \frac{0}{\sqrt{13}} \Rightarrow \theta_y = 90°$

iii $\cos\theta_z = \frac{-3}{\sqrt{13}} \Rightarrow \theta_z = -146.3°$

16 Let *A* be (2, 0, 0), *B* be (5, 0, 0) and *C* be (4, 2, 3).

$\left|\overrightarrow{AB}\right| = 3$

$\left|\overrightarrow{AC}\right| = \sqrt{2^2+2^2+3^2} = \sqrt{17} \approx 4.123$

$\left|\overrightarrow{BC}\right| = \sqrt{1^2+2^2+3^2} = \sqrt{14} \approx 3.742$

$\cos\angle ABC = \frac{9+14-17}{2\times 3\times\sqrt{14}} = 0.2672...$

$\angle ABC = 74.49...°$

Area of triangle

$= \frac{1}{2}\times 3\times\sqrt{14}\times\sin 74.49...°$

$= 5.41$

17 $\left|\overrightarrow{PQ}\right| = \sqrt{3^2+1+2^2} = \sqrt{14}$

$\left|\overrightarrow{QR}\right| = \sqrt{2^2+4^2+3^2} = \sqrt{29}$

$\left|\overrightarrow{PR}\right| = \left|\overrightarrow{PQ} + \overrightarrow{QR}\right| = \sqrt{1+3^2+5^2} = \sqrt{35}$

Using the cosine rule:

$35 = 29 + 14 - 2\sqrt{29\times 14}\cos\angle PQR$

$\cos\angle PQR = \frac{4}{\sqrt{406}} = 0.1985...$

$\angle PQR = 78.5°$ (1 d.p.)

Challenge

The vector $\mathbf{a}=-2\mathbf{i}+6\mathbf{j}-3\mathbf{k}$ will be in the same vertical plane as the vector $\mathbf{b}=-2\mathbf{i}+6\mathbf{j}$.

So the angle **a** makes with the *xy*-plane is the angle, θ, between **a** and **b**.

$$\cos\theta=\frac{|\mathbf{b}|}{|\mathbf{a}|}=\frac{\sqrt{40}}{\sqrt{49}}=0.9035...$$

$$\theta=25.4° \text{ (1 d.p.)}$$

Vectors 12C

1 a i $|\overrightarrow{OA}| = \sqrt{1+4^2+8^2} = \sqrt{81} = 9$

$|\overrightarrow{OB}| = \sqrt{4^2+4^2+7^2} = \sqrt{81} = 9$

$\Rightarrow |\overrightarrow{OA}| = |\overrightarrow{OB}|$

ii $|\overrightarrow{AC}| = |\overrightarrow{OC} - \overrightarrow{OA}| = |9\mathbf{i} + 4\mathbf{j} + 22\mathbf{k}|$

$= \sqrt{9^2+4^2+22^2} = \sqrt{581}$

$|\overrightarrow{BC}| = |\overrightarrow{OC} - \overrightarrow{OB}| = |6\mathbf{i} - 4\mathbf{j} + 23\mathbf{k}|$

$= \sqrt{6^2+4^2+23^2} = \sqrt{581}$

$\Rightarrow |\overrightarrow{AC}| = |\overrightarrow{BC}|$

b The quadrilateral *OACB* has two pairs of equal adjacent sides, so it is a kite.

2 a Let *O* be the fixed origin.

$|\overrightarrow{AB}| = |\overrightarrow{OB} - \overrightarrow{OA}| = |2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}|$

$= \sqrt{2^2+3^2+2^2} = \sqrt{17}$

$|\overrightarrow{AC}| = |\overrightarrow{OC} - \overrightarrow{OA}| = |6\mathbf{j}| = 6$

$|\overrightarrow{BC}| = |\overrightarrow{OC} - \overrightarrow{OB}| = |-2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}|$

$= \sqrt{2^2+3^2+2^2} = \sqrt{17}$

So $|\overrightarrow{AB}| = |\overrightarrow{BC}|$ and the triangle is isosceles.

b If *AC* is the base of the triangle, then the height, *h*, will be given by:

$\left(\frac{1}{2}|\overrightarrow{AC}|\right)^2 + h^2 = \left(|\overrightarrow{AB}|\right)^2$

$9 + h^2 = 17$

$h = \sqrt{8} = 2\sqrt{2}$

Area of triangle *ABC*

$= \frac{1}{2} \times 6 \times 2\sqrt{2} = 6\sqrt{2}$

c For *ABCD* to be a parallelogram, there are three possibilities:

i $\overrightarrow{AD}$ and $\overrightarrow{BC}$ are parallel and equal in magnitude.

Hence $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{BC}$

$\overrightarrow{OD} = (2\mathbf{i} + \mathbf{j} + 5\mathbf{k}) + (-2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$

$= 4\mathbf{j} + 7\mathbf{k}$

Coordinates of *D* are (0, 4, 7).

ii $\overrightarrow{CD}$ and $\overrightarrow{AB}$ are parallel and equal in magnitude.

Hence $\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{AB}$

$\overrightarrow{OD} = (2\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}) + (2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$

$= 4\mathbf{i} + 10\mathbf{j} + 3\mathbf{k}$

Coordinates of *D* are (4, 10, 3).

iii $\overrightarrow{AD}$ and $\overrightarrow{CB}$ are parallel and equal in magnitude.

Hence $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{CB}$

$\overrightarrow{OD} = (2\mathbf{i} + \mathbf{j} + 5\mathbf{k}) + (2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k})$

$= 4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

Coordinates of *D* are (4, −2, 3).

3 a Let *O* be the fixed origin.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (11\mathbf{i} + 2\mathbf{j} - 9\mathbf{k}) - (7\mathbf{i} + 12\mathbf{j} - \mathbf{k})$

$= 4\mathbf{i} - 10\mathbf{j} - 8\mathbf{k}$

$= 2(2\mathbf{i} - 5\mathbf{j} - 4\mathbf{k})$

$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$

$= (8\mathbf{i} + \mathbf{j} + 15\mathbf{k}) - (14\mathbf{i} - 14\mathbf{j} + 3\mathbf{k})$

$= -6\mathbf{i} + 15\mathbf{j} + 12\mathbf{k}$

$= -3(2\mathbf{i} - 5\mathbf{j} - 4\mathbf{k})$

$\overrightarrow{CD} = -\frac{3}{2}\overrightarrow{AB}$, so *AB* is parallel to *CD*.

$AB : CD = 2 : 3$

3 **b** $\overrightarrow{BC} = 3\mathbf{i} - 16\mathbf{j} + 12\mathbf{k}$

$\overrightarrow{AD} = \mathbf{i} - 11\mathbf{j} + 16\mathbf{k}$

BC is not parallel to *AD*. So *ABCD* is a quadrilateral with one pair of parallel sides. So it is a trapezium.

4 $(3a+b)\mathbf{i} + \mathbf{j} + ac\mathbf{k} = 7\mathbf{i} - b\mathbf{j} + 4\mathbf{k}$

Comparing coefficients of **j**:

$b = -1$

Comparing coefficients of **i**:

$3a + b = 7 \Rightarrow 3a - 1 = 7$

$a = \frac{8}{3}$

Comparing coefficients of **k**:

$ac = 4 \Rightarrow \frac{8}{3}c = 4$

$c = \frac{3}{2}$

5 $\triangle OAB$ is isosceles.

If $|\overrightarrow{OA}| = |\overrightarrow{OB}|$:

$\sqrt{10^2 + 23^2 + 10^2} = \sqrt{p^2 + 14^2 + 22^2}$

$729 = p^2 + 680$

$p^2 = 49$

$p = \pm 7$

If $|\overrightarrow{OB}| = |\overrightarrow{AB}|$:

$\overrightarrow{AB} = (p-10)\mathbf{i} + 37\mathbf{j} - 32\mathbf{k}$

$\sqrt{p^2 + 14^2 + 22^2} = \sqrt{(p-10)^2 + 37^2 + 32^2}$

$p^2 + 680 = (p-10)^2 + 1369 + 1024$

$p^2 - (p-10)^2 = 2393 - 680$

$p^2 - (p^2 - 20p + 100) = 1713$

$20p = 1813$

$p = \frac{1813}{20}$

If $|\overrightarrow{OA}| = |\overrightarrow{AB}|$:

$\sqrt{729} = \sqrt{(p-10)^2 + 37^2 + 32^2}$

$729 = (p-10)^2 + 1369 + 1024$

$0 = (p-10)^2 + 2393 - 729$

$0 = p^2 - 20p + 100 + 1664$

$0 = p^2 - 20p + 1764$

$b^2 - 4ac < 0$

So there are no solutions for *p* if $|\overrightarrow{OA}| = |\overrightarrow{AB}|$.

The three possible positions for *B* are (7, 14, −22), (−7, 14, −22) and $\left(\frac{1813}{20}, 14, -22\right)$.

6 a $|\overrightarrow{AB}| = \sqrt{7^2+1+2^2} = \sqrt{54}$

$|\overrightarrow{BC}| = \sqrt{1+5^2} = \sqrt{26}$

$|\overrightarrow{AC}| = |\overrightarrow{AB}+\overrightarrow{BC}| = \sqrt{6^2+1+7^2} = \sqrt{86}$

$$\cos\angle ABC = \frac{54+26-86}{2\times\sqrt{54}\times\sqrt{26}} = -0.080...$$

$\angle ABC = 94.59...°$

Area of triangle

$= \frac{1}{2}\times\sqrt{54}\times\sqrt{26}\times\sin 94.59...°$

$= 18.67$ (2 d.p.)

b Triangles *ABC* and *ADE* are similar with a side ratio of 1 : 3.

So area of triangle *ADE*

$= 9\times$ area of triangle *ABC*

$= 168.07$ (2 d.p.)

7 Suppose there is a point of intersection, *H*, of *OF* and *AG*.

$\overrightarrow{OH} = r\overrightarrow{OF}$ for some scalar *r*.

$\overrightarrow{AH} = s\overrightarrow{AG}$ for some scalar *s*.

But $\overrightarrow{OH} = \overrightarrow{OA}+\overrightarrow{AH} = \overrightarrow{OA}+s\overrightarrow{AG}$

so $r\overrightarrow{OF} = \overrightarrow{OA}+s\overrightarrow{AG}$ (1)

Now $\overrightarrow{OF} = \overrightarrow{OB}+\overrightarrow{BD}+\overrightarrow{DF} = \mathbf{a}+\mathbf{b}+\mathbf{c}$

and $\overrightarrow{AG} = \overrightarrow{AO}+\overrightarrow{OB}+\overrightarrow{BG} = -\mathbf{a}+\mathbf{b}+\mathbf{c}$

So (1) becomes

$r(\mathbf{a}+\mathbf{b}+\mathbf{c}) = \mathbf{a}+s(-\mathbf{a}+\mathbf{b}+\mathbf{c})$

Comparing coefficients of **a**:

$r = 1-s$

Comparing coefficients of **b**:

$r = s$

So $r = s = \frac{1}{2}$

$\overrightarrow{OH} = \frac{1}{2}\overrightarrow{OF}$ and $\overrightarrow{AH} = \frac{1}{2}\overrightarrow{AG}$

So *H* is the midpoint of *OF* and of *AG*, and the diagonals bisect each other.

8 $\overrightarrow{FP} = \overrightarrow{FB}+\overrightarrow{BO}+\overrightarrow{OA}+\overrightarrow{AP}$

$= -\mathbf{c}-\mathbf{b}+\mathbf{a}+\frac{4}{3}\overrightarrow{AM}$

But $\overrightarrow{AM} = \overrightarrow{AO}+\frac{3}{4}\overrightarrow{OE}$

$= -\mathbf{a}+\frac{3}{4}(\mathbf{a}+\mathbf{b}+\mathbf{c})$

$= -\frac{1}{4}\mathbf{a}+\frac{3}{4}\mathbf{b}+\frac{3}{4}\mathbf{c}$

So $\overrightarrow{FP} = -\mathbf{c}-\mathbf{b}+\mathbf{a}+\frac{4}{3}\left(-\frac{1}{4}\mathbf{a}+\frac{3}{4}\mathbf{b}+\frac{3}{4}\mathbf{c}\right)$

$= \frac{2}{3}\mathbf{a}$

$\overrightarrow{PE} = \overrightarrow{PA}+\overrightarrow{AG}+\overrightarrow{GE}$

$= -\frac{4}{3}\overrightarrow{AM}+\mathbf{c}+\mathbf{b}$

$= -\frac{4}{3}\left(\overrightarrow{AO}+\frac{3}{4}\overrightarrow{OE}\right)+\mathbf{c}+\mathbf{b}$

$= \frac{4}{3}\mathbf{a}-\mathbf{a} = \frac{1}{3}\mathbf{a}$

Therefore *FP* and *PE* are parallel, so *P* lies on *FE*.

$FP : PE = \frac{2}{3}|\mathbf{a}| : \frac{1}{3}|\mathbf{a}| = 2:1$

Challenge

1 $p\mathbf{a}+q\mathbf{b}+r\mathbf{c}=\begin{pmatrix}p+2q-5r\\3r\\4p-3q+r\end{pmatrix}=\begin{pmatrix}28\\-12\\-4\end{pmatrix}$

Comparing coefficients of **b**:
$r=-4$

Comparing coefficients of **a**:
$p+2q+20=28\Rightarrow p+2q=8 \quad (1)$

Comparing coefficients of **c**:
$4p-3q-4=-4\Rightarrow 4p-3q=0 \quad (2)$

Substituting for *p* in (2):
$4(8-2q)-3q=0\Rightarrow q=\frac{32}{11}$

Substituting for *q* in (1):
$p+\frac{64}{11}=8\Rightarrow p=\frac{24}{11}$

2 $\overrightarrow{OM}=\frac{1}{2}\mathbf{a}+\mathbf{b}+\mathbf{c}$

$\overrightarrow{BN}=\mathbf{a}-\mathbf{b}+\frac{1}{2}\mathbf{c}$

$\overrightarrow{AF}=-\mathbf{a}+\mathbf{b}+\mathbf{c}$

Suppose there is a point of intersection, *X*, of *OM* and *AF*.

$\overrightarrow{AX}=r\overrightarrow{AF}=r(-\mathbf{a}+\mathbf{b}+\mathbf{c})$ for scalar r.

$\overrightarrow{OX}=s\overrightarrow{OM}=s\left(\frac{1}{2}\mathbf{a}+\mathbf{b}+\mathbf{c}\right)$ for scalar s.

But $\overrightarrow{OX}=\overrightarrow{OA}+\overrightarrow{AX}=\mathbf{a}+r(-\mathbf{a}+\mathbf{b}+\mathbf{c})$

so $s\left(\frac{1}{2}\mathbf{a}+\mathbf{b}+\mathbf{c}\right)=\mathbf{a}+r(-\mathbf{a}+\mathbf{b}+\mathbf{c})$

Comparing coefficients of **a** and **b**:
$\frac{1}{2}s=1-r$ and $s=r$

So $r=s=\frac{2}{3}$

Suppose there is a point of intersection, *Y*, of *BN* and *AF***.**

$\overrightarrow{AY}=p\overrightarrow{AF}=p(-\mathbf{a}+\mathbf{b}+\mathbf{c})$ for scalar p.

$\overrightarrow{BY}=q\overrightarrow{BN}=q\left(\mathbf{a}-\mathbf{b}+\frac{1}{2}\mathbf{c}\right)$ for scalar q.

But $\overrightarrow{BY}=\overrightarrow{BA}+\overrightarrow{AY}=\mathbf{a}-\mathbf{b}+p(-\mathbf{a}+\mathbf{b}+\mathbf{c})$

so $q\left(\mathbf{a}-\mathbf{b}+\frac{1}{2}\mathbf{c}\right)=\mathbf{a}-\mathbf{b}+p(-\mathbf{a}+\mathbf{b}+\mathbf{c})$

Comparing coefficients of **a** and **c**:
$q=1-p$ and $q=2p$

So $p=\frac{1}{3},q=\frac{2}{3}$

$\overrightarrow{AX}=\frac{2}{3}\overrightarrow{AF}$ and $\overrightarrow{AY}=\frac{1}{3}\overrightarrow{AF}$

So the line segments *OM* and *BN* trisect the diagonal *AF*.

Vectors 12D

1 a $\mathbf{R} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} + 7\mathbf{i} + 4\mathbf{j} + 3\mathbf{k} - 5\mathbf{i} - 3\mathbf{j}$

$= (5\mathbf{i} - \mathbf{j} + 4\mathbf{k})\,\text{N}$

b $|\mathbf{R}| = \sqrt{5^2 + 1 + 4^2} = \sqrt{42}\,\text{N}$

2 $|\mathbf{a}| = \sqrt{4^2 + 2^2 + 3^2} = \sqrt{29}\,\text{m s}^{-2}$

Using $s = ut + \frac{1}{2}at^2$:

$s = \frac{1}{2}\sqrt{29} \times 4 = 2\sqrt{29}\,\text{m}$

3 a $\mathbf{F} = m\mathbf{a} \Rightarrow 2\mathbf{i} - 5\mathbf{j} + 3\mathbf{k} = 4\mathbf{a}$

$\mathbf{a} = \left(\frac{1}{2}\mathbf{i} - \frac{5}{4}\mathbf{j} + \frac{3}{4}\mathbf{k}\right)\text{m s}^{-2}$

b $|\mathbf{a}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{5}{4}\right)^2 + \left(\frac{3}{4}\right)^2}$

$= 1.54\,\text{m s}^{-2}$

4 $\mathbf{F}_1 + \mathbf{F}_2 = m\mathbf{a} \Rightarrow 7\mathbf{i} + 3\mathbf{j} + \mathbf{k} + \mathbf{F}_2 = 6(2\mathbf{i} - \mathbf{k})$

$\mathbf{F}_2 = (12\mathbf{i} - 7\mathbf{i} - 3\mathbf{j} - 6\mathbf{k} - \mathbf{k})$

$= (5\mathbf{i} - 3\mathbf{j} - 7\mathbf{k})\,\text{N}$

5 a Particle is in equilibrium

$\Rightarrow \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \mathbf{0}$

$(\mathbf{i} - \mathbf{j} - 2\mathbf{k}) + (-\mathbf{i} + 3\mathbf{j} + b\mathbf{k}) + (a\mathbf{j} - 2\mathbf{k}) = \mathbf{0}$

Comparing coefficients of **j**:

$-1 + 3 + a = 0 \Rightarrow a = -2$

Comparing coefficients of **k**:

$-2 + b - 2 = 0 \Rightarrow b = 4$

b $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_3 = \mathbf{i} + (a - 1)\mathbf{j} - 4\mathbf{k}$

$= (\mathbf{i} - 3\mathbf{j} - 4\mathbf{k})\,\text{N}$

c $\mathbf{F} = m\mathbf{a} \Rightarrow \mathbf{i} - 3\mathbf{j} - 4\mathbf{k} = 2\mathbf{a}$

$\mathbf{a} = \left(\frac{1}{2}\mathbf{i} - \frac{3}{2}\mathbf{j} - 2\mathbf{k}\right)\text{m s}^{-2}$

d $|\mathbf{a}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + 2^2}$

$= \frac{1}{2}\sqrt{26}\,\text{m s}^{-2}$

e $\cos\theta_\mathbf{j} = \dfrac{-\frac{3}{2}}{\frac{\sqrt{26}}{2}} = \dfrac{-3}{\sqrt{26}}$

$\theta_\mathbf{j} = 126°$

This question has been removed from the latest edition of the book.

6 a Gravitational force downwards

$= 1200 \times 9.8 = 11760\,\text{N}$

Total force on aeroplane

$= \mathbf{T} + \mathbf{L} + \mathbf{F} - 11760\mathbf{k}$

$= (1900\mathbf{i} - 1300\mathbf{j} - 460\mathbf{k})\,\text{N}$

$\mathbf{F} = m\mathbf{a} \Rightarrow 1900\mathbf{i} - 1300\mathbf{j} - 460\mathbf{k} = 1200\mathbf{a}$

$\mathbf{a} = \left(\frac{19}{12}\mathbf{i} - \frac{13}{12}\mathbf{j} - \frac{4.6}{12}\mathbf{k}\right)\text{m s}^{-2}$

$|\mathbf{a}| = \sqrt{\left(\frac{19}{12}\right)^2 + \left(\frac{13}{12}\right)^2 + \left(\frac{4.6}{12}\right)^2}$

$= 1.96\,\text{ms}^{-2}$

b As the aeroplane is initially in level flight and the acceleration in the vertical direction is $-460\,\text{m s}^{-2}$, the aeroplane must be descending.

$\cos\theta_\mathbf{k} = \dfrac{-\frac{4.6}{12}}{1.96} = -0.1956...$

$\theta_\mathbf{k} = 101.3°$

Vectors, Mixed exercise 12

1 Coordinates of M are $(3,5,4)$

Distance from M to C

$$= \sqrt{(5-3)^2 + (8-5)^2 + (7-4)^2}$$
$$= \sqrt{4+9+9} = \sqrt{22}$$

2 Distance from P to Q

$$= \sqrt{((a-2)-2)^2 + (6-3)^2 + (7-a)^2}$$
$$= \sqrt{a^2 - 8a + 16 + 9 + 49 - 14a + a^2}$$
$$= \sqrt{2a^2 - 22a + 74} = \sqrt{14}$$

$$2a^2 - 22a + 74 = 14$$
$$a^2 - 11a + 30 = 0$$
$$(a-5)(a-6) = 0$$
$$a = 5 \text{ or } a = 6$$

3 $\left|\overrightarrow{AB}\right| = \sqrt{3^2 + t^2 + 5^2} = \sqrt{t^2 + 34}$

$$\sqrt{t^2 + 34} = 5\sqrt{2}$$
$$t^2 + 34 = 50$$
$$t^2 = 16$$
$$t = 4 \quad (\text{since } t > 0)$$

So $\overrightarrow{AB} = -3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$

$$6\mathbf{i} - 8\mathbf{j} - \frac{5}{2}t\mathbf{k} = 6\mathbf{i} - 8\mathbf{j} - 10\mathbf{k}$$
$$= -2\overrightarrow{AB}$$

So $\overrightarrow{AB}$ is parallel to $6\mathbf{i} - 8\mathbf{j} - \frac{5}{2}t\mathbf{k}$

4 **a** Let O be the fixed origin.

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = -3\mathbf{i} - 8\mathbf{j} + 3\mathbf{k}$$
$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = -3\mathbf{i} - 9\mathbf{j} + 8\mathbf{k}$$
$$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = -\mathbf{j} + 5\mathbf{k}$$

b $\left|\overrightarrow{PQ}\right| = \sqrt{9+64+9} = \sqrt{82}$

$$\left|\overrightarrow{PR}\right| = \sqrt{9+81+64} = \sqrt{154}$$
$$\left|\overrightarrow{QR}\right| = \sqrt{1+25} = \sqrt{26}$$

$$\cos\angle QPR = \frac{82+154-26}{2\times\sqrt{82}\times\sqrt{154}} = 0.9343...$$
$$\angle QPR = 20.87...^\circ$$

Area of triangle

$$= \frac{1}{2}\times\sqrt{82}\times\sqrt{154}\sin 20.87...^\circ$$
$$= 20.0 \text{ (1 d.p.)}$$

5 **a** $\overrightarrow{DE} = \overrightarrow{OE} - \overrightarrow{OD} = 4\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

$$\overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE} = -3\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$$
$$\overrightarrow{FD} = \overrightarrow{OD} - \overrightarrow{OF} = -\mathbf{i} + \mathbf{j} - 8\mathbf{k}$$

b $\left|\overrightarrow{DE}\right| = \sqrt{16+9+16} = \sqrt{41}$

$$\left|\overrightarrow{EF}\right| = \sqrt{9+16+16} = \sqrt{41}$$
$$\left|\overrightarrow{FD}\right| = \sqrt{1+1+64} = \sqrt{66}$$

c Two sides are equal in length so the triangle is isosceles.

6 **a** $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = 9\mathbf{i} - 4\mathbf{j}$

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = 7\mathbf{i} + \mathbf{j} - 3\mathbf{k}$$
$$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = -2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$$

b $\left|\overrightarrow{PQ}\right| = \sqrt{81+16} = \sqrt{97}$

$$\left|\overrightarrow{PR}\right| = \sqrt{49+1+9} = \sqrt{59}$$
$$\left|\overrightarrow{QR}\right| = \sqrt{4+25+9} = \sqrt{38}$$

c $\angle QRP = 90^\circ$ so PQ is the hypotenuse.

$$\sin\angle PQR = \frac{\left|\overrightarrow{PR}\right|}{\left|\overrightarrow{PQ}\right|} = \sqrt{\frac{59}{97}} = 0.7799...$$
$$\angle PQR = 51.3^\circ$$

7 $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = -2\mathbf{j} + \mathbf{k}$

$$\left|\overrightarrow{AB}\right| = \sqrt{1+1} = \sqrt{2}$$
$$\left|\overrightarrow{BC}\right| = \sqrt{1+9+1} = \sqrt{11}$$
$$\left|\overrightarrow{AC}\right| = \sqrt{4+1} = \sqrt{5}$$

$$\cos \angle ABC = \frac{2+11-5}{2\times\sqrt{2}\times\sqrt{11}} = 0.8528...$$
$$\angle ABC = 31.5°$$

8 $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} \Rightarrow \overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$

So $\overrightarrow{BC} = \begin{pmatrix} 9 \\ 10 \\ -6 \end{pmatrix}$

$$\left|\overrightarrow{AB}\right| = \sqrt{36+4+121} = \sqrt{161}$$
$$\left|\overrightarrow{AC}\right| = \sqrt{225+64+25} = \sqrt{314}$$
$$\left|\overrightarrow{BC}\right| = \sqrt{81+100+36} = \sqrt{217}$$

$$\cos \angle ABC = \frac{161+217-314}{2\times\sqrt{161}\times\sqrt{217}} = 0.1712...$$
$$\angle ABC = 80.14...°$$

Area of triangle *ABC*

$$= \frac{1}{2}\times\sqrt{161}\times\sqrt{217}\times\sin\angle ABC$$

Area of parallelogram *ABCD*

$$= \sqrt{161}\times\sqrt{217}\times\sin\angle ABC$$
$$= \sqrt{161}\times\sqrt{217}\times\sin 80.14...°$$
$$= 184 \text{ (3 s.f.)}$$

9 **a** $\left|\overrightarrow{AB}\right| = \sqrt{4+25+9} = \sqrt{38}$

$\left|\overrightarrow{AC}\right| = \sqrt{4+25+9} = \sqrt{38}$

So *ABC* is an isosceles triangle.
Therefore *DBC* is an isosceles triangle.

So $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$ and $\overrightarrow{AC}$ is parallel to $\overrightarrow{BD}$.

Let *O* be the fixed origin.

$$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$$
$$= \overrightarrow{OC} + \overrightarrow{AB}$$
$$= \overrightarrow{OC} + \overrightarrow{OB} - \overrightarrow{OA}$$
$$= \begin{pmatrix} 4 \\ -2 \\ -5 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -7 \\ -2 \end{pmatrix}$$

Coordinates of *D* are (2, −7, −2).

b *ABCD* is a parallelogram with four sides of equal length. It is a rhombus.

c $\left|\overrightarrow{BC}\right| = \sqrt{16+36} = \sqrt{52}$

$$\cos \angle BAC = \frac{38+38-52}{2\times\sqrt{38}\times\sqrt{38}} = 0.3157...$$
$$\angle BAC = 71.59...°$$

Area of triangle *ABC*

$$= \frac{1}{2}\times\sqrt{38}\times\sqrt{38}\times\sin\angle ABC$$

Area of parallelogram *ABCD*

$$= \sqrt{38}\times\sqrt{38}\times\sin\angle ABC$$
$$= \sqrt{38}\times\sqrt{38}\times\sin 71.59...°$$
$$= 36.1 \text{ (3 s.f.)}$$

10 $\overrightarrow{OP} = \frac{1}{2}\overrightarrow{OC} = \frac{1}{2}\mathbf{c}$

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$\overrightarrow{OQ} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a})$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \frac{1}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

$\overrightarrow{OR} = \frac{1}{2}\overrightarrow{OA} = \frac{1}{2}\mathbf{a}$

$\overrightarrow{BC} = \mathbf{c} - \mathbf{b}$

$\overrightarrow{OS} = \overrightarrow{OB} + \frac{1}{2}\overrightarrow{BC} = \mathbf{b} + \frac{1}{2}(\mathbf{c} - \mathbf{b})$

$\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR} = \frac{1}{2}(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

$\overrightarrow{OT} = \frac{1}{2}\overrightarrow{OB} = \frac{1}{2}\mathbf{b}$

$\overrightarrow{AC} = \mathbf{c} - \mathbf{a}$

$\overrightarrow{OU} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC} = \mathbf{a} + \frac{1}{2}(\mathbf{c} - \mathbf{a})$

$\overrightarrow{TU} = \overrightarrow{OU} - \overrightarrow{OT} = \frac{1}{2}(\mathbf{a} - \mathbf{b} + \mathbf{c})$

Suppose there is a point of intersection, X, of PQ, RS and TU.

$\overrightarrow{PX} = r\overrightarrow{PQ} = \frac{r}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

$\overrightarrow{RX} = s\overrightarrow{RS} = \frac{s}{2}(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

$\overrightarrow{TX} = t\overrightarrow{TU} = \frac{t}{2}(\mathbf{a} - \mathbf{b} + \mathbf{c})$

for scalars $r,\ s$ and t.

But $\overrightarrow{RX} = \overrightarrow{RO} + \overrightarrow{OP} + \overrightarrow{PX}$

$= -\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{c} + \frac{r}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

so $\frac{s}{2}(-\mathbf{a} + \mathbf{b} + \mathbf{c}) = -\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{c} + \frac{r}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

$s(-\mathbf{a} + \mathbf{b} + \mathbf{c}) = (r-1)\mathbf{a} + r\mathbf{b} + (1-r)\mathbf{c}$

Comparing coefficients of **b** and **c**:
$s = r$ and $s = 1 - r$

Hence $r = s = \frac{1}{2}$

Also $\overrightarrow{TX} = \overrightarrow{TO} + \overrightarrow{OP} + \overrightarrow{PX}$

$= -\frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c} + \frac{1}{4}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

so $\frac{t}{2}(\mathbf{a} - \mathbf{b} + \mathbf{c}) = -\frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c} + \frac{1}{4}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

$t(\mathbf{a} - \mathbf{b} + \mathbf{c}) = \frac{1}{2}(\mathbf{a} - \mathbf{b} + \mathbf{c})$

Hence $t = \frac{1}{2}$

So the point X is the midpoint of all three line segments PQ, RS and TU. Therefore the line segments do meet at a point and bisect each other.

11 Total force on particle

$= \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$

$= ((b+1)\mathbf{i} + (4-b)\mathbf{j} + (7-b)\mathbf{k})$ N

$|\mathbf{F}| = \sqrt{(b+1)^2 + (4-b)^2 + (7-b)^2}$

$= \sqrt{b^2 + 2b + 1 + 16 - 8b + b^2 + 49 - 14b + b^2}$

$= \sqrt{3b^2 - 20b + 66}$

$|\mathbf{F}| = m|\mathbf{a}|$

$\Rightarrow \sqrt{3b^2 - 20b + 66} = 2 \times 3.5 = 7$

$3b^2 - 20b + 66 = 49$

$3b^2 - 20b + 17 = 0$

$(b-1)(3b-17) = 0$

$b = 1$ or $b = \frac{17}{3}$

12 a Air resistance acts in opposition to the motion of the BASE jumper. The motion downwards will be greater than the motion in the other directions.

b Gravitational force downwards
$= 50 \times 9.8 = 490$ N

Total force on BASE jumper
$= \mathbf{W} + \mathbf{F} - 490\mathbf{k}$

$= (16\mathbf{i} + 13\mathbf{j} - 40\mathbf{k})$ N

12 c $|16\mathbf{i}+13\mathbf{j}-40\mathbf{k}|=\sqrt{256+169+1600}$

$=\sqrt{2025}=45\text{ N}$

Acceleration $=\frac{45}{50}=\frac{9}{10}\text{ m s}^{-2}$

Using $s=ut+\frac{1}{2}at^2$:

$180=0+\frac{1}{2}\times\frac{9}{10}t^2$

$t^2=400$

$t=20$

The descent took 20 seconds.

Challenge

For example, if **a** = (1, 0, 0), **b** = (0, 1, 0) and **c** = (1, 1, 0) then $p = q = r = 1$ gives

a + **b** + **c** = 2**a** + 2**b**

So $s = 2 \neq p$, $t = 2 \neq q$ and $u = 0 \neq r$, and the result does not hold.

The statement is also untrue if any of the scalars p, q and r is zero. For example, with **a**, **b** and **c** as above, if $p = 0$ and $q = r = 1$, then $s = 1 \neq p$, $t = 2 \neq q$ and $u = 0 \neq r$.

Review exercise 3

1 $y=\frac{1}{2}x^2+4\cos x$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=x-4\sin x$$

When $x=\frac{\pi}{2}$:

$$y=\frac{\pi^2}{8}\ \text{ and }\ \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\pi}{2}-4=\frac{\pi-8}{2}$$

So gradient of normal is $-\frac{2}{\pi-8}$

Equation of normal is

$$y-\frac{\pi^2}{8}=-\frac{2}{\pi-8}\left(x-\frac{\pi}{2}\right)$$

$$y(8-\pi)-\frac{\pi^2}{8}(8-\pi)=2\left(x-\frac{\pi}{2}\right)$$

$$8y(8-\pi)-\pi^2(8-\pi)=16x-8\pi$$

$$8y(8-\pi)-16x-\pi^2(8-\pi)+8\pi=0$$

$$8y(8-\pi)-16x+\pi\left(\pi^2-8\pi+8\right)=0$$

2 $y=\mathrm{e}^{3x}-\ln(x^2)$

$$=\mathrm{e}^{3x}-2\ln x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=3\mathrm{e}^{3x}-\frac{2}{x}$$

When $x=2$:

$$y=\mathrm{e}^6-\ln 4\ \text{ and }\ \frac{\mathrm{d}y}{\mathrm{d}x}=3\mathrm{e}^6-1$$

Equation of tangent is

$$y-\left(\mathrm{e}^6-\ln 4\right)=\left(3\mathrm{e}^6-1\right)(x-2)$$

$$y-\mathrm{e}^6+\ln 4=\left(3\mathrm{e}^6-1\right)x-6\mathrm{e}^6+2$$

$$y-\left(3\mathrm{e}^6-1\right)x-2+\ln 4+5\mathrm{e}^6=0$$

3 $y=\frac{3}{(4-6x)^2}$

$$=3(4-6x)^{-2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=36(4-6x)^{-3}=\frac{36}{(4-6x)^3}$$

When $x=1$:

$$y=\frac{3}{4}\ \text{ and }\ \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{36}{8}=-\frac{9}{2}$$

So gradient of normal is $\frac{2}{9}$

Equation of normal is

$$y-\frac{3}{4}=\frac{2}{9}(x-1)$$

$$36y-27=8x-8$$

$$0=8x-36y+19$$

4 a $y=(2x-3)^2\mathrm{e}^{2x}$

Let $u=(2x-3)^2\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=4(2x-3)$

and $v=\mathrm{e}^{2x}\Rightarrow\frac{\mathrm{d}v}{\mathrm{d}x}=2\mathrm{e}^{2x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=u\frac{\mathrm{d}v}{\mathrm{d}x}+v\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$=2(2x-3)^2\mathrm{e}^{2x}+4(2x-3)\mathrm{e}^{2x}$$

$$=2\mathrm{e}^{2x}(2x-3)(2x-3+2)$$

$$=2\mathrm{e}^{2x}(2x-3)(2x-1)$$

b $\frac{\mathrm{d}y}{\mathrm{d}x}=0\Rightarrow 2x-3=0$ or $2x-1=0$

So $x=\frac{3}{2}$ or $\frac{1}{2}$

When $x=\frac{3}{2}$, $y=0$

When $x=\frac{1}{2}$, $y=4\mathrm{e}$

So coordinates of stationary points are $\left(\frac{3}{2},0\right)$ and $\left(\frac{1}{2},4\mathrm{e}\right)$.

5 **a** $y=\frac{(x-1)^2}{\sin x}$

Let $u=(x-1)^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=2(x-1)$

and $v=\sin x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x}=\cos x$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{v\frac{\mathrm{d}u}{\mathrm{d}x}-u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$
$$=\frac{2(x-1)\sin x-(x-1)^2\cos x}{\sin^2 x}$$
$$=\frac{(x-1)(2\sin x-x\cos x+\cos x)}{\sin^2 x}$$

b When $x=\frac{\pi}{2}$:

$y=\left(\frac{\pi}{2}-1\right)^2$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=2\left(\frac{\pi}{2}-1\right)$

Equation of tangent is

$$y-\left(\frac{\pi}{2}-1\right)^2=2\left(\frac{\pi}{2}-1\right)\left(x-\frac{\pi}{2}\right)$$
$$=(\pi-2)\left(x-\frac{\pi}{2}\right)$$
$$y=(\pi-2)x-\frac{\pi}{2}(\pi-2)+\left(\frac{\pi}{2}-1\right)^2$$
$$=(\pi-2)x-\frac{\pi^2}{2}+\pi+\frac{\pi^2}{4}-\pi+1$$
$$=(\pi-2)x+\left(1-\frac{\pi^2}{4}\right)$$

6 **a** $y=\operatorname{cosec} x=\frac{1}{\sin x}$

Let $u=\sin x \Rightarrow \frac{du}{dx}=\cos x$

and $y=\frac{1}{u} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}u}=-\frac{1}{u^2}$

Using the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}$$
$$=-\frac{1}{\sin^2 x}\times\cos x$$
$$=-\frac{1}{\sin x}\times\frac{1}{\tan x}$$
$$=-\operatorname{cosec} x\cot x$$

b $x=\operatorname{cosec} 6y$

$$\frac{\mathrm{d}x}{\mathrm{d}y}=-6\operatorname{cosec} 6y\cot 6y$$

$$\operatorname{cosec}^2 6y=1+\cot^2 6y$$
$$\Rightarrow \cot 6y=\sqrt{x^2-1}$$

$$\frac{\mathrm{d}x}{\mathrm{d}y}=-6x\sqrt{x^2-1}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{6x\sqrt{x^2-1}}$$

7 $y=\arcsin x$

So $x=\sin y$

$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y}=\cos y$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\cos y}$

$$\sin^2 y+\cos^2 y=1$$
$$\cos y=\sqrt{1-\sin^2 y}=\sqrt{1-x^2}$$

So $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\sqrt{1-x^2}}$

8 a $x = 2\cot t, y = \sin^2 t$

$$\frac{dx}{dt} = -2\,\text{cosec}^2\, t,\ \frac{dy}{dt} = 4\sin t\cos t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

$$= \frac{4\sin t\cos t}{-2\,\text{cosec}^2\, t}$$

$$= -2\sin^3 t\cos t$$

b When $t = \frac{\pi}{4}$:

$$x = 2 \text{ and } y = 2\times\left(\frac{1}{\sqrt{2}}\right)^2 = 1$$

$$\frac{dy}{dx} = -2\times\left(\frac{1}{\sqrt{2}}\right)^3 \times \left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$$

So equation of tangent is

$$y - 1 = -\frac{1}{2}(x-2)$$

$$y = -\frac{1}{2}x + 2$$

c $x = 2\cot t \Rightarrow \cot t = \frac{x}{2}$

$y = 2\sin^2 t \Rightarrow \sin^2 t = \frac{y}{2}$ and $\text{cosec}^2\, t = \frac{2}{y}$

$$\text{cosec}^2\, t = 1 + \cot^2 t$$

$$\frac{2}{y} = 1 + \left(\frac{x}{2}\right)^2$$

$$= \frac{4+x^2}{4}$$

$$\frac{y}{2} = \frac{4}{4+x^2}$$

$$y = \frac{8}{4+x^2}$$

As $0 < t \leq \frac{\pi}{2}, \cot t \geq 0$

$x = 2\cot t$ so the domain of the function is $x \geq 0$.

9 a $x = \frac{1}{1+t}, y = \frac{1}{1-t}$

Using the chain rule:

$$\frac{dx}{dt} = \frac{-1}{(1+t)^2}, \frac{dy}{dt} = \frac{1}{(1-t)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

$$= -\frac{(1+t)^2}{(1-t)^2}$$

When $t = \frac{1}{2}$:

$$x = \frac{2}{3} \text{ and } y = 2$$

$$\frac{dy}{dx} = -\frac{(\frac{3}{2})^2}{(\frac{1}{2})^2} = -\frac{\frac{9}{4}}{\frac{1}{4}} = -9$$

So equation of tangent is

$$y - 2 = -9\left(x - \frac{2}{3}\right)$$

$$y = -9x + 8$$

b $x = \frac{1}{1+t} \Rightarrow t = \frac{1}{x} - 1$

Substitute into $y = \frac{1}{1-t}$:

$$y = \frac{1}{1-\left(\frac{1}{x}-1\right)}$$

$$= \frac{1}{2-\frac{1}{x}}$$

$$= \frac{x}{2x-1}$$

10 $3x^2-2y^2+2x-3y+5=0$

Differentiating with respect to x:

$$6x-4y\frac{dy}{dx}+2-3\frac{dy}{dx}+0=0$$

Substituting $x=0,\ y=1$:

$$-4\frac{dy}{dx}+2-3\frac{dy}{dx}=0$$

$$7\frac{dy}{dx}=2$$

$$\frac{dy}{dx}=\frac{2}{7}$$

So gradient of normal at (0, 1) is $\frac{-7}{2}$

Equation of normal is

$$y-1=\frac{-7}{2}(x-0)$$

$$y=\frac{-7}{2}x+1$$

$$7x+2y-2=0$$

11 a $\sin x+\cos y=0.5$

Differentiating with respect to x:

$$\cos x-\sin y\frac{dy}{dx}=0$$

$$\Rightarrow\frac{dy}{dx}=\frac{\cos x}{\sin y}$$

b $\frac{dy}{dx}=0\Rightarrow\cos x=0\Rightarrow x=\pm\frac{\pi}{2}$

When $x=\frac{\pi}{2}$:

$1+\cos y=0.5\Rightarrow\cos y=-0.5$

$y=\frac{2\pi}{3}$ or $y=\frac{-2\pi}{3}$

When $x=-\frac{\pi}{2}$:

$-1+\cos y=0.5\Rightarrow\cos y=1.5$

(no solutions)

So the only stationary points in the given range are at $\left(\frac{\pi}{2},\frac{2\pi}{3}\right)$ and $\left(\frac{\pi}{2},\frac{-2\pi}{3}\right)$.

12 $y=x^2e^{-x}$

$$\frac{dy}{dx}=-x^2e^{-x}+2xe^{-x}=e^{-x}\left(2x-x^2\right)$$

$$\frac{d^2y}{dx^2}=e^{-x}\left(2-2x\right)-e^{-x}\left(2x-x^2\right)$$

$$=e^{-x}\left(x^2-4x+2\right)$$

For C to be convex, $\frac{d^2y}{dx^2}\geq 0$.

$e^{-x}>0$, and for all $x<0,\ x^2-4x+2>0$

So $\frac{d^2y}{dx^2}>0$ for all $x<0$.

Hence C is convex for all $x<0$.

13 a $V=\frac{4}{3}\pi r^3\Rightarrow\frac{dV}{dr}=4\pi r^2$

b Using the chain rule:

$$\frac{dV}{dt}=\frac{dV}{dr}\times\frac{dr}{dt}=4\pi r^2\times\frac{dr}{dt}$$

$$\frac{dV}{dt}=\frac{1000}{(2t+1)^2}=4\pi r^2\times\frac{dr}{dt}$$

So $\frac{dr}{dt}=\frac{1000}{4\pi(2t+1)^2r^2}$

$$=\frac{250}{\pi(2t+1)^2r^2}$$

14 a $g(x)=x^3-x^2-1$

$g(1.4)=1.4^3-1.4^2-1=-0.216<0$

$g(1.5)=1.5^3-1.5^2-1=0.125>0$

The change of sign implies that the root α is in $[1.4,1.5]$.

b $g(1.4655)=-0.00025...<0$

$g(1.4665)=0.00326...>0$

The change of sign implies that the root α satisfies $1.4655<\alpha<1.4665$, and so $\alpha=1.466$ correct to 3 decimal places.

15 a $\mathrm{p}(x) = \cos x + \mathrm{e}^{-x}$

$$\mathrm{p}(1.7) = \cos 1.7 + \mathrm{e}^{-1.7} = 0.054... > 0$$
$$\mathrm{p}(1.8) = \cos 1.8 + \mathrm{e}^{-1.8} = -0.062... < 0$$

The change of sign implies that the root α is in $[1.7, 1.8]$.

b $\mathrm{p}(1.7455) = \cos 1.7455 + \mathrm{e}^{-1.7455}$

$$= 0.00074... > 0$$

$\mathrm{p}(1.7465) = \cos 1.7465 + \mathrm{e}^{-1.7465}$

$$= -0.00042... < 0$$

The change of sign implies that the root α satisfies $1.7455 < \alpha < 1.7465$, and so $\alpha = 1.746$ correct to 3 decimal places.

16 a $\mathrm{f}(x) = \mathrm{e}^{x-2} - 3x + 5 = 0$

$$\mathrm{e}^{x-2} = 3x - 5$$
$$x - 2 = \ln(3x - 5)$$
$$x = \ln(3x - 5) + 2, \text{ for } 3x - 5 > 0 \Rightarrow x > \frac{5}{3}$$

b Using $x_0 = 4$:

$$x_1 = \ln 7 + 2 = 3.9459$$
$$x_2 = \ln(3 \times 3.9459 - 5) + 2 = 3.9225$$
$$x_3 = \ln(3 \times 3.9225 - 5) + 2 = 3.9121$$

All correct to 4 decimal places.

17 a $\mathrm{f}(x) = \dfrac{1}{(x-2)^3} + 4x^2$

$$\mathrm{f}(0.2) = \frac{1}{(0.2-2)^3} + 4 \times 0.2^2$$
$$= -0.011... < 0$$
$$\mathrm{f}(0.3) = \frac{1}{(0.3-2)^3} + 4 \times 0.3^2$$
$$= 0.156... > 0$$

The change of sign implies that the root α is in $[0.2, 0.3]$.

b $\mathrm{f}(x) = \dfrac{1}{(x-2)^3} + 4x^2 = 0$

$$\frac{1}{(x-2)^3} = -4x^2$$
$$(x-2)^3 = -\frac{1}{4x^2}$$
$$x - 2 = \sqrt[3]{\frac{-1}{4x^2}}$$
$$x = \sqrt[3]{\frac{-1}{4x^2}} + 2$$

c Using $x_0 = 1$:

$$x_1 = \sqrt[3]{\frac{-1}{4}} + 2 = 1.3700$$
$$x_2 = \sqrt[3]{\frac{-1}{4 \times 1.3700^2}} + 2 = 1.4893$$
$$x_3 = \sqrt[3]{\frac{-1}{4 \times 1.4893^2}} + 2 = 1.5170$$
$$x_4 = \sqrt[3]{\frac{-1}{4 \times 1.5170^2}} + 2 = 1.5228$$

All correct to 4 decimal places.

d $\mathrm{f}(1.5235) = \dfrac{1}{(1.5235-2)^3} + 4 \times 1.5235^2$

$$= 0.0412... > 0$$

$\mathrm{f}(1.5245) = \dfrac{1}{(1.5245-2)^3} + 4 \times 1.5245^2$

$$= -0.0050... < 0$$

The change of sign implies that the root α satisfies $1.5235 < \alpha < 1.5245$, and so $\alpha = 1.524$ correct to 3 decimal places.

18 a $\mathrm{f}(x) = \dfrac{1}{10} x^2 \mathrm{e}^x - 2x - 10$

As A is a stationary point, the gradient at A is zero. So $\mathrm{f}'(a) = 0$.

The Newton–Raphson process uses $\mathrm{f}'(x)$ as a denominator. Division by zero is undefined so $x_0 = a$ cannot be used to find an approximation for α.

18 b $\text{f}'(x)=x\text{e}^x(0.1x+0.2)-2$

Using $x_0=2.9$:

$$\begin{aligned}x_1&=x_0-\frac{\text{f}(x_0)}{\text{f}'(x_0)}\\&=2.9-\frac{\text{f}(2.9)}{\text{f}'(2.9)}\\&=2.9+\frac{0.5155}{23.825}\\&=2.922\ (3\text{ d.p.})\end{aligned}$$

19 a $\text{f}(x)=\frac{3}{10}x^3-x^{\frac{2}{3}}+\frac{1}{x}-4$

i $$\begin{aligned}\text{f}(0.2)&=\frac{3}{10}\times0.2^3-0.2^{\frac{2}{3}}+\frac{1}{0.2}-4\\&=0.660...>0\end{aligned}$$

$$\begin{aligned}\text{f}(0.3)&=\frac{3}{10}\times0.3^3-0.3^{\frac{2}{3}}+\frac{1}{0.3}-4\\&=-1.107...<0\end{aligned}$$

The change of sign implies that there is a root α in $[0.2,0.3]$.

ii $$\begin{aligned}\text{f}(2.6)&=\frac{3}{10}\times2.6^3-2.6^{\frac{2}{3}}+\frac{1}{2.6}-4\\&=-0.233...<0\end{aligned}$$

$$\begin{aligned}\text{f}(2.7)&=\frac{3}{10}\times2.7^3-2.7^{\frac{2}{3}}+\frac{1}{2.7}-4\\&=0.336...>0\end{aligned}$$

The change of sign implies that there is a root α in $[2.6,2.7]$.

b $$\text{f}(x)=\frac{3}{10}x^3-x^{\frac{2}{3}}+\frac{1}{x}-4=0$$

$$\frac{3}{10}x^3=4+x^{\frac{2}{3}}-\frac{1}{x}$$

$$x^3=\frac{10}{3}\left(4+x^{\frac{2}{3}}-\frac{1}{x}\right)$$

$$x=\sqrt[3]{\frac{10}{3}\left(4+x^{\frac{2}{3}}-\frac{1}{x}\right)}$$

c Using $x_0=2.5$:

$$x_1=\sqrt[3]{\frac{10}{3}\left(4+2.5^{\frac{2}{3}}-\frac{1}{2.5}\right)}=2.6275$$

$$x_2=\sqrt[3]{\frac{10}{3}\left(4+2.6275^{\frac{2}{3}}-\frac{1}{2.6275}\right)}=2.6406$$

$$x_3=\sqrt[3]{\frac{10}{3}\left(4+2.6406^{\frac{2}{3}}-\frac{1}{2.6406}\right)}=2.6419$$

$$x_3=\sqrt[3]{\frac{10}{3}\left(4+2.6419^{\frac{2}{3}}-\frac{1}{2.6419}\right)}=2.6420$$

All correct to 4 decimal places.

d $\text{f}'(x)=\frac{9}{10}x^2-\frac{2}{3}x^{-\frac{1}{3}}-\frac{1}{x^2}$

Using $x_0=0.3$:

$$\begin{aligned}x_1&=x_0-\frac{\text{f}(x_0)}{\text{f}'(x_0)}\\&=0.3-\frac{\text{f}(0.3)}{\text{f}'(0.3)}\\&=0.3-\frac{-1.10671}{-12.02598}\\&=0.208\ (3\text{ d.p.})\end{aligned}$$

20 a $$\begin{aligned}\text{v}(x)&=0.12\cos\left(\frac{2x}{5}\right)-0.35\sin\left(\frac{2x}{5}\right)+120\\&=R\left(\cos\frac{2x}{5}+\alpha\right)+120\end{aligned}$$

$$\begin{aligned}&R\left(\cos\frac{2x}{5}+\alpha\right)\\&=R\left(\cos\frac{2x}{5}\cos\alpha-\sin\frac{2x}{5}\sin\alpha\right)\\&=0.12\cos\left(\frac{2x}{5}\right)-0.35\sin\left(\frac{2x}{5}\right)\end{aligned}$$

So $R\cos\alpha = 0.12$ and $R\sin\alpha = 0.35$

$\tan\alpha = \frac{0.35}{0.12} \Rightarrow \alpha = 1.2405$ (4 d.p.)

$R^2\left(\cos^2\theta + \sin^2\theta\right) = 0.12^2 + 0.35^2$

$R^2 = 0.1369$ so $R = 0.37$

20 b $$\text{v}(x) = 0.37\cos\left(\frac{2x}{5} + 1.2405\right) + 120$$

$$\text{v}'(x) = -\frac{2}{5} \times 0.37\sin\left(\frac{2x}{5} + 1.2405\right)$$

$$= -0.148\sin\left(\frac{2x}{5} + 1.2405\right)$$

c $$\text{v}'(4.7) = -0.148\sin\left(\frac{9.4}{5} + 1.2405\right)$$

$$= -0.0031... < 0$$

$$\text{v}'(4.8) = -0.148\sin\left(\frac{9.6}{5} + 1.2405\right)$$

$$= 0.0028... > 0$$

The change of sign implies that there is a stationary point in the interval $[4.7, 4.8]$.

d $$\text{v}''(x) = -\frac{2}{5} \times 0.148\cos\left(\frac{2x}{5} + 1.2405\right)$$

Using $x_0 = 12.6$:

$$x_1 = 12.6 - \frac{\text{v}'(12.6)}{\text{v}''(12.6)}$$

$$= 12.6 - \frac{0.148\sin\left(\frac{25.2}{5} + 1.2405\right)}{\frac{2}{5} \times 0.148\cos\left(\frac{25.2}{5} + 1.2405\right)}$$

$$= 12.6 + \frac{0.0003974...}{0.5920...}$$

$$= 12.607 \text{ (3 d.p.)}$$

e $$\text{v}'(12.60665)$$

$$= -0.148\sin\left(\frac{25.2133}{5} + 1.2405\right)$$

$$= 0.0000037... > 0$$

$$\text{v}'(12.60675)$$

$$= -0.148\sin\left(\frac{25.2135}{5} + 1.2405\right)$$

$$= -0.0000022... < 0$$

The change of sign implies that there is a stationary point at $x = 12.6067$ correct to 4 decimal places.

21 $$\int_a^3 (12 - 3x)^2 \, \text{d}x = 78$$

$$\left[-\frac{1}{9}(12 - 3x)^3\right]_a^3 = -\frac{27}{9} + \frac{1}{9}(12 - 3a)^3$$

$$-3 + \frac{1}{9}(12 - 3a)^3 = 78$$

$$\frac{1}{9}(12 - 3a)^3 = 81$$

$$(12 - 3a)^3 = 729$$

$$12 - 3a = 9$$

$$a = 1$$

22 a $$\cos(5x + 2x) = \cos 5x \cos 2x - \sin 5x \sin 2x$$

$$\cos(5x - 2x) = \cos 5x \cos 2x + \sin 5x \sin 2x$$

Adding:

$$\cos 7x + \cos 3x = 2\cos 5x \cos 2x$$

b $$\int 6\cos 5x \cos 2x \, \text{d}x$$

$$= 3\int (\cos 7x + \cos 3x)\text{d}x$$

$$= \frac{3}{7}\sin 7x + \sin 3x + c$$

23 Consider $y = \text{e}^{x^4} \Rightarrow \frac{\text{d}y}{\text{d}x} = 4x^3\,\text{e}^{x^4}$

So $$\int_0^m mx^3\,\text{e}^{x^4}\,\text{d}x = \left[\frac{m}{4}\text{e}^{x^4}\right]_0^m$$

$$= \frac{m}{4}\text{e}^{m^4} - \frac{m}{4}$$

So $$\frac{m}{4}\text{e}^{m^4} - \frac{m}{4} = \frac{3}{4}(\text{e}^{81} - 1)$$

$$\frac{m}{4}(\text{e}^{m^4} - 1) = \frac{3}{4}(\text{e}^{81} - 1)$$

$$m = 3$$

24 Let $I=\int_1^5 \frac{3x}{\sqrt{2x-1}}\,dx$

Let $u^2=2x-1 \Rightarrow 2u\frac{du}{dx}=2$

So replace dx with u du.

$\sqrt{2x-1}=u$ and $x=\frac{u^2+1}{2}$

x	u
1	1
5	3

$$\begin{aligned}
\text{So } I &= \int_1^3 \frac{3}{2}\times\frac{u^2+1}{u}\times u\,du\\
&= \int_1^3\left(\frac{3}{2}u^2+\frac{3}{2}\right)du\\
&= \left[\frac{1}{2}u^3+\frac{3}{2}u\right]_1^3\\
&= \left(\frac{27}{2}+\frac{9}{2}\right)-\left(\frac{1}{2}+\frac{3}{2}\right)\\
&= 18-2\\
&= 16
\end{aligned}$$

25 Let $I=\int_0^{\frac{1}{2}} \frac{x^3}{(1-x^2)^{\frac{1}{2}}}\,dx$

Let $u=1-x^2 \Rightarrow \frac{du}{dx}=-2x$

So replace x dx with $-\frac{du}{2}$.

$x^2=1-u$

$$\begin{aligned}
\text{So } \int\frac{x^3}{(1-x^2)^{\frac{1}{2}}}dx &= \int\frac{x^2}{(1-x^2)^{\frac{1}{2}}}x\,dx\\
&= \int\frac{1-u}{u^{\frac{1}{2}}}\left(-\frac{du}{2}\right)\\
&= -\frac{1}{2}\int\frac{1-u}{u^{\frac{1}{2}}}du\\
&= -\frac{1}{2}\int\left(u^{-\frac{1}{2}}-u^{\frac{1}{2}}\right)du
\end{aligned}$$

x	u
$\frac{1}{2}$	$\frac{3}{4}$
0	1

$$\begin{aligned}
\text{So } I &= \left[-u^{\frac{1}{2}}+\frac{1}{3}u^{\frac{3}{2}}\right]_1^{\frac{3}{4}}\\
&= \left(-\frac{\sqrt{3}}{2}+\frac{1}{3}\times\frac{3\sqrt{3}}{4\sqrt{4}}\right)-\left(-1+\frac{1}{3}\right)\\
&= \left(-\frac{3\sqrt{3}}{8}\right)-\left(-\frac{2}{3}\right)\\
&= \frac{2}{3}-\frac{3\sqrt{3}}{8}
\end{aligned}$$

26 Let $I=\int_1^{\mathrm{e}}(x^2+1)\ln x\,\mathrm{d}x$

Let $u=1\mathrm{n}\,x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x}$

and $\frac{\mathrm{d}v}{\mathrm{d}x}=x^2+1 \Rightarrow v=\frac{x^3}{3}+x$

Using the integration by parts formula:

$$I=\left[\left(\frac{x^3}{3}+x\right)1\mathrm{n}\,x\right]_1^{\mathrm{e}}-\int_1^{\mathrm{e}}\frac{1}{x}\left(\frac{x^3}{3}+x\right)\mathrm{d}x$$

$$=\left(\frac{\mathrm{e}^3}{3}+\mathrm{e}\right)\times 1-\left(\frac{1}{3}+1\right)\times 0-\int_1^{\mathrm{e}}\left(\frac{x^2}{3}+1\right)\mathrm{d}x$$

$$=\frac{\mathrm{e}^3}{3}+\mathrm{e}-0-\left[\frac{x^3}{9}+x\right]_1^{\mathrm{e}}$$

$$=\frac{\mathrm{e}^3}{3}+\mathrm{e}-\left(\left(\frac{\mathrm{e}^3}{9}+\mathrm{e}\right)-\left(\frac{1}{9}+1\right)\right)$$

$$=\frac{2\mathrm{e}^3}{9}+\frac{10}{9}$$

$$=\frac{1}{9}(2\mathrm{e}^3+10)$$

27 a $\frac{5x+3}{(2x-3)(x+2)}\equiv\frac{A}{2x-3}+\frac{B}{x+2}$

$$\equiv\frac{A(x+2)+B(2x-3)}{(2x-3)(x+2)}$$

$$5x+3\equiv A(x+2)+B(2x-3)$$

Let $x=-2$: $\quad -7=B(-7)$ so $B=1$

Let $x=\frac{3}{2}$: $\quad \frac{21}{2}=A\left(\frac{7}{2}\right)$ so $A=3$

So $\frac{5x+3}{(2x-3)(x+2)}\equiv\frac{3}{2x-3}+\frac{1}{x+2}$

b $\int_2^6\frac{5x+3}{(2x-3)(x+2)}\mathrm{d}x$

$$=\int_2^6\frac{3}{2x-3}\mathrm{d}x+\int_2^6\frac{1}{x+2}\mathrm{d}x$$

$$=\left[\frac{3}{2}1\mathrm{n}(2x-3)+1\mathrm{n}(x+2)\right]_2^6$$

$$=\left(\frac{3}{2}1\mathrm{n}\,9+1\mathrm{n}\,8\right)-\left(\frac{3}{2}\ln 1+1\mathrm{n}\,4\right)$$

$$=\ln 9^{\frac{3}{2}}+\ln 8-0-\ln 4$$

$$=1\mathrm{n}\,9^{\frac{3}{2}}+1\mathrm{n}\frac{8}{4}$$

$$=1\mathrm{n}\,27+1\mathrm{n}\,2$$

$$=1\mathrm{n}\,54$$

28 Area $R=\int_0^{\frac{\pi}{2}}8\sin x\cos^3 x\,\mathrm{d}x$

Consider $y=\cos^4 x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=-4\sin x\cos^3 x$

So $\int_0^{\frac{\pi}{2}}8\sin x\cos^3 x\,\mathrm{d}x=\left[-2\cos^4 x\right]_0^{\frac{\pi}{2}}$

$$=0-(-2)$$

$$=2$$

29 a When $x=1$:

$y=\sqrt{1+\sin 1}=1.3570$ (4 d.p.) $x=2$:

When $x=2$:

$y=\sqrt{1+\sin 2}=1.3818$ (4 d.p.)

b $I=\int_0^2 \sqrt{1+\sin x}\,\mathrm{d}x$

$$\approx \frac{1}{2}\times 0.5(1+2(1.216+1.357+1.413)+1.382)$$
$$=0.25\times 10.354$$
$$=2.5885$$
$$=2.589 \text{ (4 s.f.)}$$

30 a Let $I=\int_0^1 x\mathrm{e}^{2x}\mathrm{d}x$

Let $u=x\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=1$

and $\frac{\mathrm{d}v}{\mathrm{d}x}=\mathrm{e}^{2x}\Rightarrow v=\frac{1}{2}\mathrm{e}^{2x}$

Using the integration by parts formula:

$$I=\left[\frac{1}{2}x\mathrm{e}^{2x}\right]_0^1-\int_0^1 1\times\frac{1}{2}\mathrm{e}^{2x}\mathrm{d}x$$
$$=\left(\frac{1}{2}\mathrm{e}^2-0\right)-\left[\frac{1}{4}\mathrm{e}^{2x}\right]_0^1$$
$$=\frac{1}{2}\mathrm{e}^2-\left(\frac{1}{4}\mathrm{e}^2-\frac{1}{4}\right)$$
$$=\frac{1}{4}\mathrm{e}^2+\frac{1}{4}$$

b When $x = 0.4$, $y = 0.89022$

When $x = 0.8$, $y = 3.96243$

c Area of R

$$\approx\frac{1}{2}\times 0.2(0+2(0.29836+0.89022+1.99207+3.96243)+7.38906)$$
$$=0.1\times 21.67522$$
$$=2.168 \text{ (4s.f.)}$$

d Percentage error in answer from part **c**

$$=\frac{\frac{1}{4}\mathrm{e}^2+\frac{1}{4}-2.168}{\frac{1}{4}\mathrm{e}^2+\frac{1}{4}}\times 100\%=3.37\%$$

31 a $\frac{2x-1}{(x-1)(2x-3)}\equiv\frac{A}{x-1}+\frac{B}{2x-3}$

$2x-1\equiv A(2x-3)+B(x-1)$

Let $x=\frac{3}{2}$: $2=B\left(\frac{1}{2}\right)\Rightarrow B=4$

Let $x=1$: $1=A(-1)\Rightarrow A=-1$

So $\frac{2x-1}{(x-1)(2x-3)}\equiv\frac{-1}{x-1}+\frac{4}{2x-3}$

b $(2x-3)(x-1)\frac{\mathrm{d}y}{\mathrm{d}x}=(2x-1)y$

Separating the variables:

$$\int\frac{1}{y}\mathrm{d}y=\int\frac{2x-1}{(2x-3)(x-1)}\mathrm{d}x$$

So $1\mathrm{n}\, y=\int\frac{-1}{x-1}\mathrm{d}x+\int\frac{4}{2x-3}\mathrm{d}x$

$$=-1\mathrm{n}|x-1|+21\mathrm{n}|2x-3|+c$$
$$=-1\mathrm{n}|x-1|+1\mathrm{n}(2x-3)^2+1\mathrm{n}\, A$$
$$=1\mathrm{n}\, A\frac{(2x-3)^2}{x-1}$$

So the general solution is

$$y=\frac{A(2x-3)^2}{x-1}$$

c $y=\frac{A(2x-3)^2}{x-1}$

When $x=2,\ y=10$ so

$$10=\frac{A(4-3)^2}{2-1}\Rightarrow A=10$$

So the particular solution is

$$y=\frac{10(2x-3)^2}{(x-1)}$$

32 a $V=\frac{4}{3}\pi r^3 \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}r}=4\pi r^2$

Using the chain rule:

$$\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}r}\times\frac{\mathrm{d}r}{\mathrm{d}t}$$
$$=4\pi r^2\times\frac{\mathrm{d}r}{\mathrm{d}t}$$

So $\frac{k}{V}=4\pi r^2\times\frac{\mathrm{d}r}{\mathrm{d}t}$

$$\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{k}{\frac{4}{3}\pi r^3}\times\frac{1}{4\pi r^2}$$
$$=\frac{3k}{16\pi^2 r^5}$$

So $B=\frac{3k}{16\pi^2}$

b Separating the variables:

$$\int r^5\mathrm{d}r=\int\frac{3k}{16\pi^2}\mathrm{d}t$$
$$\frac{r^6}{6}=\frac{3k}{16\pi^2}t+A$$
$$r^6=\frac{9k}{8\pi^2}t+A'$$
$$r=\left(\frac{9k}{8\pi^2}t+A'\right)^{\frac{1}{6}}$$

33 a Rate of change of volume is $\frac{\mathrm{d}V}{\mathrm{d}t}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$

Increase is 20 $\mathrm{cm}^3\,\mathrm{s}^{-1}$
Decrease is $kV\,\mathrm{cm}^3\,\mathrm{s}^{-1}$, where k is a constant of proportionality.

So the overall rate of change is

$$\frac{\mathrm{d}V}{\mathrm{d}t}=20-kV$$

b Separating the variables:

$$\int\frac{1}{20-kV}\mathrm{d}V=\int 1\mathrm{d}t$$

So $-\frac{1}{k}\ln|20-kV|=t+c$

When $t=0, V=0$ so

$$-\frac{1}{k}\ln 20=c$$

Combining the ln terms:

$$-\frac{1}{k}\ln\frac{20-kV}{20}=t$$
$$\ln\frac{20-kV}{20}=-kt$$
$$\frac{20-kV}{20}=\mathrm{e}^{-kt}$$
$$kV=20-20\mathrm{e}^{-kt}$$
$$V=\frac{20}{k}-\frac{20}{k}\mathrm{e}^{-kt}$$

So $A=\frac{20}{k}$ and $B=-\frac{20}{k}$

33 c $V = \frac{20}{k} - \frac{20}{k}\mathrm{e}^{-kt} \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}t} = 20\,\mathrm{e}^{-kt}$

Substitute $\frac{\mathrm{d}V}{\mathrm{d}t} = 10$ when $t = 5$:

$10 = 20\,\mathrm{e}^{-5k} \Rightarrow \mathrm{e}^{-5k} = \frac{1}{2}$

Taking natural logarithms:

$-5k = 1\mathrm{n}\frac{1}{2}$ or $5k = \ln 2$

$k = \frac{1}{5}1\mathrm{n}\,2 = 0.1386$ (4 d.p.)

So $V = \frac{100}{1\mathrm{n}\,2} - \frac{100}{1\mathrm{n}\,2}\left(\frac{1}{2}\right)^{\frac{t}{5}}$

When $t = 10$:

$V = \frac{100}{1\mathrm{n}\,2} - \frac{100}{1\mathrm{n}\,2} \times \frac{1}{4}$

$= \frac{75}{1\mathrm{n}\,2}$

$= 108.2$ (1 d.p.)

So the volume is 108 cm^3 (3 s.f.).

34 a $\frac{\mathrm{d}C}{\mathrm{d}t}$ is the rate of change of concentration.

The concentration is decreasing so the rate of change is negative.

So $-\frac{\mathrm{d}C}{\mathrm{d}t} \propto C$ or $\frac{\mathrm{d}C}{\mathrm{d}t} = -kC,$
where k is a positive constant of proportionality.

b Separating the variables:

$\int \frac{1}{C}\,\mathrm{d}C = -\int k\,\mathrm{d}t$

so $\ln C = -kt + \ln A,$
where 1n A is a constant.

So $\ln\frac{C}{A} = -kt$

$\frac{C}{A} = \mathrm{e}^{-kt}$

So the general solution is $C = A\mathrm{e}^{-kt}$.

c When $t = 0, C = C_0$ so $A = C_0$

So $C = C_0\,\mathrm{e}^{-kt}$

When $t = 4, C = \frac{1}{10}C_0$ so

$\frac{1}{10}C_0 = C_0\,\mathrm{e}^{-4k}$

$\mathrm{e}^{4k} = 10$

$4k = 1\mathrm{n}10$

$k = \frac{1}{4}1\mathrm{n}10$

35 $\left|\overrightarrow{PQ}\right| = \sqrt{(8-(-1)^2 + (-4-4)^2 + (k-6)^2}$

$= \sqrt{9^2 + (-8)^2 + (k-6)^2} = 7\sqrt{5}$

$81 + 64 + (k-6)^2 = 245$

$(k-6)^2 = 100$

$k - 6 = \pm 10$

$k = -4$ or $k = 16$

36 $\left|\overrightarrow{AB}\right| = \sqrt{1+36+16} = \sqrt{53}$

$\left|\overrightarrow{AC}\right| = \sqrt{25+4+9} = \sqrt{38}$

$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} = 6\mathbf{i} - 8\mathbf{j} - 7\mathbf{k}$

$\left|\overrightarrow{BC}\right| = \sqrt{36+64+49} = \sqrt{149}$

$\cos\angle BAC = \frac{53+38-149}{2\times\sqrt{53}\times\sqrt{38}} = -0.6462...$

$\angle BAC = 130.3°$ (1 d.p.)

37 a Let O be the fixed origin.

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = 10\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$

b $\left|\overrightarrow{PQ}\right| = \sqrt{100+25+4} = \sqrt{129}$

Unit vector in direction of $\overrightarrow{PQ}$

$= \frac{10}{\sqrt{129}}\mathbf{i} - \frac{5}{\sqrt{129}}\mathbf{j} - \frac{2}{\sqrt{129}}\mathbf{k}$

c $\cos\theta_z = \frac{-2}{\sqrt{129}} = -0.1761$

$\theta_z = 101.1°$ (1 d.p.)

37 d $\overrightarrow{AB} = 30\mathbf{i} - 15\mathbf{j} + 6\mathbf{k}$

There is no scalar, say m, for which $\overrightarrow{AB} = m\overrightarrow{PQ}$, so $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are not parallel.

38 $\overrightarrow{MN} = 10\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$

$$\left|\overrightarrow{MN}\right| = \sqrt{10^2 + 5^2 + 4^2} = \sqrt{141}$$

$$\overrightarrow{MP} = (k+2)\mathbf{i} - 2\mathbf{j} - 11\mathbf{k}$$

$$\left|\overrightarrow{MP}\right| = \sqrt{(k+2)^2 + 2^2 + 11^2}$$

$$= \sqrt{(k+2)^2 + 125}$$

$$\overrightarrow{NP} = (k-8)\mathbf{i} + 3\mathbf{j} - 7\mathbf{k}$$

$$\left|\overrightarrow{NP}\right| = \sqrt{(k-8)^2 + 3^2 + 7^2}$$

$$= \sqrt{(k-8)^2 + 56}$$

If $\left|\overrightarrow{MN}\right| = \left|\overrightarrow{MP}\right|$ then

$$\sqrt{141} = \sqrt{(k+2)^2 + 125}$$

$$(k+2)^2 = 16$$

$$k + 2 = \pm 4$$

$k = 2$ or $k = -6$

$\Rightarrow k = 2$ (since k is positive)

If $\left|\overrightarrow{MN}\right| = \left|\overrightarrow{NP}\right|$ then

$$\sqrt{141} = \sqrt{(k-8)^2 + 56}$$

$$(k-8)^2 = 85$$

So there are no integer solutions for k if $\left|\overrightarrow{MN}\right| = \left|\overrightarrow{NP}\right|$

If $\left|\overrightarrow{MP}\right| = \left|\overrightarrow{NP}\right|$ then

$$\sqrt{(k+2)^2 + 125} = \sqrt{(k-8)^2 + 56}$$

$$k^2 + 4k + 129 = k^2 - 16k + 122$$

$$20k = -7$$

So there are no positive solutions for k if $\left|\overrightarrow{MP}\right| = \left|\overrightarrow{NP}\right|$

So $k = 2$

39 $-6\mathbf{i} + 40\mathbf{j} + 16\mathbf{k} = 3p\mathbf{i} + (8 + qr)\mathbf{j} + 2pr\mathbf{k}$

Comparing coefficients of **i**:
$-6 = 3p \Rightarrow p = -2$

Comparing coefficients of **k**:
$16 = 2pr \Rightarrow pr = 8 \Rightarrow r = -4$

Comparing coefficients of **j**:
$40 = 8 + qr \Rightarrow qr = 32 \Rightarrow q = -8$

$p = -2, q = -8, r = -4$

40 a Particle is in equilibrium so
$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \mathbf{0}$

Comparing coefficients of **i**:
$a - 9 + 3 = 0 \Rightarrow a = 6$

Comparing coefficients of **j:**
$2 + 5 + b = 0 \Rightarrow b = -7$

Comparing coefficients of **k**:
$-4 + c + 5 = 0 \Rightarrow c = -1$

b $\mathbf{R} = \mathbf{F}_2 + \mathbf{F}_3$

$$= (-9+3)\mathbf{i} + (5-7)\mathbf{j} + (-1+5)\mathbf{k}$$

$$= (-6\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})\,\mathrm{N}$$

c $\mathbf{F} = m\mathbf{a} \Rightarrow -6\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} = 2\mathbf{a}$

Acceleration $\mathbf{a} = (-3\mathbf{i} - \mathbf{j} + 2\mathbf{k})\,\mathrm{m\,s^{-2}}$

d $|\mathbf{a}| = \sqrt{9 + 1 + 4} = \sqrt{14}\,\mathrm{m\,s^{-2}}$

Challenge

1 a $ay+x^2+4xy=y^2$

Differentiating with respect to x:

$$a\frac{\mathrm{d}y}{\mathrm{d}x}+2x+4\left(x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)=2y\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}(a+4x-2y)=-4y-2x$$

$$=\frac{-4y-2x}{a+4x-2y}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=0\Rightarrow x=-2y$$

Substituting for x in the original equation:

$$ay+4y^2-8y^2=y^2$$

$$ay-5y^2=0$$

$$y(a-5y)=0\Rightarrow y=0 \text{ or } y=\frac{a}{5}$$

When $y=0$, $x=-2y=0$

When $y=\frac{a}{5}$, $x=-2y=\frac{-2a}{5}$

So $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ at $(0,0)$ and at $\left(-\frac{2a}{5},\frac{a}{5}\right)$.

b $\frac{\mathrm{d}x}{\mathrm{d}y}=\frac{a+4x-2y}{-4y-2x}$

$$\frac{\mathrm{d}x}{\mathrm{d}y}=0\Rightarrow a+4x-2y=0\Rightarrow y=2x+\frac{a}{2}$$

Substituting for y in the original equation:

$$a\left(2x+\frac{a}{2}\right)+x^2+4x\left(2x+\frac{a}{2}\right)=\left(2x+\frac{a}{2}\right)^2$$

$$2ax+\frac{a^2}{2}+x^2+8x^2+2ax=4x^2+2ax+\frac{a^2}{4}$$

$$5x^2+2ax+\frac{a^2}{4}=0$$

$$\text{‘}b^2-4ac\text{’}=4a^2-\frac{20a^2}{4}=-a^2$$

$-a^2<0$ (as $a\neq 0$) so $5x^2+2ax+\frac{a^2}{4}=0$ has no solutions.

Hence $\frac{\mathrm{d}x}{\mathrm{d}y}\neq 0$ for all x.

2 $y=\sin x+2$ and $y=\cos 2x+2$

Curves intersect when
$\sin x+2=\cos 2x+2$

$$\begin{aligned}\sin x&=\cos 2x\\&=1-2\sin^2 x\end{aligned}$$
$$2\sin^2 x+\sin x-1=0$$
$$(2\sin x-1)(\sin x+1)=0$$

So $\sin x=\frac{1}{2}$ or $\sin x=-1$

So the intersections are at
$x=\frac{\pi}{6}$, $x=\frac{5\pi}{6}$ and $x=\frac{3\pi}{2}$

Shaded area up to $x=\frac{\pi}{6}$ is

$$\int_0^{\frac{\pi}{6}}\left(\cos 2x+2-(\sin x+2)\right)\mathrm{d}x$$
$$=\int_0^{\frac{\pi}{6}}(\cos 2x-\sin x)\mathrm{d}x$$
$$=\left[\frac{1}{2}\sin 2x+\cos x\right]_0^{\frac{\pi}{6}}$$
$$=\left(\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{2}\right)-(0+1)$$
$$=\frac{3\sqrt{3}}{4}-1$$

Shaded area between $x=\frac{\pi}{6}$ and $\frac{5\pi}{6}$ is

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}(\sin x-\cos 2x)\mathrm{d}x$$
$$=\left[-\cos x-\frac{1}{2}\sin 2x\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$
$$=\left(\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{4}\right)-\left(-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{4}\right)$$
$$=\frac{3\sqrt{3}}{2}$$

Shaded area between $x=\frac{5\pi}{6}$ and $\frac{3\pi}{2}$ is

$$\int_{\frac{5\pi}{6}}^{\frac{3\pi}{2}}(\cos 2x-\sin x)\mathrm{d}x$$
$$=\left[\frac{1}{2}\sin 2x+\cos x\right]_{\frac{5\pi}{6}}^{\frac{3\pi}{2}}$$
$$=(0+0)-\left(-\frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{2}\right)$$
$$=\frac{3\sqrt{3}}{4}$$

So the total shaded area is
$$\frac{3\sqrt{3}}{4}-1+\frac{3\sqrt{3}}{2}+\frac{3\sqrt{3}}{4}$$
$$=3\sqrt{3}-1$$

Exam-style practice: Paper 1

1 $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\frac{2\sec^2 t}{2\sin t\cos t}=\frac{1}{\sin t\cos^3 t}=\operatorname{cosec} t\sec^3 t$

2 **a** $2(7x-5)-6x<10x-7$

$\Rightarrow 14x-10-6x<10x-7$

$\Rightarrow -3<2x$

So $x>-\frac{3}{2}$

b $|2x+5|-3>0$

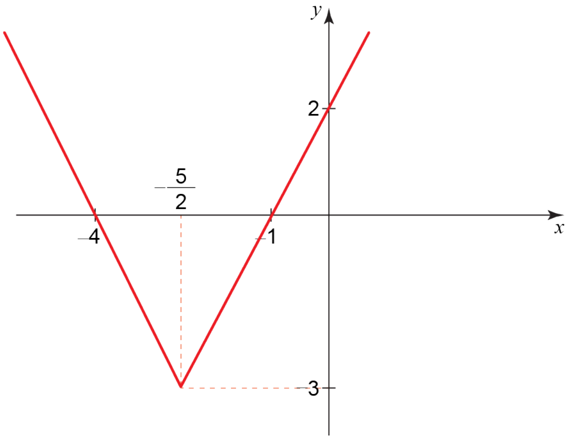
Solve $|2x+5|-3=0$

$x<-\frac{5}{2}$: $-(2x+5)-3=0$

$\Rightarrow 2x=-8\Rightarrow x=-4$

$x>-\frac{5}{2}$: $(2x+5)-3=0$

$\Rightarrow 2x=-2\Rightarrow x=-1$



From the graph, we see that the inequality holds when $x<-4$ or $x>-1$

c For both inequalities to hold, x must satisfy both $x>-\frac{3}{2}$ and $x<-4$ or $x>-1$ so the solution is $x>-1$

3 **a** $2x+y-3=0\Rightarrow y=3-2x$

Substitute this equation for y in the equation of the circle

$x^2+kx+(3-2x)^2+4(3-2x)=4$

$5x^2+kx-20x+17=0$

If this equation has solutions, the line will intersect the circle. As the equation is a positive quadratic, there will be no solutions, and the line will not intersect the circle, if

$5x^2+kx-20x+17>0$

b As there are no solutions to the equation

$5x^2+(k-20)x+17=0$

the discriminant must be less than zero

$\Rightarrow (k-20)^2-4(5)(17)<0$

$\Rightarrow k^2-40k+400-340<0$

$\Rightarrow k^2-40k+60<0$

Solve $k^2-40k+60=0$

Using the quadratic formula,

$$k=\frac{40\pm\sqrt{(-40)^2-4(1)(60)}}{2(1)}=20\pm2\sqrt{85}$$

Since $\mathrm{f}(k)=k^2-40k+60$ is a positive quadratic in k, the set of values of k for which $\mathrm{f}(k)$ is negative must be

$20-2\sqrt{85}<k<20+2\sqrt{85}$

4 Let $\mathrm{f}(\theta)=\cos\theta$

$$\begin{aligned}\mathrm{f}'(\theta)&=\lim_{h\to0}\frac{\mathrm{f}(\theta+h)-\mathrm{f}(\theta)}{h}\\&=\lim_{h\to0}\frac{\cos(\theta+h)-\cos\theta}{h}\\&=\lim_{h\to0}\frac{\cos\theta\cos h-\sin\theta\sin h-\cos\theta}{h}\\&=\lim_{h\to0}\left[\left(\frac{\cos h-1}{h}\right)\cos\theta-\left(\frac{\sin h}{h}\right)\sin\theta\right]\\&=-\sin\theta\end{aligned}$$

5 **a** Using the binomial expansion, the coefficient of x^2 in the expansion of $(3+px)^6$ is

$$\frac{6(6-1)}{2!}(3)^4 p^2 = 1215p^2$$

As $1215p^2 = 19440 \Rightarrow p^2 = 16$

So solutions are $p = 4$, $p = -4$

b The coefficient of x^5 is

$$\frac{6(6-1)(6-2)(6-3)(6-4)}{5!}3p^5 = 1215p^2$$

Since this negative, use $p = -4$, so the coefficient is

$$6 \times 3 \times (-4)^5 = -18\,432$$

6 First find the y-coordinate of R

$y = (2)^2 + 4(2) - 2 = 10$ so $R(2,10)$

To find the normal line to the curve at R, find the gradient at R

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x+4, \text{ so at } x = 2, \frac{\mathrm{d}y}{\mathrm{d}x} = 8$$

At R, the normal line will therefore have a gradient of $-\frac{1}{8}$

So the equation of the normal line at R is

$$(y-10) = -\frac{1}{8}(x-2) \Rightarrow y = -\frac{x}{8} + \frac{41}{4}$$

To find T, solve

$$-\frac{x}{8} + \frac{41}{4} = x^2 + 4x - 2$$

$$\Rightarrow 8x^2 + 33x - 98 = 0$$

In factorising this equation, remember that $x = 2$ is a solution

So $(x-2)(8x+49) = 0$

The normal also meets the curve at $x = -\frac{49}{8}$

And when $y = -\frac{-\frac{49}{8}}{8} + \frac{41}{4} = \frac{705}{64}$

Required coordinates are $\left(-\frac{49}{8}, \frac{705}{64}\right)$

7 **a** $u_1 = a, u_2 = ar = 96, S_\infty = 600$

$$a = \frac{96}{r}, S_\infty = \frac{\frac{96}{r}}{1-r} = 600$$

$$\Rightarrow 600 - 600r = \frac{96}{r}$$

$$\Rightarrow 600r^2 - 600r + 96 = 0$$

$$\Rightarrow 25r^2 - 25r + 4 = 0 \quad \text{(dividing by 24)}$$

b $25r^2 - 25r + 4 = 0$

Factorise $(5r - 1)(5r - 4) = 0$

Or use the quadratic formula

$$r = \frac{-(-25) \pm \sqrt{(-25)^2 - 4(4)(25)}}{2(25)}$$

$$= \frac{25 \pm 15}{50}$$

Solutions are $r = \frac{1}{5} = 0.2, r = \frac{4}{5} = 0.8$

c The larger value of r is $r = 0.8$. The corresponding value of a is

$$a = \frac{96}{0.8} = 120$$

d $S_n = \dfrac{a(1-r^n)}{1-r} > 599.9$

$$\Rightarrow \frac{120\left(1-(0.8)^n\right)}{1-0.8} > 599.9$$

$$\Rightarrow 1-(0.8)^n > \frac{599.9}{600}$$

$$\Rightarrow (0.8)^n < \frac{0.1}{600}$$

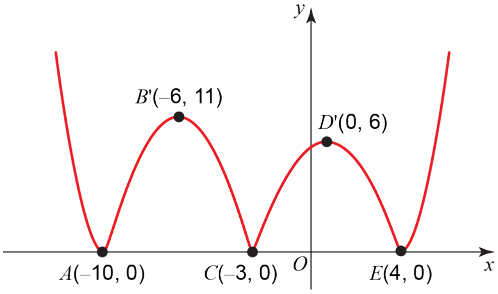
$$\Rightarrow n \ln 0.8 < \ln\frac{0.1}{600}$$

$$\Rightarrow n > \frac{\ln\frac{0.1}{600}}{\ln 0.8} \quad \text{(as ln0.8 is negative)}$$

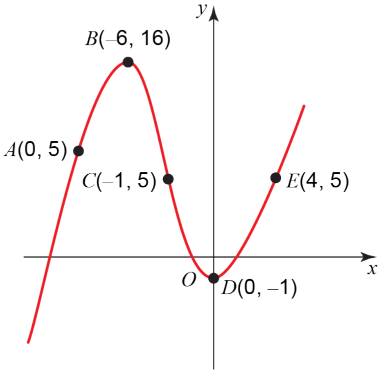
$\Rightarrow n > 38.986$ (3 d.p.)

So $n = 39$

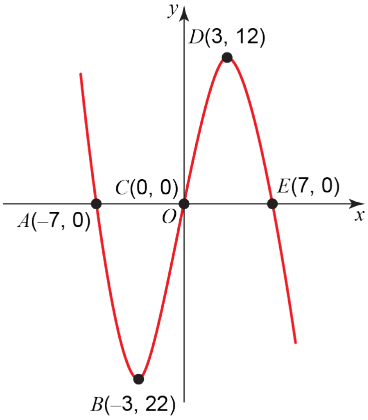
8 a Reflect graph of f(x) the x-axis in regions where f(x) < 0, i.e. $-10 < x < -3$ and $x > 4$



b First reflect the graph in the x-axis to obtain $y = -\mathrm{f}(x)$ then translate this graph by the vector $\begin{pmatrix}0\\5\end{pmatrix}$



c Translate by the vector $\begin{pmatrix}3\\0\end{pmatrix}$ to obtain $y = \mathrm{f}(x - 3)$ and then stretch in the y-direction by a scale factor of 2



9 $31 - 25\cos x = 19 - 12\sin^2 x$

$\Rightarrow 31 - 25\cos x = 19 - 12(1 - \cos^2 x)$

$\Rightarrow 12\cos^2 x + 25\cos x - 24 = 0$

$$\Rightarrow \cos x = \frac{-25 \pm \sqrt{(25)^2 - 4(12)(-24)}}{2(12)}$$

$= 0.7148, -2.7981$ (4 d.p.)

$\cos x = -2.798...$ has no solutions,

since $|\cos x| \leq 1$

So there are two solutions in the required interval

$\cos^{-1}(0.7148) = 0.77$ (2 d.p.)

and $2\pi - \cos^{-1}(0.7148) = 5.51$ (2 d.p.)

10 a Let the constant of proportionality be $-k$, where $k > 0$. Therefore

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -kV \Rightarrow V = Ce^{-kt}$$

where C is a constant

$V = V_0$ at $t = 0 \Rightarrow C = V_0$

So $V = V_0 e^{-kt}$

b $25000 = V_0 e^{-2k}$ (1)

$15000 = V_0 e^{-5k}$ (2)

Dividing equations (1) and (2)

$\frac{5}{3} = e^{3k} \Rightarrow k = \frac{1}{3}\ln\left(\frac{5}{3}\right)$

Substituting this value of k into (1)

$V_0 = 25000e^{\frac{2}{3}\ln\frac{5}{3}} = 35143.0$ (6 s.f.)

So $V_0 = 35\,100$ to the nearest hundred

c $V_0 e^{-kt} = 5000$

$$\Rightarrow e^{kt} = \frac{V_0}{5000}$$

$$\Rightarrow kt = \ln\left(\frac{V_0}{5000}\right)$$

$$\Rightarrow t = 3\frac{\ln\left(\frac{35143}{5000}\right)}{\ln\left(\frac{5}{3}\right)} = 11.45 \text{ years (2 d.p.)}$$

d k should be changed to a smaller value e.g. 0.1 (any value smaller than 0.17 acceptable)

11 a Apply the cosine rule to the triangles ABC and ACD to find $\angle BCA$ and $\angle ACD$

$$\cos(\angle BCA) = \frac{21^2 + 19^2 - 8^2}{2(21)(19)}$$
$$= 0.9248 \text{ (4 d.p.)}$$
$$\Rightarrow \angle BCA = 0.3903 \text{ rad (4 d.p.)}$$
$$\cos(\angle ACD) = \frac{14^2 + 21^2 - 11^2}{2(14)(21)}$$
$$= 0.8776 \text{ (4 d.p.)}$$
$$\Rightarrow \angle ACD = 0.5001 \text{ rad (4 d.p.)}$$

So $\angle BCD = \angle BCA + \angle ACD = 0.890$ rad

Now apply the cosine rule to triangle BCD

$$\cos(\angle BCD) = \frac{14^2 + 19^2 - |BD|^2}{2(14)(19)}$$
$$|BD| = \sqrt{14^2 + 19^2 - 2(14)(19)\cos(\angle BCD)}$$
$$= \sqrt{196 + 361 - 334.847} = 14.9 \text{ (1 d.p.)}$$

b The shortest distance between two points is a straight line, so any other route will be longer.

12 a $y = -0.01x^2 + 0.22x + 1.58$

$$= -0.01(x^2 - 22x - 158)$$
$$= -0.01((x-11)^2 - 279)$$
$$= 2.79 - 0.01(x-11)^2$$

b The ball reaches its highest point when its horizontal distance from the goal is 11 metres. Its maximum height is 2.79 metres.

c The ball is kicked when $y = 0$

$$2.79 - 0.01(x-11)^2 = 0$$
$$\Rightarrow (x-11)^2 = \frac{2.79}{0.01} = 279$$
$$\Rightarrow x - 11 = \pm 16.703$$

$x > 0$, so $x = 27.7$ m (1 d.p.)

d At $x = 0$, $y = 2.79 - 0.01(-11)^2 = 1.58$

As $1.5 < 1.58 < 2.44$, the ball will go not be saved by the keeper but it will go under the crossbar, so it will enter the goal

13 a Surface area of box

$$= 2x^2 + 2(2xh + xh) = 2x^2 + 6xh$$

Surface area of lid

$$= 2x^2 + 2(6x + 3x) = 2x^2 + 18x$$

Total surface area

$$= 4x^2 + 6xh + 18x = 5356$$

So $h = \dfrac{5356 - 18x - 4x^2}{6x}$

$$V = 2x^2h = \frac{2}{3}\left(2678x - 9x^2 - 2x^3\right)$$

b $V = \dfrac{2}{3}(2678x - 9x^2 - 2x^3)$

$$\Rightarrow \frac{dV}{dx} = \frac{2}{3}(2678 - 18x - 6x^2)$$

$\dfrac{dV}{dx} = 0$ at a stationary point so

$$6x^2 + 18x - 2678 = 0$$

Since $x > 0$

$$x = \frac{-18 + \sqrt{(18)^2 - 4(6)(-2678)}}{2(6)}$$
$$= 19.68 \text{ cm (2 d.p.)}$$

c $\dfrac{d^2V}{dx^2} = \dfrac{2}{3}(-18 - 12x)$

Since $x > 0$, $\dfrac{d^2V}{dx^2} < 0 \Rightarrow$ maximum

d $x = 19.68 \Rightarrow V = 22\,648.7 \text{ cm}^3$ (1 d.p.)

e From part **a**,

surface area of lid $= 2x^2 + 18x$

So percentage of cardboard in the lid is

$$\frac{2(19.68)^2 + 18(19.68)}{5356} \times 100$$
$$= 21.1\% \text{ (1 d.p.)}$$

Exam-style practice: Paper 2

1 $y = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$

Maximum at $(-2, 8)$

$\Rightarrow \frac{b}{2a} = 2 \Rightarrow b = 4a \quad (1)$

and $c - \frac{b^2}{4a} = 8 \quad (2)$

Passes through $(-4, 4)$

$\Rightarrow 16a - 4b + c = 4 \quad (3)$

$(3) - (2)$: $16a - 4b + \frac{b^2}{4a} = -4$

Substitute expression for b from (1) gives

$16a - 16a + 4a = -4$

$\Rightarrow a = -1 \Rightarrow b = 4a = -4$

From (3) $\Rightarrow c = 4 + 4b - 16a$

$= 4 + 4(-4) - 16(-1) = 4$

So $a = -1,\ b = -4,\ c = 4$

2 **a** l_1 passes through $P(6, 4)$ and $Q(0, 28)$

Gradient $= m_1 = \frac{28-4}{0-6} = -4$

So $(y-4) = (-4)(x-6)$

$\Rightarrow y = -4x + 28$

b Let l_2 have gradient m_2

Since l_1 and l_2 are perpendicular,

$m_1 m_2 = -1 \Rightarrow m_2 = \frac{1}{4}$

So $(y-4) = \frac{1}{4}(x-6)$

$\Rightarrow y = \frac{1}{4}x + \frac{5}{2}$

c R is positioned where l_2 crosses the x-axis

$\frac{1}{4}x + \frac{5}{2} = 0 \Rightarrow x = -10$

So $R(-10, 0)$

d ΔPQR is a right-angled triangle.

Area $= \frac{1}{2} \times (\text{base}) \times (\text{height})$

Using Pythagoras' theorem,

$|PQ| = \sqrt{(6)^2 + (24)^2} = \sqrt{612} = 2\sqrt{153}$

$|RP| = \sqrt{(16)^2 + (4)^2} = \sqrt{272} = 2\sqrt{68}$

Area $= \frac{1}{2}\left(2\sqrt{153}\right)\left(2\sqrt{68}\right)$

$= 2\sqrt{10\,404} = 204$

So the area of triangle PQR is 204 units^2

3 $\mathrm{f}(x) = \mathrm{e}^{3x} - 1, \quad x \in \mathbb{R}$

$y = \mathrm{e}^{3x} - 1$

$\Rightarrow y + 1 = \mathrm{e}^{3x}$

$\Rightarrow 3x = \ln(y+1)$

$\Rightarrow x = \frac{1}{3}\ln(y+1)$

So $\mathrm{f}^{-1}(x) = \frac{1}{3}\ln(x+1), \quad x > -1$

4 **a** The student did not apply the laws of logarithms correctly in moving from the first line to the second line:

$\log_a x + \log_a y = \log_a xy$

b $\log_4(x+3) + \log_4(x+4) = \frac{1}{2}$

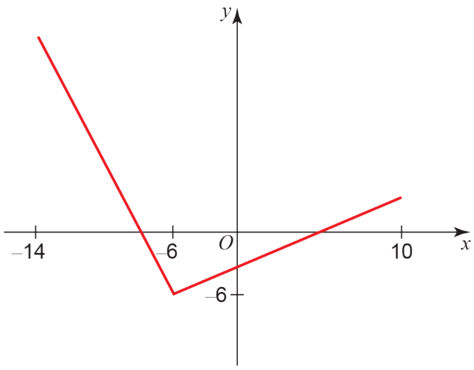
$\Rightarrow (x+3)(x+4) = 4^{\frac{1}{2}} = 2$

$\Rightarrow x^2 + 7x + 10 = (x+5)(x+2) = 0$

So $x = -2$

Note that $x = -5$ is not a solution since the function $\log_4 x$ is defined only on the domain $x > 0$, so $\log_4(x+4)$ is undefined when $x = -5$.

5 a



b $-6 \leq y \leq 18$

c Consider each section of the domain separately. There will be a solution in each section, because 18 > –3 > –6 and –6 < –3 < 2.

First find the equation of the line when $-14 \leqslant x \leqslant -6$ and solve for $y = -3$

$$(y-(-6)) = \frac{-6-18}{-6-(-14)}(x-(-6))$$

$$\Rightarrow y+6 = -3(x+6)$$

$$\Rightarrow y = -3x-24$$

$$-3 = -3a-24 \Rightarrow a = -7$$

Now find the equation of the line when $-6 \leqslant x \leqslant 10$ and solve for $y = -3$

$$(y-(-6)) = \frac{2-(-6)}{10-(-6)}(x-(-6))$$

$$\Rightarrow y+6 = \frac{1}{2}(x+6)$$

$$\Rightarrow y = \frac{1}{2}x-3$$

$$-3 = \frac{1}{2}a-3 \Rightarrow a = 0$$

Solutions are $a = -7$, $a = 0$

6 a $f(x) = x^3 - kx^2 - 10x + k$

$(x+2)$ is a factor of $f(x)$ so $f(-2) = 0$

$$\Rightarrow f(-2) = (-2)^3 - k(-2)^2 - 10(-2) + k$$
$$= -8 - 4k + 20 + k = 0$$

$$\Rightarrow 3k = 12 \Rightarrow k = 4$$

b $x^3 - 4x^2 - 10x + 4 = 0$

First take out the known factor $(x + 2)$

$$\Rightarrow (x+2)(x^2-6x+2) = 0$$

So $x = -2$ or $x^2 - 6x + 2 = 0$

$$x^2 - 6x + 2 = 0$$

$$\Rightarrow (x-3)^2 - 9 + 2 = 0$$

$$\Rightarrow (x-3)^2 = y \Rightarrow$$

$$x = 3 \pm \sqrt{7}$$

Solutions are $x = -2$, $x = 3 + \sqrt{7}$ and $x = 3 - \sqrt{7}$

7 a Area $= \frac{1}{2}(x-3)(x-10)\sin 30° = 11$

$$\Rightarrow \frac{1}{2}(x-3)(x-10) \times \frac{1}{2} = 11$$

$$\Rightarrow (x-3)(x-10) = 44$$

$$\Rightarrow x^2 - 13x + 30 = 44$$

$$\Rightarrow x^2 - 13x - 14 = 0$$

b $x^2 - 13x - 14 = (x-14)(x+1) = 0$

So $x = 14$, $x = -1$, but $x > 3$ as the lengths of the sides of this triangle must be positive. So solution is $x = 14$.

8 a $x = 6\sin t + 5 \Rightarrow \sin t = \frac{x-5}{6}$

$$y = 6\cos t - 2 \Rightarrow \cos t = \frac{y+2}{6}$$

Since $\sin^2 t + \cos^2 t = 1$,

$$\left(\frac{x-5}{6}\right)^2 + \left(\frac{y+2}{6}\right)^2 = 1$$

$$\Rightarrow (x-5)^2 + (y+2)^2 = 36$$

So $h = -5$, $k = 2$, $c = 36$

8 b $c = 36 = (\text{radius})^2 \Rightarrow \text{radius} = 6$

t parameterises the circle and takes values

$$-\frac{\pi}{3} \leq t < \frac{3\pi}{4}$$

So the angle that subtends the arc is

$$\frac{3\pi}{4} - \left(-\frac{\pi}{3}\right) = \frac{13\pi}{12} = \theta$$

So C is an arc of radius 6, and its length is $\frac{\theta}{2\pi} \times (\text{circumference of circle radius 6})$

$$\text{Length of } C = \frac{1}{2\pi} \times \frac{13\pi}{12} \times 2\pi \times 6 = \frac{13\pi}{2}$$

9 a $$\frac{4x^2 + 7x}{(x-2)(x+4)} = A + \frac{B}{x-2} + \frac{C}{x+4}$$

So $4x^2 + 7x$

$$= A(x-2)(x+4) + B(x+4) + C(x-2)$$

Set $x = 2$: $30 = 6B \Rightarrow B = 5$

Set $x = -4$: $36 = -6C \Rightarrow C = -6$

Compare coefficients of $x^2 \Rightarrow A = 4$

So $A = 4,\ B = 5,\ C = -6$

b $$\frac{4x^2 + 7x}{(x-2)(x+4)} = 4 + 5(x-2)^{-1} - 6(x+4)^{-1}$$

So to find the expansion as far as the term in x^2, only need to find the expansions of $(x-2)^{-1}$ and $(x+4)^{-1}$ as far as the term in x^2

$$(x-2)^{-1} = -\frac{1}{2}\left(1 - \frac{x}{2}\right)^{-1}$$

$$= -\frac{1}{2}\left(1 + (-1)\left(-\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(-\frac{x}{2}\right)^2 + \dots\right)$$

$$= -\frac{1}{2} - \frac{1}{4}x - \frac{1}{8}x^2 + \dots$$

$$(x+4)^{-1} = \frac{1}{4}\left(1 + \frac{x}{4}\right)^{-1}$$

$$= \frac{1}{4}\left(1 + (-1)\left(\frac{x}{4}\right) + \frac{(-1)(-2)}{2!}\left(\frac{x}{4}\right)^2 + \dots\right)$$

$$= \frac{1}{4} - \frac{1}{16}x + \frac{1}{64}x^2 + \dots$$

So $\frac{4x^2 + 7x}{(x-2)(x+4)}$

$$= 4 + 5\left(-\frac{1}{2} - \frac{1}{4}x - \frac{1}{8}x^2 + \dots\right)$$

$$-6\left(= \frac{1}{4} - \frac{1}{16}x + \frac{1}{64}x^2 + ..\right)$$

$$= \left(4 - \frac{5}{2} - \frac{3}{2}\right) + \left(-\frac{5}{4} + \frac{3}{8}\right)x$$

$$+ \left(-\frac{5}{8} - \frac{3}{32}\right)x^2 + \dots$$

$$= -\frac{7}{8}x - \frac{23}{32}x^2 \dots$$

10 $\overrightarrow{MN}=\overrightarrow{MB}+\overrightarrow{BN}=\frac{1}{2}\overrightarrow{OB}+\frac{1}{2}\overrightarrow{BA}$

So $\overrightarrow{MN}=\frac{1}{2}\mathbf{b}+\frac{1}{2}(-\mathbf{b}+\mathbf{a})=\frac{1}{2}\mathbf{a}$

Therefore $\overrightarrow{OA}$ and $\overrightarrow{MN}$ are parallel

and $\overrightarrow{MN}=\frac{1}{2}\overrightarrow{OA}$ as required

11 a At $x=\frac{\pi}{2}$,

$$y=\left(\frac{\pi}{2}\right)^2\left(\sin\frac{\pi}{2}+\cos\frac{\pi}{2}\right)$$

$$=2.46740 \text{ (5 d.p.)}$$

b $\int_a^b y\mathrm{d}x\approx\frac{h}{2}(y_0+2(y_1+y_2+\ldots+y_{n-1})+y_n)$

So $\int_0^{\frac{3\pi}{4}} y\mathrm{d}x$

$$\approx\frac{h}{2}(y_0+2(y_1+y_2+\ldots+y_5)+y_6)$$

$$=\frac{1}{2}\left(\frac{\frac{3\pi}{4}-0}{6}\right)2(0.20149+0.87239$$

$$+1.81340+2.46740+2.08648)$$

$$=2.922 \quad \text{(3d.p.)}$$

c Use integration by parts twice. First let

$u=x^2, \frac{\mathrm{d}v}{\mathrm{d}x}=\sin x+\cos x$

$\Rightarrow v=\sin x-\cos x$

$$\int_0^{\frac{3\pi}{4}} x^2(\sin x+\cos x)$$

$$=\left[x^2(\sin x-\cos x)\right]_0^{\frac{3\pi}{4}}$$

$$-2\int_0^{\frac{3\pi}{4}} x(\sin x-\cos x)\mathrm{d}x$$

Use integration by parts again, letting

$u=x, \frac{\mathrm{d}v}{\mathrm{d}x}=\sin x-\cos x$

$\Rightarrow v=\sin x+\cos x$

This gives

$$\int_0^{\frac{3\pi}{4}} x^2(\sin x+\cos x)$$

$$=\left(\frac{3\pi}{4}\right)^2\left(\sqrt{2}\right)$$

$$-2\left\{-\left[x(\sin x+\cos x)\right]_0^{\frac{3\pi}{4}}\right.$$

$$\left.+\int_0^{\frac{3\pi}{4}}(\sin x+\cos x)\mathrm{d}x\right\}$$

$$=\left(\frac{9\pi^2\sqrt{2}}{16}\right)-2\left\{0+\left[\sin x-\cos x\right]_0^{\frac{3\pi}{4}}\right\}$$

$$=\left(\frac{9\pi^2\sqrt{2}}{16}\right)-2\left(\sqrt{2}+1\right)$$

$$=3.023 \quad \text{(3d.p.)}$$

d $\frac{3.023-2.922}{3.023}\times 100=3.3\% \quad \text{(1d.p.)}$

12 a $u_n=a+(n-1)d$

$a=1000,\ d=150$

So $u_{18}=1000+(18-1)(150)=3550$

In the 18th year Ruth saves £3550

b $S_n=\frac{n}{2}\left(2a+(n-1)d\right)$

$S_{18}=\frac{18}{2}(2(1000)+(18-1)(150))=40950$

So in 18 years, the total amount that Ruth will have saved is £40 950

12 c The sequence is now geometric with $a = 1000,\ r = 1.1$

$$S_{18} = \frac{1000(1-(1.1)^{18})}{1-1.1} = 45599.17313$$

So after 18 years, Ruth will have saved £45 599.17 (2 d.p.) under this new scheme

13 a $R\cos(x-\alpha) = R\cos x\cos\alpha + R\sin x\sin\alpha$

$R\cos\left(x-\alpha\right) = 0.09\cos x + 0.4\sin x$

$R\cos\alpha = 0.09,\ R\sin\alpha = 0.4$

$\Rightarrow R^2 = (0.09)^2 + \left(0.4\right)^2$

(as $\sin^2\alpha + \cos^2\alpha \equiv 1$)

So $R = \sqrt{(0.09)^2 + \left(0.4\right)^2} = 0.41 \quad (R > 0)$

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{0.4}{0.09} = \frac{40}{9}$$

$$\Rightarrow \alpha = \tan^{-1}\left(\frac{40}{9}\right) = 1.3495 \text{ rad (4 d.p.)}$$

So $R = 0.41,\ \alpha = 1.3495$

b Use part **a** to write equation as

$$h = \frac{16.4}{0.41\cos\left(\frac{t}{2}-\alpha\right)}$$

$$\Rightarrow h = \frac{40}{\cos\left(\frac{t}{2}-\alpha\right)}$$

So the minimum value of h occurs when

$$\frac{t}{2} - \alpha = 0 \Rightarrow t = 2\alpha$$

$\Rightarrow t = 2\times 1.3495 = 2.70$ seconds (2 d.p.)

$$h = \frac{40}{\cos 0} = 40\,\text{cm}$$

c $$h = \frac{40}{\cos\left(\frac{t}{2}-\alpha\right)} = 100$$

$$\Rightarrow \cos\left(\frac{t}{2}-\alpha\right) = \frac{2}{5}$$

This has two solutions in the interval

$$-1.3495 \leqslant \frac{t}{2} - 1.3496 \leqslant 1.3505$$

$$\frac{t}{2} - \alpha = 1.1593, -1.1593$$

$t = 2\times(1.1593 + 1.3495) = 5.02$ seconds

$t = 2\times(-1.1593 + 1.3495) = 0.38$ seconds

14 a $\mathrm{h}(t) = -10\mathrm{e}^{-0.3(t-6.4)} - 10\mathrm{e}^{0.8(t-6.4)} + 70$

$\mathrm{h}'(t) = -10(-0.3)\mathrm{e}^{-0.3(t-6.4)} - 10(0.8)\mathrm{e}^{0.8(t-6.4)}$

$\Rightarrow \mathrm{h}'(t) = 3\mathrm{e}^{-0.3(t-6.4)} - 8\mathrm{e}^{0.8(t-6.4)}$

b From part **a**, when $\mathrm{h}'(t) = 0$

$$\frac{3}{8}\mathrm{e}^{-0.3(t-6.4)} = \mathrm{e}^{0.8(t-6.4)}$$

$$\Rightarrow \ln\left(\frac{3}{8}\mathrm{e}^{-0.3(t-6.4)}\right) = 0.8(t-6.4)$$

$$\Rightarrow \frac{5}{4}\ln\left(\frac{3}{8}\mathrm{e}^{-0.3(t-6.4)}\right) = t - 6.4$$

$$\Rightarrow t = \frac{5}{4}\ln\left(\frac{3}{8}\mathrm{e}^{-0.3(t-6.4)}\right) + 6.4$$

c $$t_{n+1} = \tfrac{5}{4}\ln\left(\frac{3\mathrm{e}^{-0.3(t_n-6.4)}}{8}\right) + 6.4$$

$$t_1 = \tfrac{5}{4}\ln\left(\frac{3\mathrm{e}^{-0.3(5-6.4)}}{8}\right) + 6.4 = 5.6990$$

$$t_2 = \tfrac{5}{4}\ln\left(\frac{3\mathrm{e}^{-0.3(t_1-6.4)}}{8}\right) + 6.4 = 5.4369$$

$$t_3 = \tfrac{5}{4}\ln\left(\frac{3\mathrm{e}^{-0.3(t_2-6.4)}}{8}\right) + 6.4 = 5.5351$$

$$t_4 = \tfrac{5}{4}\ln\left(\frac{3\mathrm{e}^{-0.3(t_3-6.4)}}{8}\right) + 6.4 = 5.4983$$

All answers are to 4 decimal places.

14 d $h'(5.5075) = 0.000360$ (6 d.p.)

$h'(5.5085) = -0.000702$ (6 d.p.)

The sign change implies slope change,
which implies a turning point at
$t = 5.508$ (3 d.p.)